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Weighted Dirichlet Norm Estimate-Bound for Cauchy Transform, $\widehat{K}$

Berhanu Kidane

University of North Georgia
Department of Mathematics

Abebaw Tadesse

Langston University
Department of Mathematics

Abstract

The purpose of this paper is to give estimates for Cauchy Transform of weighted Dirichlet space norm of weight 0.5; that is find an estimate for $\|\widehat{K}\|_{0.5}$, where $\widehat{K}$ is the Cauchy Transform. In general, Estimate $\|\widehat{K}\|_{\alpha}$, $\alpha > 0$ is instrumental for proving the Coronal problems in weighted Dirichlet spaces; [KT], [TT1] and [TT2].

1. Preliminaries

Notations, in this paper:

- $\mathbb{D}$, denotes the open unit disc
- $Hol(\mathbb{D})$, denotes the set holomorphic functions on $\mathbb{D}$
- $\mathbb{D}_{1+\epsilon} = \{z \in \mathbb{C}: |z| \leq 1 + \epsilon\}$
- $C^2(\mathbb{D}_{1+\epsilon})$ the space of functions on $\mathbb{D}_{1+\epsilon}$ whose first and second derivatives are continuous.
- $\mathcal{D}_{0.5}(\mathbb{D})$ or $\mathcal{D}_{0.5}$, weighted Dirichlet space of weight 0.5
- $\|\widehat{K}\|_{0.5} = \|\widehat{K}\|_{\mathcal{D}_{0.5}(\mathbb{D})}$

Definition 1.1. (Integral operator): Suppose (X, μ) is a measure space and K is a function on X × X, then $Tf(x) = \int_X K(x,y) f(y) d\mu(y)$ is an integral operator with kernel K.

Theorem 1.1. (Shur's Theorem)
Suppose K is a non-negative measurable function on X × X. Let T be the integral operator with kernel K and $1 < p < \infty$, $with\ \frac{1}{p} + \frac{1}{q} = 1$. If there are positive constants C_1, C_2 and positive measurable functions h and l on X such that

$$\int_X K(x,y)[h(y)]^q d\mu(y) \leq C_1 [l(x)]^q \ a.e\ x \in X$$

and

$$\int_X K(x,y)[l(x)]^p d\mu(x) \leq C_2 [h(y)]^p \ a.e\ y \in X$$

then T is a bounded linear operator on $L^p(X, d\mu)$ with norm $\leq C_1^{1/q}\, C_2^{1/p}$.
Proof: [KZ] ∎

Lemma 1.1. Suppose $z \in \mathbb{D}, c \in \mathbb{R}, t > -1$, and $I_{c,t}(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-z\,\bar{w}|^{2+t+c}} dA(w)$, then the following hold:

i. If $c < 0$, then the function $I_{c,t}(z)$ is bounded from above and below on $\mathbb{D}$.
ii. If $c > 0$, then $I_{c,t}(z) \sim \frac{1}{(1-|z|^2)^c}$, $|z| \to 1^-$

Proof: [HKZ] ∎

Lemma 1.2. If $c > 0$ and $|z| \to 1^-$, then $\sum_1^\infty n^{c-1} |z|^{2n} \sim \frac{1}{(1-|z|^2)^c}$.
Proof: [KZ]

Theorem 1.2. (Pompeiu's Formula).
Let Ω be a bounded domain. If $f \in C^1(\bar{\Omega})$ and $w \in \Omega$, then

$$f(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-w} dz - \frac{1}{\pi} \int_{\Omega} \frac{\partial f}{\partial \bar{u}}(u) \frac{dA(u)}{u-w}$$

Proof: [AN] ∎

Lemma 1.3. Let $\varphi \in C^2(\mathbb{D}_{1+\epsilon})$, then

$$\varphi(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{\varphi(u)}{u-z} du - \frac{1}{\pi} \int_{\mathbb{D}} \frac{\partial\varphi/\partial\bar{u}}{u-w}\ dA(u). \ \textbf{(1.3.1)}$$

And if $\varphi_1(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{\varphi(u)}{u-z} du$ and $\varphi_2(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{\partial\varphi/\partial\bar{u}}{u-z}\ dA(u)$, then

$\varphi_1(e^{it}) = \sum_0^\infty \varphi_n e^{int}$, where $\varphi_n = \int_{-\pi}^{\pi} \varphi(e^{it}) e^{-int} d\sigma(t) = \langle \varphi, e^{int} \rangle$ **(1.3.2)**

and

$\varphi_2(e^{it}) = \sum_1^\infty \varphi_{-n} e^{-int}$, where $\varphi_{-n} = \int_{-\pi}^{\pi} \varphi(e^{it}) e^{int} d\sigma(t)$ **(1.3.3).**

Proof: **(1.3.1)** follows from **Theorem 1.2.**

To prove **(1.3.2)**, let $u = e^{it}$, then

$$\begin{aligned}\varphi_1(z) &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{\varphi(e^{it})}{e^{it}-z} ie^{t} dt \\ &= \int_{-\pi}^{\pi} \frac{\varphi(e^{it})}{1-ze^{-it}} d\sigma, \text{ where } d\sigma = dt/2\pi \\ &= \sum_0^\infty \int_{-\pi}^{\pi} [\varphi(e^{it}) e^{-int} d\sigma(t)] z^n \\ &= \sum_0^\infty \varphi_n z^n, \text{ where } \varphi_n = \int_{-\pi}^{\pi} \varphi(e^{it}) e^{-int} d\sigma(t) = \langle \varphi, e^{int}\rangle.\end{aligned}$$

To prove **(1.3.3)**, note $\varphi \in C^2(\mathbb{D}_{1+\epsilon})$ has Fourier series representation.

$$\begin{aligned}\varphi(e^{i\theta}) &= \sum_{-\infty}^{\infty} \varphi_n e^{in\theta} \\ &= \sum_0^\infty \varphi_n e^{in\theta} + \sum_1^\infty \varphi_{-n} e^{-in\theta} \\ &= \varphi_1(e^{i\theta}) + \sum_1^\infty \varphi_{-n} e^{-in\theta}\end{aligned}$$

Thus,

$$\begin{aligned}-\frac{1}{\pi}\int_{\mathbb{D}} \frac{\partial\varphi/\partial\bar{u}}{u - e^{-i\theta}}\, dA(u) &= \varphi_2(e^{i\theta}) \\ &= \sum_1^\infty \varphi_{-n} e^{-in\theta}, \text{ where } \varphi_{-n} = \int_{-\pi}^{\pi} \varphi(e^{it}) e^{int} d\sigma(t). \ \blacksquare\end{aligned}$$

2. $\|\widehat{K}\|_{0.5}$ Estimate (Bound)

Definition 2.1. Let $z \in \mathbb{D}$, and K smooth on $\overline{\mathbb{D}}$. Then the Cauchy transform $\widehat{K}$ is defined by $\widehat{K}(z) = \int_{\mathbb{D}} \frac{k(w)}{z-w} dA(w)$, where $dA(w) = \frac{1}{\pi} dm$.

Theorem 2.1. $\bar{\partial}_z \widehat{K} = k \ in\ \mathbb{D}$.

Proof: [AM] ■

Corollary 2.1.1. In **Lemma 1.3.** $\widehat{\frac{\partial\varphi}{\partial u}}(e^{i\theta}) = \sum_1^\infty \varphi_{-n} e^{-in\theta}$.

Theorem 2.2. Let $k \in C^2(\mathbb{D}_{1+\epsilon})$ and $\widehat{K}(z) = \int_{\mathbb{D}} \frac{k(w)}{z-w} dA(w)$, where $z \in \mathbb{D}$ and $dA(w) = \frac{1}{\pi} dm$. Then

$$\|\hat{k}\|_{0.5}^2 \sim \int_{\mathbb{D}} \int_{\mathbb{D}} k(w)\overline{k(z)} \frac{1}{(1-w\,\bar{z})^{1.5}} dA(w) dA(z)$$

Proof: First note that $\int_{-\pi}^{\pi} e^{i(n-1-k)t} d\sigma(t) = \begin{cases} 0, & k \neq n-1 \\ 1, & k = n-1 \end{cases}$

$\widehat{K}(e^{i\theta}) = \sum_1^\infty \widehat{K}_{-n} e^{-in\theta}$, where $\widehat{K}_{-n} = \int_{-\pi}^{\pi} \widehat{K}(e^{it}) e^{int} d\sigma(t) = \langle \widehat{K}, e^{-int}\rangle$

$$\int_{-\pi}^{\pi} \widehat{K}(e^{it}) e^{int} d\sigma(t) = \int_{-\pi}^{\pi} \left(-\int_{\mathbb{D}} \frac{k(w)}{w-e^{it}} dA(w)\right) e^{int} d\sigma(t)$$

$$= \int_{\mathbb{D}} \left(\int_{-\pi}^{\pi} \frac{e^{i(n-1)t}}{1-we^{-it}} d\sigma(t)\right) k(w) dA(w)$$
$$= \int_{\mathbb{D}} \left(\int_{-\pi}^{\pi} \sum_{k=0}^{\infty} e^{i(n-1)t}\, e^{-ikt} w^k d\sigma(t)\right) k(w) dA(w)$$
$$= \int_{\mathbb{D}} \sum_{k=0}^{\infty} \int_{-\pi}^{\pi} e^{i(n-1-k)t}\, d\sigma(t) w^k k(w) dA(w)$$
$$= \int_{\mathbb{D}} k(w) w^{n-1} dA(w)$$

On the other hand, for $z \in \partial\mathbb{D}$, the weighted Dirichlet norm for $\widehat{K}$ is ([KB], [KT])

$$\left\|\widehat{K}\right\|_{0.5}^2 = \sum_1^{\infty}(|-n|+1)^{0.5}\left|\widehat{K}_{-n}\right|^2$$
$$= \sum_1^{\infty}(n+1)^{0.5}\left|\int_{\mathbb{D}} k(w) w^{n-1} dA(w)\right|^2$$
$$= \sum_1^{\infty}(n+1)^{0.5}\int_{\mathbb{D}}\int_{\mathbb{D}} k(w)\overline{k(z)}\, w^{n-1}\bar{z}^{n-1} dA(w) dA(z)$$

By Stirling's formula:

$$\sum_1^{\infty}(n+1)^{0.5} \sim \sum_1^{\infty}\frac{\Gamma(n+1.5)}{\Gamma(1.5)\Gamma(n+1)}$$

Thus,

$$\left\|\widehat{K}\right\|_{0.5}^2 \sim \sum_1^{\infty}\frac{\Gamma(n+1.5)}{\Gamma(1.5)\Gamma(n+1)}\int_{\mathbb{D}}\int_{\mathbb{D}} k(w)\overline{k(z)}\, w^{n-1}\bar{z}^{n-1} dA(w) dA(z)$$

And

$$\sum_1^{\infty}\frac{\Gamma(n+1.5)}{\Gamma(1.5)\Gamma(n+1)}\int_{\mathbb{D}}\int_{\mathbb{D}} k(w)\overline{k(z)}\, w^{n-1}z^{n-1} dA(w) dA(z)$$
$$= \int_{\mathbb{D}}\int_{\mathbb{D}} k(w)\overline{k(z)}\left(\sum_1^{\infty}\frac{\Gamma(n+1.5)}{\Gamma(1.5)\Gamma(n+1)}(w\bar{z})^{n-1}\right) dA(w) dA(z)$$
$$= \int_{\mathbb{D}}\int_{\mathbb{D}} k(w)\overline{k(z)}\,\frac{1}{(1-w\bar{z})^{1.5}} dA(w) dA(z)$$

Hence

$$\left\|\widehat{K}\right\|_{0.5}^2 \sim \int_{\mathbb{D}}\int_{\mathbb{D}} k(w)\overline{k(z)}\,\frac{1}{(1-w\bar{z})^{1.5}} dA(w) dA(z) \;\blacksquare$$

Definition 2.2. Define the norm $\left\|\widehat{K}\right\|_{0.5}^2$ by:

$$\left\|\widehat{K}\right\|_{0.5}^2 = \int_{\mathbb{D}}\int_{\mathbb{D}} k(w)\overline{k(z)}\,\frac{1}{(1-w\bar{z})^{1.5}} dA(w) dA(z)$$

Lemma 2.1. Let $k(z)$ be smooth on $\mathbb{D}_{1+\epsilon}$ and $L(z) = k(z)(1-|z|^2)^{\beta/2}$, where $\beta \in (0,2)$. Let $K(z,w) = \frac{1}{(1-|w|^2)^{\beta/2}(1-|z|^2)^{\beta/2}(1-w\bar{z})^{1.5}}$ and $\mathrm{T}L(w) = \int_{\mathbb{D}} |L(z)| K(z,w) dA(z)$. If there is a positive function P such that $\int_{\mathbb{D}} P(z) K(z,w) dA(z) \leq C_0 P(w)$ for some constant C_0, then the integral operator T is bounded with $\|\mathrm{T}\| \leq C_0$.

Proof: Immediate consequence of Shur's Theorem

Theorem 2.3. Let $\widehat{\mathrm{K}}$ be as in Theorem 2.2. Then there is a constant C_0 such that

$$\|\widehat{K}\|_{0.5}^2 \leq C_0 \int_{\mathbb{D}} \left|k(z)(1-|z|^2)^{1.5/2}\right|^2 dA(z)$$

Proof: $\|\widehat{K}\|_{0.5}^2 = \int_{\mathbb{D}} \int_{\mathbb{D}} k(w)\overline{k(z)} \frac{1}{(1-w\bar{z})^{1.5}} dA(w)dA(z)$

$$= \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{k(w)(1-|w|^2)^{\beta}}{(1-|w|^2)^{\beta}} \frac{\overline{k(z)}\,(1-|z|^2)^{\beta}}{(1-|z|^2)^{\beta}(1-w\bar{z})^{1.5}} dA(w)dA(z)$$

Let

$$K(z,w) = \frac{1}{(1-|w|^2)^{\beta/2}(1-|z|^2)^{\beta/2}(1-w\bar{z})^{1.5}} \text{ and } L(z) = k(z)(1-|z|^2)^{\beta/2}$$

Then

$$\begin{aligned}\|\widehat{K}\|_{0.5}^2 &= \int_{\mathbb{D}} \int_{\mathbb{D}} L(w)\overline{L(z)}\ K(z,w)dA(w)dA(z)\\ &\leq \int_{\mathbb{D}} |L(w)| \int_{\mathbb{D}} |L(z)|\, K(z,w)dA(z)dA(w)\\ &\leq C_0 \int_{\mathbb{D}} |L(z)|^2 dA(z) \qquad \textbf{(2.3.1)}\end{aligned}$$

To prove **(2.3.1)** define

$$Tf(w) = \int_{\mathbb{D}} f(z)K(z,w)dA(z)$$

Then

$$\begin{aligned}\|\widehat{K}\|_{0.5}^2 &\leq \int_{\mathbb{D}} |L(w)| \int_{\mathbb{D}} |L(z)|\, K(z,w)dA(z)dA(w)\\ &\leq \int_{\mathbb{D}} |L(w)|T(|L(w)|)dA(w) = \langle |L|,\ T(|L|)\rangle_2\\ &\leq \|T\|\|L\|^2\\ &\leq\ \|T\| \int_{\mathbb{D}} |L(z)|^2 dA(z)\end{aligned}$$

Then by **Lemma 2.1.**

$$\|\widehat{K}\|_{0.5}^2 \leq C_0 \int_{\mathbb{D}} |L(z)|^2 dA(z).\ \blacksquare$$

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