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Maximal Rank in Elliptic Curve $y^2 = x^3 - px$ II

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Abstract

Set E_{-p} as an elliptic curve $y^2 = x^3 - px$ where p is a prime then, we shall regard maximal rank with $p = At^4 \pm Bt^3u \pm Ctu^3 + Dt^2u^2 + Fu^4$ and $A > 1$. And some examples will be submitted.

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1 Introduction

In [5], the rank of elliptic curve $y^2 = x^3 - px$ was treated where primes were the forms $p = t^4 + 4t^3u + 4tu^3 + 6t^2u^2 + 2u^4$ and $p = t^4 - 4t^3u - 4tu^3 + 6t^2u^2 + 2u^4$. Its forms were different from general forms $p = Hu^8 + Iu^4v^4 + Kv^8$, $p = Hu^4 + Iu^2v^2 + Kv^4$. The crucial factor determines the primes of forms was bigness of coefficient A in $p = At^4 \pm Bt^3u \pm Ctu^3 + Dt^2u^2 + Fu^4$. And this is originally correlated to quartic of an integer M in solution (M, e, N). Our concern is looking for more consequences of primes in $p = At^4 \pm Bt^3u \pm Ctu^3 + Dt^2u^2 + Fu^4$. Henceforth, we pursue integers whose quartic is bigger than 1. In this paper, we shall investigate the forms of primes as $p = At^4 \pm Bt^3u \pm Ctu^3 + Dt^2u^2 + Fu^4$ where A is larger than 1. Namely, we succeed the consequences of [5]. And for this we should fix the coefficient F as 2 this is because as pursing bigger value in A other one must be maintained with unchanged which will give ease to find examples of primes. As we did in [5] we only have to regard the solvability of two relating equations $2)N^2 = -M^4 + pe^4$ for Γ and $2)N^2 = 2M^4 + 2pe^4$ for $\bar{\Gamma}$. The notations $w.i.t.u.1$ and $r4.4$ are in [4] and [3] respectively.

2 Prime with Rank 2

Our aim is finding the solutions of equations $2)N^2 = -M^4 + pe^4$ for Γ and $2)N^2 = 2M^4 + 2pe^4$ for $\bar{\Gamma}$.

Theorem 2.1. Consider prime as $p = 81t^4 \pm 108t^3u \pm 12tu^3 + 54t^2u^2 + 2u^4$ $w.i.t.u.$ 1 and $p \equiv 1(mod\ 8)$ in $E_{-p}: y^2 = x^3 - px$ then, it deduces that

$$rank(E_{-(81t^4 \pm 108t^3u \pm 12tu^3 + 54t^2u^2 + 2u^4)}(Q)) = 2.$$

Proo$f$. We regard first case where prime is $p = 81t^4 + 108t^3u + 12tu^3 + 54t^2u^2 + 2u^4$.
There are relating equations

$2)N^2 = -M^4 + (81t^4 + 108t^3u + 12tu^3 + 54t^2u^2 + 2u^4)e^4$ for Γ and

$2)N^2 = 2M^4 + 2(81t^4 + 108t^3u + 12tu^3 + 54t^2u^2 + 2u^4)e^4$ for $\bar{\Gamma}$.

Consider equation for Γ.
The square term $81t^4$ exists in coefficient of e^4.
Owing to term $2u^4$ if we take $M = u$ then, we have that

$$-u^4 + 2u^4 = u^4.$$

Thus, we select components of N as $9t^2$ and u^2.
There are non-square terms

$$108t^3u \text{ and } 12tu^3.$$

These are factored as

$$108t^3u = 2 \cdot 9t^2 \cdot 6tu \text{ and}$$

$$12tu^3 = 2 \cdot 6tu \cdot u^2.$$

Common factor $6tu$ is in both factorizations.
And the term $54t^2u^2$ can be expressed as

$$2 \cdot 9t^2u^2 + 36t^2u^2.$$

Here $2 \cdot 9t^2u^2$ can be derived from $9t^2$ and u^2.
If we decide last component as $6tu$ then, from squaring of this there comes $36t^2u^2$.
Therefore, we obtain as

$$N = 9t^2 + 6tu + u^2.$$

And this selection is done under the supposition that $e = 1$.
Accordingly, the solution is induced as

$$(u, 1, 9t^2 + 6tu + u^2).$$

Besides, we can get one more solution.
In above $9t^2 + 6tu + u^2$ is square of $3t + u$.
Thus, quartic of $3t + u$ is given as

$$81t^4 + 108t^3u + 12tu^3 + 54t^2u^2 + u^4.$$

Hence, taking M and e as $3t + u$ and 1 then, we are confronted with

$$-(3t + u)^4 + (81t^4 + 108t^3u + 12tu^3 + 54t^2u^2 + 2u^4) = u^4.$$

Thereby, the solution is deduced as

$$(3t + u, 1, u^2).$$

And it implies that $\#\alpha(\Gamma) = 4$.
In the next step, we regard second equation.
The square $4u^4$ is found in coefficient of e^4.
In numerical value $2M^4 + 162t^4e^4$ take $e = 1$.
From $2(M^4 + 81t^4)$ if an integer M is selected as $3t$ then, we are faced with

$$2(81t^4 + 81t^4) = 324t^4.$$

Thus, two squares $324t^4$ and $4u^4$ are given.
The remaining terms are

$$216t^3u \text{ and } 24tu^3 \text{ and } 108t^2u^2.$$

Now by common factor 4 we obtain the multiplication

$$4(81t^4 + 54t^3u + 6tu^3 + 27t^2u^2 + u^4).$$

It is square of $2(9t^2 + 3tu + u^2)$.
Whence, the triple

$$(3t, 1, 2(9t^2 + 3tu + u^2))$$

satisfies the solution.
For this reason, there induced that $r4.4$.

On this account, rank of $E_{-(81t^4+108t^3u+12tu^3+54t^2u^2+2u^4)}$ is gotten as 2.
Second, we treat the case where prime is such that $p = 81t^4 - 108t^3u - 12tu^3 + 54t^2u^2 + 2u^4$.
Hence, the relating equations are gotten as follows:

$2)N^2 = -M^4 + (81t^4 - 108t^3u - 12tu^3 + 54t^2u^2 + 2u^4)e^4$ for Γ and

$2)N^2 = 2M^4 + 2(81t^4 - 108t^3u - 12tu^3 + 54t^2u^2 + 2u^4)e^4$ for $\bar{\Gamma}$.

The difference with first treatment is symbols of terms for t^3u and tu^3.
Whence, it only needs to change the symbols in triples.
Thus, solutions of equation for Γ are deduced as

$$(u, 1, 9t^2 - 6tu + u^2) \text{ and}$$

$$(3t - u, 1, u^2).$$

Thereby, we take that $\#\alpha(\Gamma) = 4$.
And that of 2) for $\bar{\Gamma}$ is given as

$$(3t, 1, 2(9t^2 - 3tu + u^2)).$$

Consequentially, we face the conclusion $r4.4$.
Lastly, we have the consequence

$$rank(E_{-(81t^4-108t^3u-12tu^3+54t^2u^2+2u^4)}(Q)) = 2. \qquad \square$$

As we said in previous section the crucial factor that decides the form $p = 81t^4 \pm 108t^3u \pm 12tu^3 + 54t^2u^2 + 2u^4$ was coefficient 81 which is quartic of number 3.

Remark 2.2. It is possible to get more consequences of rank 2 when primes are the forms $p = At^4 \pm Bt^3u \pm Ctu^3 + Dt^2u^2 + Fu^4$.

Remark 2.3. In curve $E_{4p}: y^2 = x^3 + 4px$ we got the rank 2 where prime is such that $p = t^4 + 16$ with an odd integer t in [2]. If we apply above form $p = At^4 \pm Bt^3u \pm Ctu^3 + Dt^2u^2 + Fu^4$ to E_{4p} then, there also can be derived rank 2.

3 Examples

Now we consider examples of theorem 2.1.
Prime is checked by [1].

There are examples as follows:

$$(p, t, u)_{+108,+12}:$$

(257, 1, 1) and (641, 1, 2) and (2657, 1, 4) and

(14657, 3, 2).

$$(p, t, u)_{-108,-12}:$$

(17, 1, 1) and (2417, 3, 2) and (130337, 7, 2).

In above, we needed to regard examples (257, 1, 1), (17, 1, 1), (2417, 3, 2). Each one is result of $p = 81t^4 + 108t^3u + 12tu^3 + 54t^2u^2 + 2u^4$ and $p = 81t^4 - 108t^3u - 12tu^3 + 54t^2u^2 + 2u^4$ respectively. Simultaneously it is result of $p = t^4 - 4t^3u - 4tu^3 + 6t^2u^2 + 2u^4$ in [5]. Besides, prime 17 is also the consequence of $p = t^4 + 4t^3u + 4tu^3 + 6t^2u^2 + 2u^4$ in [5].

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