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The First Homology Group of a Complete Flag Complex

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Abstract

The geometry of a flag complex is explored and the homology group H_1 computed for the complete flag $F(s)$ by viewing it as a simplicial complex. The homomorphisms between the vertices are extended to the free abelian groups generated by the vertices and it is proved that $H_1(F(s)) \cong \underbrace{\mathbb{Z} + \mathbb{Z} + \cdots + \mathbb{Z}}_{k \text{ times}}$, $k \geq 1$ where $k = s - r$, r the rank of a matrix A associated to $F(s)$.

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1 Introduction

The computations of homology groups of topological objects is an important problem in algebraic topology. Homology is a functor serving as a link from topology to algebra(page 13 of [5]). In this work, the homology of a complete flag is investigated by way of looking at such as a simplicial complex, with the aim of computing its first homology group.

Let $\mathscr{P}^2$ be the family of topological pairs (i.e. $(X, A) \in \mathscr{P}^2$ where $A \subseteq X$). A homology theory h_* consists of a family $h = \{h_n : n \in \mathbb{Z}\}$ of covariant functors from the category $\mathscr{P}^2$ of topological pairs to the category of R - modules where R is a commutative ring with identity; together with a family of natural transformations $\partial = \{\partial_n : n \in \mathbb{Z}\}$ $\partial_n : h_n(X, A) \longrightarrow h_{n-1}(A)$ called boundary homomorphisms such that some 7 axioms are satisfied (pages 10-12 of [2]). It happens that the composition of any two successive natural transformations is trivial, i.e. $\partial_n \circ \partial_{n+1} = 0$ which is equivalent to the statement that $ker\partial_n \subset im\partial_{n+1}$. The elements of $ker\partial_n$ are called cycles while the elements of $im\partial_{n+1}$ are called boundaries. We then get homology of $H(X, A)$ as the family of quotients $\{H_n(X, A) = ker\partial_n / im\partial_{n+1}\}$

In this work we shall compute the first homology group $H_1(F(s))$ for a flag complex $F(s)$.

Definition 1.1 *Let $\mathbb{F}$ be the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Suppose $n_1, \cdots, n_s$ are fixed positive integers $\ni$ $n_1 + \cdots + n_s$. A "flag" or more precisely a "$(n_1 \cdots n_s)$ - flag over $\mathbb{F}$" is a collection σ of mutually orthogonal subspaces $(\sigma_1, \cdots, \sigma_s)$ of $\mathbb{F}^n$ $\ni$ $dim\sigma_i = n_i$. The space of all such flags form a compact smooth manifold (respectively complex manifold) for $\mathbb{F} = \mathbb{R}$ (respectively $\mathbb{F} = \mathbb{C}$) called the generalized real flag manifold (respectively complex flag manifold) denoted by $G_{\mathbb{F}}(n_1, \cdots, n_s)$ or simply G([3]).*

In what follows, we view the complete flag as a simplicial complex and then compute it's homology.

Definition 1.2 *A simplicial complex K ([5]) consists of a set $\{v\}$ of vertices and a set $\{s\}$ of finite non-empty subsets of $\{v\}$ called simplexes such that*

(a) Any set consisting of exactly one vertex is a simplex.

(b) Any non-empty subset of a simplex is a simplex.

2 The Flag as a Simplicial Complex

As seen above, a flag is a collection

$$\sigma = (\sigma_1, \cdots, \sigma_s)$$

Flag complexes are (abstract) simplicial complexes $\triangle$ satisfying the property that every set of vertices of $\triangle$ that are pairwise connected by edges form a face of $\triangle$[1].

Consequently, each σ_i in this collection can be taken as a vertex in a simplex σ; this is how the flag is to be viewed here. Hence the set σ now constitute the set of all simplexes; and the result is a simplicial complex; which we call a flag complex and denote by $F(s)$.

The flag above as earlier pointed out is in generalized form (i.e. the flag dimensions $dim\sigma_i = n_i$ are arbitrary integers as in [3]); this work is hereafter concerned with the case where $dim\sigma_i = i$ i.e. the complete flag whose geometry is treated in [4].

Theorem 2.1 *Let $F(s)$ be a complete flag complex with an associated matrix of rank $r < s$. Then* $H_1(F(s)) \cong \underbrace{\mathbb{Z}+\mathbb{Z}+\cdots+\mathbb{Z}}_{k\ times}$, $k \geq 1$ *where* $k = s - r$.

Proof: Denote by C_0 the free abelian group generated by the vertices $\sigma_1, \sigma_2, \cdots, \sigma_s$ and C_1 the free abelian group generated by the edges $e_1, e_2, \cdots e_s$ linking them

Without loss of generality, assume the boundaries are:

$\partial(e_1) = \sigma_2 - \sigma_1$

$\partial(e_2) = \sigma_3 - \sigma_2$

$\vdots \qquad\qquad (*)$

$\partial(e_{s-1}) = \sigma_s - \sigma_{s-1}$

$\partial(e_s) = \sigma_1 - \sigma_s$

The homomorphism $\partial : C_1 \longrightarrow C_0$ extends $(*)$

Since

$\partial(e_1+e_2+\cdots+e_s) = (\sigma_2-\sigma_1)+(\sigma_3-\sigma_2)+\cdots+(\sigma_s-\sigma_{s-1})+(\sigma_1-\sigma_s) = 0$

An element (i.e. a chain) t in C_1 is a cycle if and only if $\partial(t) = 0$

i.e. $\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_s e_s$ is a cycle $\Longleftrightarrow$

$\partial(\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_s e_s) = 0$

Now

$\partial(\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_s e_s) = \alpha_1\partial(e_1) + \alpha_2\partial(e_2) + \cdots + \alpha_s\partial(e_s)$

$= \alpha_1(\sigma_2 - \sigma_1) + \alpha_2(\sigma_3 - \sigma_2) + \alpha_3(\sigma_4 - \sigma_3) + \cdots$

$+\alpha_{s-3}(\sigma_{s-2} - \sigma_{s-3}) + \alpha_{s-2}(\sigma_{s-1} - \sigma_{s-2}) + \alpha_{s-1}(\sigma_s - \sigma_{s-1}) + \alpha_s(\sigma_1 - \sigma_s)$

$= \alpha_1\sigma_2 - \alpha_1\sigma_1 + \alpha_2\sigma_3 - \alpha_2\sigma_2 + \alpha_3\sigma_4 - \alpha_3\sigma_3 + \cdots + \alpha_{s-3}\sigma_{s-2} - \alpha_{s-3}\sigma_{s-3} + \alpha_{s-2}\sigma_{s-1} - \alpha_{s-2}\sigma_{s-2} + \alpha_{s-1}\sigma_s - \alpha_{s-1}\sigma_{s-1} + \alpha_s\sigma_1 - \alpha_s\sigma_s$

$= (-\alpha_1 + \alpha_s)\sigma_1 + (\alpha_1 - \alpha_2)\sigma_2 + (\alpha_2 - \alpha_3)\sigma_3 + \cdots + (\alpha_{s-3} - \alpha_{s-2})\sigma_{s-2} + (\alpha_{s-2} - \alpha_{s-1})\sigma_{s-1} + (\alpha_{s-1} - \alpha_s)\sigma_s = 0 \Longleftrightarrow$

$-\alpha_1 + \alpha_s = 0$

$\alpha_1 - \alpha_2 = 0$

$\alpha_2 - \alpha_3 = 0$

$\vdots$

$\alpha_{s-3} - \alpha_{s-2} = 0$

$\alpha_{s-2} - \alpha_{s-1} = 0$

$\alpha_{s-1} - \alpha_s = 0$

whence

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{s-2} \\ \alpha_{s-1} \\ \alpha_s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus $A\underline{\alpha} = 0$ where A is the matrix associated with $F(s)$.
Row operations on A ultimately leads to $A \sim A_r$
$rank(A_r) = r, \ (r < s)$.
Thus $A\underline{\alpha} = 0$ has $s - r = k$ linearly independent solutions.
Hence $H_1(F(s)) \cong \underbrace{\mathbb{Z} + \mathbb{Z} + \cdots + \mathbb{Z}}_{k \text{ times}}, \ k \geq 1$.

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