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Reproducing Kernel Hilbert Spaces (RKHS)

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Abstract

The purpose of this paper is to give some general outlines, characterizations/ properties and basic examples of well-known RKHS on the unit disc $\mathbb{D}$. The ideas are gleaned from different sources, primarily [BK], [KT], [TT1], [TT2], [HKZ] and put together for quick references. RKHS are extensively used in the proof of the Corona Theorems; for Example, [KT], [TT1] and [TT2], as such this paper focus purely on the points of views of RKHS, as it is used on those papers.

1. Preliminaries

Notations, in this paper:

- $\mathbb{D}$, denotes the open unit disc
- $Hol(\mathbb{D})$, denotes the set holomorphic functions on $\mathbb{D}$
- $\mathcal{A}_\alpha(\mathbb{D})$ or $\mathcal{A}$, denotes Hilbert Spaces of holomorphic functions of weight α on $\mathbb{D}$

At the beginning of the 20$^{\text{th}}$ century the development of Integral Equations and Functional Analysis by David Hilbert (1862 – 1943, [CR]) [DH1], [DH2], and the concepts of Orthogonal Expansions, by Erhard Schmidt (1876 -1959) [SE] lays the foundation for the development of Reproducing Kernel Hilbert Spaces. By the middle of the 20th century Nachman Aronszajn (1907-1980) on his paper, "Theory of Reproducing Kernel" [NA], systematically constructed and formally defined Reproducing Kernel Hilbert Spaces.

2. Definition and Properties of RKHS

Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real numbers and complex numbers respectively. We will use $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$. Given a non - empty set E, let $\mathcal{F}(\mathrm{E},\mathbb{F})$ denotes the set of all functions from E to $\mathbb{F}$.

Theorem 2.1: $\mathcal{F}(E,\mathbb{F})$ with the usual operations of addition of functions and scalar multiplication is a vector space over $\mathbb{F}$.

Definition 2.1: Given a set $\mathrm{E} \neq \emptyset$ (from now on set E is assumed nonempty), then $\mathcal{H} = \mathcal{H}(\mathrm{E})$ is a reproducing kernel Hilbert space on E over $\mathbb{F}$ if:

i. $\mathcal{H}$ is a subspace of $\mathcal{F}(\mathrm{E},\mathbb{F})$
ii. $\mathcal{H}$ endowed with an inner product $\langle\ ,\ \rangle$ is a Hilbert space
iii. For each $x \in \mathrm{E}$, the linear evaluation functional $\ell_x: \mathcal{H} \to \mathbb{F}$, given by $\ell_x(f) = f(x)$, is bounded.

Theorem 2.2: Let $\mathcal{H}$ be a RKHS on E, then for each $y \in \mathrm{E}$, there exists a unique $k_y \in \mathcal{H}$ such that $\ell_y(f) = \langle f,\ k_y\rangle$.
Proof: Consequence of Riesz Representation Theorem ■

Definition 2.2: The function k_y is called the Reproducing Kernel for point y.

Definition 2.3: The two-variable function $\mathrm{K}: \mathrm{E} \times \mathrm{E} \to \mathbb{F}$ defined by $\mathrm{K}(x,y) = k_y(x)$ is called the Reproducing Kernel for the Hilbert space $\mathcal{H}$.

2.1. Properties of RKHS

Property 2.1.1: The reproducing kernel of a Hilbert space is unique.

Proof: Let $\mathcal{H}$ be a Hilbert space and $\mathrm{K}: \mathrm{E} \times \mathrm{E} \to \mathbb{F}$ be a reproducing kernel for $\mathcal{H}$. By Reisz representation for each $y \in E$ there is a unique $k_y \in \mathcal{H}$; hence, K is unique. ■

Property 2.1.2: Let $\mathcal{H}$ be a RKHS on E with reproducing kernel $\mathrm{K}(x,y)$ then

i. $\mathrm{K}(x,y) = \langle k_y, k_x \rangle$
ii. $\mathrm{K}(x,x) \geq 0$
iii. $\mathrm{K}(x,y) = \overline{\mathrm{K}(y,x)}$
iv. $|\mathrm{K}(x,y)|^2 \leq \mathrm{K}(x,x)\mathrm{K}(y,y)$

Proof:

i. $k_y \in \mathcal{H}$ and $k_y(x) = \langle k_y, k_x \rangle$
ii. $\mathrm{K}(x,x) = \langle k_x, k_x \rangle$
iii. $\langle k_y, k_x \rangle = \overline{\langle k_x, k_y \rangle}$
iv. $\left|\langle k_y, k_x \rangle\right| \leq \|k_x\| \left\|k_y\right\|$ and $\|k_x\|^2 = \langle k_x, k_x \rangle$ ■

Property 2.1.3: Suppose $\mathcal{H}$ is a RKHS on set E with reproducing kernel $\mathrm{K}(x,y)$. Let $\{e_n\}_0^\infty$ be an orthonormal basis of $\mathcal{H}$. Then for each $y \in$ E, $k_y \in \mathcal{H}$ and it follows that

i. $k_y = \sum_0^\infty \alpha_n e_n$, where $\sum_0^\infty |\alpha_n|^2 < \infty$
ii. $\langle e_n, k_y \rangle = \overline{\alpha_n} \leftrightarrow \alpha_n = \langle k_y,\ e_n \rangle$
iii. $\sum_0^\infty |e_n(y)|^2 < \infty$

Proof

i. $\{e_n\}_0^\infty$ is an orthonormal basis for the Hilbert space $\mathcal{H}$
ii. $\langle e_n, k_y \rangle = \langle e_n,\ \sum_0^\infty \alpha_j e_j \rangle = \overline{\alpha_n}$
iii. $\alpha_n = \overline{\langle e_n, k_y \rangle} = \overline{e_n(y)}$ ■

Property 2.1.4: Suppose $\mathcal{H}$ is a Hilbert space of functions on E with orthonormal basis $\{e_n\}_0^\infty$. If $\forall y \in$ E, $\sum_0^\infty |e_n(y)|^2 < \infty$, then $\mathcal{H}$ is a reproducing RKHS with reproducing kernel given by: $\mathrm{K}(x,y) = \sum_0^\infty e_n(x)\, \overline{e_n(y)}$.

Proof: Converse of Property 2.1.3

Let $y \in$ E and define: $h_y = \sum_0^\infty \overline{e_n(y)}\, e_n$, clearly $h_y \in \mathcal{H}$.

Let $f \in \mathcal{H}$, then $f = \sum_0^\infty \langle f, e_n \rangle e_n$ and

$$\begin{aligned} \langle f, h_y \rangle &= \langle\, \sum_0^\infty \langle f, e_n \rangle e_n\,, \sum_0^\infty \overline{e_j(y)}\, e_j \,\rangle \\ &= \sum_0^\infty \langle f, e_n \rangle\, e_n(y) \\ &\leq \|f\| \left\|h_y\right\| \end{aligned}$$

That means evaluation at y is a bounded linear functional; hence $\mathcal{H}$ is a RKHS with reproducing kernel at y, $h_y = k_y$ (by uniqueness). ■

3. RKHS on the Unit Circle, $\mathbb{D}$

Definition 3.1: Consider the sequence of positive real numbers $\{\alpha_n\}_0^\infty$ such that $\liminf_{n\to\infty} (\alpha_n)^{1/n} = 1$, then define $H_{\alpha_n}(\mathbb{D})$ by:

$$H_{\alpha_n}(\mathbb{D}) = \{ f \in Hol(\mathbb{D}): f(z) = \sum_0^\infty f_n\, z^n \text{ and } \sum_0^\infty |f_n|^2 \alpha_n < \infty\}$$

If f and g are in, $H_{\alpha_n}(\mathbb{D})$ then define $\langle f, g\rangle = \sum_0^\infty f_n\, \overline{g_n}\, \alpha_n$ and the norm, $\|f\|_{\alpha_n}^2$ by

$$\|f\|_{\alpha_n}^2 = \sum_0^\infty |f_n|^2 \alpha_n$$

Property 3.2: $H_{\alpha_n}(\mathbb{D})$ is a RKHS on $\mathbb{D}$.

Proof: Let $z \in \mathbb{D}$ and $e_n(z) = \frac{1}{\sqrt{\alpha_n}} z^n$.

Then $\{e_n\}_0^n$ forms an orthonormal basis for $H_{\alpha_n}(\mathbb{D})$ and

$$\sum_0^\infty |e_n(z)|^2 = \sum_0^\infty \frac{1}{\alpha_n} |z|^{2n} < \infty.$$

Thus, $H_{\alpha_n}(\mathbb{D})$ is a RKHS on $\mathbb{D}$ with reproducing Kernel given by

$$\begin{aligned} \mathrm{K}(z, w) &= \sum_0^\infty e_n(z)\, \overline{e_n(w)} \\ &= \sum_0^\infty (1/\alpha_n)(z\overline{w})^n \quad \blacksquare \end{aligned}$$

Definition 3.2: let $\alpha \in \mathbb{R}$, $\alpha_n = (n+1)$, then define, the weighted holomorphic spaces of weight α, $\mathcal{A}_\alpha(\mathbb{D})$ or simply $\mathcal{A}_\alpha$, by

$$\mathcal{A}_\alpha(\mathbb{D}) = \{ f \in Hol(\mathbb{D}): f(z) = \sum_0^\infty f_n\, z^n \text{ and } \sum_0^\infty (n+1)^\alpha |f_n|^2 < \infty\}$$

and the norm of $f \in \mathcal{A}_\alpha(\mathbb{D})$ by

$$\|f\|_{\mathcal{A}_\alpha}^2 = \sum_0^\infty (n+1)^\alpha |f_n|^2$$

Examples 3.1: Special Classes of $\mathcal{A}_\alpha(\mathbb{D})$.

i. **Hardy Space**, $H^2(\mathbb{D})$: The case $\alpha = 0$

$$H^2(\mathbb{D}) = \{f \in Hol(\mathbb{D}): f(z) = \sum_0^\infty f_n\, z^n \text{ and } \sum_0^\infty |f_n|^2 < \infty\}$$

If $f \in H^2(\mathbb{D})$, then the norm $\|f\|_{H^2(\mathbb{D})}^2 = \sum_0^\infty |f_n|^2$

$\{e_n(z)\}_0^\infty = \{z^n\}_0^\infty$ is an orthonormal basis for $H^2(\mathbb{D})$; furthermore $H^2(\mathbb{D})$ is a RKHS with reproducing kernel

$$\begin{aligned} K(z, w) &= \sum_0^\infty e_n(z)\, \overline{e_n(w)} \\ &= \sum_0^\infty (z\overline{w})^n \\ &= 1/(1 - z\,\overline{w}) \end{aligned}$$

ii. **Bergman Space**, $A^2(\mathbb{D})$: The case $\alpha = -1$

$$A^2(\mathbb{D}) = \{ f \in Hol(\mathbb{D}): f(z) = \sum_0^\infty f_n\, z^n \text{ and } \sum_0^\infty |f_n|^2/(n+1) < \infty\}$$

If $f \in A^2(\mathbb{D})$, then the norm $\|f\|_{A^2(\mathbb{D})}^2 = \sum_0^\infty |f_n|^2/(n+1)$

$\{e_n(z)\}_0^\infty = \left\{\sqrt{n+1}\, z^n\right\}_0^\infty$ is an orthonormal basis for $A^2(\mathbb{D})$; furthermore $A^2(\mathbb{D})$ is a RKHS with reproducing kernel

$$\begin{aligned} K(z,w) &= \textstyle\sum_0^\infty e_n(z)\,\overline{e_n(w)} \\ &= \textstyle\sum_0^\infty (n+1)(z\overline{w})^n \\ &= 1/(1-z\overline{w})^2 \end{aligned}$$

iii. **Dirichlet Space**, $\mathcal{D}(\mathbb{D})$: The case $\alpha = 1$

$\mathcal{D}(\mathbb{D}) = \{\, f \in Hol(\mathbb{D}): f(z) = \sum_0^\infty f_n\, z^n \;\; and \;\; \sum_0^\infty\; (n+1)|f_n|^2 < \infty\}$

If $f \in \mathcal{D}(\mathbb{D})$, then the norm $\|f\|_{\mathcal{D}}^2 \; = \sum_0^\infty (n+1)|f_n|^2$

$\{e_n(z)\}_0^\infty = \left\{z^n/\sqrt{n+1}\right\}_0^\infty$ is an orthonormal basis for $\mathcal{D}(\mathbb{D})$; furthermore$\mathcal{D}(\mathbb{D})$ is a RKHS with reproducing kernel

$$\begin{aligned} K(z,w) &= \textstyle\sum_0^\infty e_n(z)\,\overline{e_n(w)} \\ &= \textstyle\sum_0^\infty (1/(n+1))\,(z\overline{w})^n \\ &= \frac{1}{\overline{w}z}\log(1/(1-\,\overline{w}z)) \end{aligned}$$

iv. **Weighted Bergman Spaces,** $A_\alpha^2(\mathbb{D})$: The case $\alpha < 0$

$A_\alpha^2(\mathbb{D}) = \left\{f \in Hol(\mathbb{D}): f(z) = \sum_0^\infty f_n\, z^n \;\; and \;\; \sum_0^\infty \frac{1}{(n+1)^\alpha}|f_n|^2 < \infty\right\}$

If $f \in A_\alpha^2(\mathbb{D})$, then the norm $\|f\|_{A_\alpha^2}^2 = \sum_0^\infty \frac{1}{(n+1)^\alpha}|f_n|^2\,,\;\alpha > 0$

$\{e_n(z)\}_0^\infty = \left\{(n+1)^{\alpha/2}\, z^n\right\}_0^\infty$ is an orthonormal basis for $A_\alpha^2(\mathbb{D})$; furthermore $A_\alpha^2(\mathbb{D})$ is a RKHS with reproducing kernel

$$\begin{aligned} K(z,w) &= \textstyle\sum_0^\infty e_n(z)\,\overline{e_n(w)} \\ &= \textstyle\sum_0^\infty (n+1)^\alpha (z\overline{w})^n \end{aligned}$$

v. **Weighted Dirichlet Spaces**, $\mathcal{D}_\alpha(\mathbb{D})$: Thae case $\alpha > 0$

$\mathcal{D}_\alpha(\mathbb{D}) = \{\, f \in Hol(\mathbb{D}): f(z) = \sum_0^\infty f_n\, z^n$ and $\sum_0^\infty |f_n|^2 (n+1)^\alpha < \infty\}$

If $f \in \mathcal{D}_\alpha(\mathbb{D})$, then the norm $\|f\|_{\mathcal{D}_\alpha}^2 \; = \sum_0^\infty (n+1)^\alpha |f_n|^2$

$\{e_n(z)\}_0^\infty = \left\{z^n/\sqrt{(n+1)^\alpha}\right\}_0^\infty$ is an orthonormal basis for $\mathcal{D}_\alpha(\mathbb{D})$; furthermore $\mathcal{D}_\alpha(\mathbb{D})$ is a RKHS with reproducing kernel

$$\begin{aligned} K(z,w) &= \textstyle\sum_0^\infty e_n(z)\,\overline{e_n(w)} \\ &= \textstyle\sum_0^\infty\; (n+1)^{-\alpha}(z\overline{w})^n \end{aligned}$$

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