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A Containment Association Between Holomorphic Spaces on $\mathbb{D}$

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Abstract

In [BK] and [KT], the Weighted Dirichlet Spaces of weight $\alpha \in (0,1), \mathcal{D}_\alpha(\mathbb{D})$, were discussed.
In this paper, we discuss containment relations between analytic spaces on $\mathbb{D}$, in general, and presents special cases of materials covered on [BK]. We make particular emphases on the weighted Dirichlet space of weight 0.5. We produce an example of the containment relationships between the multiplier algebras of the weighted Dirichlet space of weight 0.5, and the space of bounded holomorphic functions on the open unit disc.

1. Preliminaries

1.1. Weighted Analytic Spaces

Notation: In this paper

- $\mathbb{D}$ denotes the open unit disc
- $\mathcal{D}_{0.5}$ or $\mathcal{D}_{1/2}$ denotes the weighted Dirichlet space of weight 0.5.
- $H^\infty(\mathbb{D})$, the space of bounded holomorphic on the open unit disc.
- M_φ^*, conjugate operator of the multiplier M_φ
- $\mathcal{B}(\mathcal{D}_\alpha)$, the space of bounded linear operators on $\mathcal{D}_\alpha(\mathbb{D}), \alpha \in (0,1)$

Definition 1.1.1: Weighted Analytic Spaces of weight α

Let $\alpha \in \mathbb{R}$; $\mathcal{A}_\alpha(\mathbb{D})$ or simply $\mathcal{A}_\alpha$ will denote a weighted analytic space of weight α on the open unit disc $\mathbb{D}$, and is defined by:

$\mathcal{A}_\alpha(\mathbb{D}) = \{f \epsilon\ Hol(\mathbb{D}) : f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $\sum_{n=0}^{\infty}(n+1)^\alpha |f_n|^2 < \infty\}$
For $f \in \mathcal{A}_\alpha(\mathbb{D})$, the square of the norm, $\|f\|_{\mathcal{A}_\alpha}^2$ is defined by:
$\|f\|_{\mathcal{A}_\alpha}^2 = \sum_{n=0}^{\infty}(n+1)^\alpha |f_n|^2$.

Note: If $\alpha = -1, \mathcal{A}_{-1} = \mathcal{A}$ is Bergman space
If $\alpha < 0, \mathcal{A}_\alpha$ is weighted Bergman space
If $\alpha = 0, \mathcal{A}_0 = H^2(\mathbb{D})$ is Hardy space
If $\alpha = 1, \mathcal{A}_1 = \mathcal{D}(\mathbb{D}) = \mathcal{D}$ is Dirichlet space
If $\alpha > 0, \mathcal{A}_\alpha = \mathcal{D}_\alpha(\mathbb{D}) = \mathcal{D}_\alpha$ is weighted Dirichlet space

Fact 1.1.2: Let α and $\beta \in \mathbb{R}$, such that $\alpha < \beta$, then it holds that $(n+1)^{\beta-\alpha} > 1$, for $n = 1, 2, 3, \ldots$. This implies that
$(n+1)^\alpha |f_n|^2 < (n+1)^\beta |f_n|^2$,
consequently
$\mathcal{A}_\beta(\mathbb{D}) \subseteq \mathcal{A}_\alpha(\mathbb{D})$.

Definition 1.1.3: Weighted Dirichlet space of weight $\alpha = 0.5$,
The weighted Dirichlet space of weight $\alpha = 0.5$ is denoted by $\mathcal{D}_{0.5}(\mathbb{D})$ or simply $\mathcal{D}_{0.5}$ or $\mathcal{D}_{1/2}$ is defined as
$\mathcal{D}_{0.5}(\mathbb{D}) = \left\{f \epsilon\ Hol(\mathbb{D}) : f(z) = \sum_{n=0}^{\infty} f_n z^n \text{ and } \sum_{n=0}^{\infty}(n+1)^{1/2}|f_n|^2 < \infty\right\}$
If $f \in \mathcal{D}_{0.5}(\mathbb{D})$, then the square of the norm, $\|f\|_{\mathcal{D}_{0.5}}^2$ is defined by:
$\|f\|_{\mathcal{D}_{0.5}}^2 = \sum_{n=0}^{\infty}(n+1)^{1/2}|f_n|^2$

2. Multiplier Algebras

Definition 2.1. (Multiplier)
Let fix $\varphi \epsilon \mathcal{D}_{0.5}(\mathbb{D})$. Define M_φ on $\mathcal{D}_{0.5}$ by,
$M_\varphi(f) = \varphi f,\ \forall f \epsilon\ \mathcal{D}_{0.5}(\mathbb{D})$.
If $\varphi f \epsilon \mathcal{D}_{0.5}(\mathbb{D})\ \forall f \epsilon \mathcal{D}_{0.5}(\mathbb{D})$, then φ or M_φ is called a *multiplier* for the space $\mathcal{D}_{0.5}(\mathbb{D})$.

Definition 2.2. (Multiplier Algebras)
We denote the multiplier algebra of operators on $\mathcal{D}_{0.5}(\mathbb{D})$ by, $\mathcal{M}(\mathcal{D}_{0.5})$ and $\mathrm{M}(\mathcal{D}_{0.5})$, and define each as follows:
$\mathcal{M}(\mathcal{D}_{0.5}) = \{\varphi \epsilon \mathcal{D}_{0.5} : g\varphi \epsilon \mathcal{D}_{0.5},\ \forall g \epsilon \mathcal{D}_{0.5}\}$
$\mathrm{M}(\mathcal{D}_{0.5}) = \{M_\varphi : \mathcal{D}_{0.5} \to \mathcal{D}_{0.5} : M_\varphi(f) = \varphi f, \forall f \epsilon \mathcal{D}_{0.5}\}$

Fact 2.3. $\mathcal{D}_{0.5}(\mathbb{D})$ is a reproducing kernel Hilbert space (RKHS), thus $\forall x \in \mathbb{D}$, by Riesz Representation Theorem, there is a unique vector (function) $k_x \in \mathcal{D}_{0.5}$ such that $f(x) = \langle f, k_x \rangle, \forall f \in \mathcal{D}_{0.5}$.

Definition 2.4. (*Reproducing Kernel*)
The function $k_x \in \mathcal{D}_{0.5}$ is called the Reproducing Kernel for the point *x*. The two-variable function $K(y, x)$, defined by $K(y, x) = k_x(y)$ is called the reproducing kernel for $\mathcal{D}_{0.5}(\mathbb{D})$.

Theorem 2.5. (*Criterion for closedness*)
Let X, Y be normed spaces. The mapping $\mathrm{T}: \mathrm{X} \to \mathrm{Y}$ is a closed linear operator if and only if for every sequence $\{x_n\} \in \mathrm{X}$ whenever $x_n \to x$ and $\mathrm{T}(x_n) \to y \,\epsilon\, \mathrm{Y}$, we have that $x \in \mathrm{X}$ and $\mathrm{T}(x) = y$.

Theorem 2.6. M_φ is a closed linear operator on $\mathcal{D}_{0.5}(\mathbb{D})$
Proof: First note that $\mathcal{D}_{1/2}(\mathbb{D})$ is a Hilbert space.
That M_φ linear is clear. To prove M_φ is closed:
Let f_n be a sequence in $\mathcal{D}_{1/2}(\mathbb{D})$, that converges to $f \in \mathcal{D}_{1/2}(\mathbb{D})$, then

$$\|M_\varphi f_n - M_\varphi f\| = \|M_\varphi(f_n - f)\| \leq \|M_\varphi\| \|f_n - f\|.$$

Hence, M_φ is closed

Theorem 2.7. (Closed Graph Theorem)
Let X and Y be Banach spaces and $T: X \to Y$ a closed linear operator. If X is closed, then T is bounded.

Corollary 2.7.1. Since $\mathcal{D}_{0.5}(\mathbb{D})$ is a Hilbert space, by the *Closed Graph Theorem* the multiplier operator $M_\varphi: \mathcal{D}_{0.5}(\mathbb{D}) \to \mathcal{D}_{0.5}(\mathbb{D})$ is a bounded linear operator**;** that is, $M_\varphi \in \mathcal{B}(\mathcal{D}_{0.5})$, space of bounded linear operators on $\mathcal{D}_{0.5}(\mathbb{D})$.

Theorem 2.8. If M_φ is a multiplier on $\mathcal{D}_{0.5}$, then $M_\varphi^*(k_\omega) = \overline{\varphi(\omega)} k_\omega$
Proof: let $f \in \mathcal{D}_{0.5}$ be any, then

$$\begin{aligned} \langle f,\ M_\phi^* k_\omega \rangle &= \langle M_\phi f,\ k_\omega \rangle \\ &= \langle \phi f,\ k_\omega \rangle \\ &= \phi(\omega) f(\omega) \\ &= \langle f,\ \overline{\phi(\omega)}\, k_\omega \rangle, \forall f \in \mathcal{D}_{0.5} \end{aligned}$$

Corollary 2.8.1: If M_φ is a multiplier on $\mathcal{D}_{1/2}$, then $|\varphi(\omega)| \leq \|M_\varphi\|$ $\forall \omega \,\epsilon\, \mathbb{D}$
Proof: $\|\overline{\varphi(\omega)}\, k_\omega\| = \|M_\varphi^*(k_\omega)\| \leq \|M_\varphi^*\| \|k_\omega\|_{\mathcal{D}_{1/2}}$

$$\Rightarrow |\overline{\varphi(\omega)}| \|k_\omega\|_{\mathcal{D}_{1/2}} \leq \|M_\varphi^*\| \|k_\omega\|_{\mathcal{D}_{1/2}}$$

$\Rightarrow |\ \varphi(\omega)\ | \leq \left\| M_{\varphi}^{*} \right\|$

$\Rightarrow |\ \varphi(\omega)\ | \leq \left\| M_{\varphi} \right\| \ \forall \omega \epsilon \mathbb{D}$ #

A consequence of *Corollary 2.8.1* is that: If φ is a multiplier on $\mathcal{D}_{0.5}(\mathbb{D})$, then

$\varphi \in H^{\infty}(\mathbb{D})$

Giving

$\mathcal{M}(\mathcal{D}_{0.5}) \subseteq H^{\infty}(\mathbb{D})$

Fact 2.9.: $\mathcal{M}(\mathcal{D}_{0.5})$ is strictly contained in $H^{\infty}(\mathbb{D})$

proof: Consider the function $g(z) = \sum_{n=0}^{\infty} \frac{z^{n^{11}}}{n^3}, z \in \mathbb{D}$

$$|\ g(z)\ | \leq \sum_{n=0}^{\infty} \frac{|\ z^{n^{11}}|}{n^3} \leq \sum_{n=0}^{\infty} \frac{1}{n^2}, \forall z \in \mathbb{D}$$

Implies that,

$g \in H^{\infty}(\mathbb{D})$

Now we show that $g \notin \mathcal{M}(\mathcal{D}_{0.5})$

Recall, if $f \in \mathcal{D}_{0.5}$, then $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $\sum_{n=0}^{\infty}(n+1)^{1/2}|f_n|^2 < \infty$

$$g(z) = \sum_{n=1}^{\infty} \frac{z^{n^{11}}}{n^3} = \sum_{n=0}^{\infty} g_n z^n \text{ , where } g_0 = 0 \text{ , but}$$

$$\begin{aligned} \|g\|_{\mathcal{D}_{0.5}} &= \sum_{n=0}^{\infty}(n+1)^{\frac{1}{2}}|g_n|^2 \\ &= \sum_{n=1}^{\infty}(n^{11}+1)^{\frac{1}{2}}\left(\frac{1}{n^3}\right)^2 \\ &= \sum_{n=1}^{\infty}(n^{11}+1)^{\frac{1}{2}}\left(\frac{1}{n^{12}}\right)^{1/2} \\ &= \sum_{n=1}^{\infty}\left(\frac{n^{11}+1}{n^{12}}\right)^{\frac{1}{2}} \\ &= \sum_{n=1}^{\infty}\left(\frac{1}{n} + \frac{1}{n^{12}}\right)^{\frac{1}{2}} \\ &> \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{\frac{1}{2}} \end{aligned}$$

This shows that

$g \notin \mathcal{M}(\mathcal{D}_{0.5})$

Hence, $\mathcal{M}(\mathcal{D}_{0.5})$ is strictly contained in $H^{\infty}(\mathbb{D})$

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