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General L_p Blaschke Bodies and Applications

Feixiang Chen

School of Mathematics and Statistics
Chongqing Three Gorges University, Wanzhou, 404000, China

Zhanfei Zuo

School of Mathematics and Statistics
Chongqing Three Gorges University, Wanzhou, 404000, China

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Abstract

In this paper, based on the L_p Minkowski problem for $p > 1$, we define the general L_p Blaschke addition and general L_p Blaschke bodies, reapectively, and obtain the extremal values of their volume and L_p affine surface area. Further, as the applications, we study two negative forms of the Shephard problems.

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1 Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space R^n. For the set of convex bodies containing the origin, the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in R^n, we write $\mathcal{K}_o^n$, $\mathcal{K}_{(0)}^n$ and $\mathcal{K}_{os}^n$, respectively. $\mathcal{F}_o^n$ denotes the set of all bodies in $\mathcal{K}_{(0)}^n$ which have a positive continuous curvature function. $\mathcal{S}_o^n$ denotes the set of star bodies (about the origin) in R^n. Let S^{n-1} denote the unit sphere in R^n, and let $V(K)$ denote the n-dimensional volume of a body K. Let B denote the standard Euclidean unit ball in R^n and write $\omega_n = V(B)$ for its volume.

A convex body is uniquely determined by its support function. The support function of $K \in \mathcal{K}^n, h(K, \cdot)$, is defined on S^{n-1} by

$$h(K, u) = \max\{u \cdot x : \ x \in K\}. \tag{1.1}$$

Let δ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$, $\delta(K, L) = |h_K - h_L|_\infty$, where $| \cdot |_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

Based on the classical Minkowski problem, the Blaschke sum $S(K\#L, \cdot)$ (Usually when working with Blaschke sum, translations do not matter) of $K, L \in \mathcal{K}^n$ is defined by (see [21])

$$S(K\#L, \cdot) = S(K, \cdot) + S(L, \cdot), \tag{1.2}$$

where $S(K, \cdot)$ denotes the surface area of K.

In [4], the Blaschke body of $K \in \mathcal{K}^n$ is defined by

$$S(\nabla K, \cdot) = \frac{1}{2} S(K, \cdot) + \frac{1}{2} S(-K, \cdot). \tag{1.3}$$

With respect to the Blaschke body ∇K, Firey get the following inequality in [3]:

Theorem 1.A. *If $K \in \mathcal{K}_o^n$, then*

$$V(\nabla K) \geq V(K),$$

with equality holds if and only if K is origin-symmetric.

Similarly, in [18] Petty also get a corresponding result for affine surface area as follows:

Theorem 1.B. *If $K \in \mathcal{K}_o^n$, then*

$$\Omega(\nabla K) \geq \Omega(K),$$

with equality holds if and only if K is origin-symmetric.

The notion of L_p Blaschke additon was given by Lutwak (see [15]). For $K, L \in \mathcal{K}_{(os)}^n$, $n \neq p \geq 1$, L_p Blaschke additon, $K\#_p L \in \mathcal{K}_{(os)}^n$, of K and L is defined by

$$S_p(K\#_p L, \cdot) = S_p(K, \cdot) + S_p(L, \cdot), \tag{1.4}$$

where $S_p(K, \cdot)$ denotes the L_p-surface area measure of $K \in \mathcal{K}_o^n$.

The L_p Minkowski problem for $p > 1$ with even case was proved by Lutwak in [15]. In the general case, non-even case, the following results was proved by Chou and Wang [1] and Hug, Lutwak, Yang and Zhang [10], with different methods, respectively (also see [21]).

Theorem 1.C. *Let $n \neq p > 1$. Let ϕ be a finite Borel measure on S^{n-1} which is positive on each open hemisphere of S^{n-1}. Then there exists a unique convex body $K \in \mathcal{K}_o^n$ such that*

$$d\phi = h_K^{1-p} dS(K, \cdot).$$

More over, if ϕ is discrete, then K is a polytope in $\mathcal{K}_{(o)}^n$, and for general ϕ, if $p > n$, then $K \in \mathcal{K}_{(o)}^n$.

Combining with the definition of L_p Blaschke combination, Lutwak in [15] gave the concept of L_p Blaschke body as follows: For $K \in \mathcal{K}_0^n$, the Blaschke body $\nabla_p K \in \mathcal{K}_{os}^n$ is given by

$$S_p(\nabla_p K, \cdot) = \frac{1}{2} S_p(K, \cdot) + \frac{1}{2} S_p(-K, \cdot). \tag{1.5}$$

Lutwak [15] also get the following result:

Theorem 1.D. *If $K \in \mathcal{K}_o^n$, $n \neq p > 1$, then*

$$V(\nabla_p K) \geq V(K),$$

with equality holds if and only if K is origin-symmetric.

Combining with the Theorem 1.C and (1.2), we define the general L_p Blaschke addition as follows: For $K, L \in \mathcal{K}_o^n$, $p \geq 1$, the general L_p Blaschke addition, $K \sharp_p L \in \mathcal{K}_o^n$, is defined by

$$S_p(K \sharp_p L, \cdot) = S_p(K, \cdot) + S_p(L, \cdot), \tag{1.6}$$

where $S_p(K, \cdot)$ denotes the L_p-surface area measure of $K \in \mathcal{K}_o^n$.

Now, by combining with the definition of general L_p Blaschke additon and Theorem 1.C, we define the general L_p Blaschke bodies as follows: For $K \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, the general L_p Blaschke body, $\nabla_p^\tau K$, of K is defined by

$$S_p(\nabla_p^\tau K, \cdot) = f_1(\tau) S_p(K, \cdot) + f_2(\tau) S_p(-K, \cdot), \tag{1.7}$$

where

$$f_1(\tau) = \frac{1+\tau}{2}, \qquad f_2(\tau) = \frac{1-\tau}{2}. \tag{1.8}$$

If $p = 1$, we denote $\nabla^\tau K = \nabla_1^\tau K$ and (1.7) is

$$S(\nabla^\tau K, \cdot) = \frac{1+\tau}{2} S(K, \cdot) + \frac{1-\tau}{2} S(-K, \cdot). \tag{1.9}$$

From (1.8), we have that

$$f_1(\tau) + f_2(\tau) = 1; \tag{1.10}$$

$$f_1(-\tau) = f_2(\tau), \qquad f_2(-\tau) = f_1(\tau). \tag{1.11}$$

From (1.7), it easily follows that

$$\nabla_p^\tau K = f_1(\tau) \cdot K \sharp_p f_2(\tau) \cdot (-K). \tag{1.12}$$

Besides, by (1.5), (1.7) and (1.8), we see that if $\tau = 0$, then $\nabla_p^0 K = \nabla_p K$; if $\tau = \pm 1$, then $\nabla_p^{+1} K = K$, $\nabla_p^{-1} K = -K$.

The main results of this paper can be stated as follows: First, we give the extremal values of the volume of general L_p Blaschke bodies.

Theorem 1.1. *If $K \in \mathcal{K}_o^n$, $n > p > 1$, $\tau \in [-1, 1]$, then*

$$V(\nabla_p K) \geq V(\nabla_p^\tau K) \geq V(K). \tag{1.13}$$

If $\tau \neq 0$, equality holds in the left inequality if and only if K is origin-symmetric, if $\tau \neq \pm 1$, then equality holds in the right inequality if and only if K is also origin-symmetric.

Moreover, based on the L_p affine surface area (see (2.2)), we give another class of extremal values for general L_p Blaschke bodies.

Theorem 1.2. *If $K \in \mathcal{K}_o^n$, $n > p > 1$, $\tau \in [-1, 1]$, then*

$$\Omega_p(\nabla_p K) \geq \Omega_p(\nabla_p^\tau K) \geq \Omega_p(K). \tag{1.14}$$

If $\tau \neq 0$, equality holds in the left inequality if and only if K is origin-symmetric, if $\tau \neq \pm 1$, then equality holds in the right inequality if and only if K is also origin-symmetric.

Theorems 1.1-1.2 belong to part of a new and rapidly evolving asymmetric L_p Brunn-Minkowski theory that has its origins in the work of Ludwig, Haberl and Schuster (see [5, 6, 7, 8, 13, 14]). For the studies of asymmetric L_p Brunn-Minkowski theory, also see [2, 9, 19].

The notion of L_p-projection body was introduced by Lutwak, Yang and Zhang [17]. For each $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p-projection body, $\Pi_p K$, of K is the origin-symmetric convex body whose support function is given by

$$h_{\Pi_p K}^p(u) = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} | u \cdot v |^p \, dS_p(K, v), \tag{1.15}$$

for all $u \in S^{n-1}$, and

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}.$$

Here $S_p(K, \cdot)$ denotes the L_p-surface area measure of $K \in \mathcal{K}_o^n$. Lutwak [10] showed that the measure $S_p(K, \cdot)$ is absolutely continuous with respect to the classical surface area measure $S(K, \cdot)$ of K, and has Radon-Nikodym derivative

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h_K^{1-p}.$$

The unusual normalization of definition (1.15) is chosen so that for the unit ball, B, we have $\Pi_p B = B$. In particular, for $p = 1$, the convex body $\Pi_1 K$ is a dilate of the classical projection body ΠK of K and $\Pi_1 B = B$.

For further results on L_p-projection bodies, see also [11, 12, 20].

As the application of Theorem 1.1, we extend the scope of negative solutions of Shephard problems from origin-symmetric star bodies to star bodies.

Theorem 1.3. *Let $K \in \mathcal{K}_o^n$ and $p > 1$, if K is not origin-symmetric, then there exists $L \in \mathcal{K}_o^n$ such that*

$$\Pi_p L \subset \Pi_p K,$$

but

$$V(L) > V(K).$$

Let $p \to 1$ in Theorem 1.3, we have that

Corollary 1.1. *Let $K \in \mathcal{K}_o^n$, if K is not origin-symmetric, then there exists $L \in \mathcal{K}_o^n$ such that*

$$\Pi L \subset \Pi K,$$

but

$$V(L) > V(K).$$

Remark 1.1. For the Shephard problems, Petty (see [18]) gave that negative solution $L \in \mathcal{K}_{os}^n$. Obviously, Corollary 1.1 extend Petty's result.

Similarly, applying Theorem 1.2, we get the form of L_p affine surface areas for the negative solutions of Shephard problems.

Theorem 1.4. *For $K \in \mathcal{K}_o^n$, $p > 1$, if K is not origin-symmetric, then there exists $L \in \mathcal{K}_o^n$, such that*

$$\Pi_p L \subset \Pi_p K,$$

but

$$\Omega_p(L) > \Omega_p(K).$$

Let $p \to 1$ in Theorem 1.4, we have that

Corollary 1.2. *Let $K \in \mathcal{K}_o^n$, if K is not origin-symmetric, then there exists $L \in \mathcal{K}_o^n$ such that*

$$\Pi L \subset \Pi K,$$

but

$$\Omega(L) > \Omega(K).$$

Remark 1.2. For the Shephard problems for affine surface area, Petty (see [18]) gave that negative solution $L \in \mathcal{K}_{os}^n$. Obviously, Corollary 1.2 extend Petty's result.

In this paper, the proofs of Theorems 1.1-1.4 will be given in Section 4. In Section 3, we obtain some properties of general L_p Blaschke bodies.

2 Preliminary Notes

2.1 L_p Mixed Volume Let $K_1, K_2 \in \mathcal{K}_0^n, p \geq 1$, and $\lambda_1, \lambda_2 \geq 0$ (not both 0). The L_p Minkowski addition $\lambda_1 \cdot K_1 +_p \lambda_2 \cdot K_2$ is a convex body whose support function is given by (see [21])

$$h(\lambda_1 \cdot K_1 +_p \lambda_2 \cdot K_2, \cdot)^p = \lambda_1 h(K_1, \cdot)^p + \lambda_2 h(K_2, \cdot)^p.$$

For $p \geq 1$, the L_p-mixed volume $V_p(K, L)$ of $K, L \in \mathcal{K}_o^n$, can be defined by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

In [15], Lutwak has shown that for $p \geq 1$, and each $K \in \mathcal{K}_o^n$, there exists a positive Borel measure $S_p(K, \cdot)$ on S^{n-1}, such that the L_p-mixed volume $V_p(K, L)$ has the following integral representation:

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h^p(L, u) dS_p(K, u),$$

for all $L \in \mathcal{K}_o^n$. The L_p-Minkowski inequality states that for $K, L \in \mathcal{K}_o^n$ and $p \geq 1$

$$V_p(K, L) \geq V(K)^{(n-p)/n} V(L)^{p/n}, \tag{2.1}$$

with equality for $p = 1$ if and only if K and L are homothetic, for $p > 1$ if and only if K and L are dilates.

2.2 L_p Affine Surface Area

In [16], Lutwak defined the L_p-affine surface area $\Omega_p(K)$ by using the Brunn-Minkowski-Fiery theory as follows:

$$n^{-\frac{p}{n}} \Omega_p(K)^{\frac{n+p}{n}} = \inf\{n V_p(K, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_o^n\}. \tag{2.2}$$

Here E^* is the polar set of a non-empty set E which be defined by (see [21])

$$E^* = \{x \in R^n : x \cdot y \leq 1, \text{for all } y \in E\}.$$

Lemma[21] **2.1.** *For $K \in \mathcal{K}_o^n$, $p \geq 1$ then*

$$\Omega_p(\phi K) = \Omega_p(K)$$

for all $\phi \in SL(n)$.

An immediate consequence of Lemma 2.1 is

$$\Omega_p(-K) = \Omega_p(K).$$

A convex body $K \in \mathcal{K}_o^n$ is said to have a L_p-curvature function [16] $f_p(K, \cdot) : S^{n-1} \to R$, if its L_p-surface area measure $S_p(K, \cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$\frac{dS_p(K, \cdot)}{dS} = f_p(K, \cdot). \tag{2.3}$$

In [16], Lutwak proved that if $K \in \mathcal{F}_o^n$ and $p \geq 1$, then the L_p-affine surface area of K have the integral representation

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K, u)^{\frac{n}{n+p}} dS(u). \tag{2.4}$$

Wang and Leng in [22] defined the ith L_p-mixed affine surface area as follows: For $K, L \in \mathcal{F}_o^n$, $p \geq 1$ and real i, the ith L_p-mixed affine surface area, $\Omega_{p,i}(K, L)$, of K and L is defined by

$$\Omega_{p,i}(K, L) = \int_{S^{n-1}} f_p(K, u)^{\frac{n-i}{n+p}} f_p(L, u)^{\frac{i}{n+p}} dS(u). \tag{2.5}$$

In the case $i = -p$, we write $\Omega_{p,-p}(K, L) = \Omega_{-p}(K, L)$ and see by (2.11) that

$$\Omega_{-p}(K, L) = \int_{S^{n-1}} f_p(K, u) f_p(L, u)^{-\frac{p}{n+p}} dS(u). \tag{2.6}$$

Obviously,

$$\Omega_{-p}(K, K) = \Omega_p(K). \tag{2.7}$$

For the ith L_p-mixed affine surface area, Wang and Leng in [22] proved the following Minkowski inequality. **Theorem 2.A.** *If $K, L \in \mathcal{F}_o^n$, $p \geq 1$, $i \in R$, then for $i < 0$ or $i > n$,*

$$\Omega_{p,i}(K, L)^n \geq \Omega_p(K)^{n-i} \Omega_p(L)^i; \tag{2.8}$$

for $0 < i < n$, inequality (2.8) is reversed. In every case, equality holds for $p = 1$ if and only if K and L are homothetic, for $n \neq p > 1$ if and only if K and L are dilates. For $i = 0$ or $i = n$, (2.8) is an identity.

For $i = -p$ in (2.8), we get that *if $K, L \in \mathcal{F}_o^n$, $p \geq 1$, then*

$$\Omega_{-p}(K, L)^n \geq \Omega_p(K)^{n+p} \Omega_p(L)^{-p}, \tag{2.9}$$

with equality for $p = 1$ if and only if K and L are homothetic, for $n \neq p > 1$ if and only if K and L are dilates.

3 Some Properties of General L_p Blaschke Bodies

In this section, we give some properties of general L_p Blaschke bodies.

Theorem 3.1. *If $K \in \mathcal{K}_o^n$, $p > 1$ and $\tau \in [-1, 1]$, then*

$$\nabla_p^{-\tau} K = \nabla_p^{\tau}(-K) = -\nabla_p^{\tau} K.$$

Proof. From (1.10) and (1.11), we obtain that for $p > 1$ and $\tau \in [-1, 1]$, we have that for any $u \in S^{n-1}$,

$$\begin{aligned} S_p(\nabla_p^{-\tau} K, u) &= f_1(-\tau) S_p(K, u) + f_2(-\tau) S_p(-K, u) \\ &= f_2(\tau) S_p(K, u) + f_1(\tau) S_p(-K, u) \\ &= S_p(\nabla_p^{\tau}(-K), u) \end{aligned}$$

Hence, we get

$$\nabla_p^{-\tau} K = \nabla_p^{\tau}(-K).$$

Further, we have that for any $u \in S^{n-1}$,

$$\begin{aligned} S_p(-\nabla_p^{\tau} K, u) &= S_p(\nabla_p^{\tau} K, -u) \\ &= f_1(\tau) S_p(K, -u) + f_2(\tau) S_p(K, -u) \\ &= f_1(\tau) S_p(-K, u) + f_2(\tau) S_p(-K, u) \\ &= S_p(\nabla_p^{\tau}(-K), u). \end{aligned}$$

Hence, we get

$$\nabla_p^{\tau}(-K) = -\nabla_p^{\tau} K.$$

Theorem 3.2. *For $K \in \mathcal{K}_o^n$, $p > 1$ and $\tau \in [-1, 1]$, if $\tau \neq 0$, then $\nabla_p^{\tau} K = \nabla_p^{-\tau} K$ if and only if $K \in \mathcal{K}_{os}^n$.*

Proof. From (1.7), we get that for all $u \in S^{n-1}$

$$S_p(\nabla_p^{\tau} K, u) = f_1(\tau) S_p(K, u) + f_2(\tau) S_p(-K, u), \tag{3.1}$$

$$S_p(\nabla_p^{-\tau} K, u) = f_2(\tau) S_p(K, u) + f_1(\tau) S_p(-K, u). \tag{3.2}$$

Hence, if $K \in \mathcal{K}_{os}^n$, i.e., $K = -K$. This combining with (3.1) and (3.2), we get

$$S_p(\nabla_p^{\tau} K, u) = S_p(\nabla_p^{-\tau} K, u).$$

Thus,

$$\nabla_p^{\tau} K = \nabla_p^{-\tau} K.$$

Conversely, if $\nabla_p^{\tau}K=\nabla_p^{-\tau}K$, then together with (3.1) and (3.2) to yield

$$[f_1(\tau)-f_2(\tau)]S_p(K,u)=[f_1(\tau)-f_2(\tau)]S_p(-K,u).$$

Since $f_1(\tau)-f_2(\tau)\neq 0$ when $\tau\neq 0$, thus it follows that $S_p(K,u)=S_p(-K,u)$ for all $u\in K^{n-1}$, i.e., $K\in\mathcal{K}_{os}^n$.

From Theorem 3.2, it immediately obtains the following corollary.

Corollary 3.1. *For $K\in\mathcal{K}_o^n$, $p>1$ and $\tau\in[-1,1]$, If K is not origin-symmetric, then $\nabla_p^{\tau}K=\nabla_p^{-\tau}K$ if and only if $\tau=0$.*

Theorem 3.3. *If $K\in\mathcal{K}_{os}^n$, $p>1$ and $\tau\in[-1,1]$, then*

$$\nabla_p^{\tau}K=K.$$

Proof. Since $K\in\mathcal{K}_{os}^n$, i.e., $K=-K$, by (1.10), we know for any $u\in S^{n-1}$

$$S_p(\nabla_p^{\tau}K,u)=f_1(\tau)S_p(K,u)+f_2(\tau)S_p(-K,u)=S_p(K,u).$$

That is

$$\nabla_p^{\tau}K=K.$$

Theorem 3.3. *If $K\in\mathcal{K}_o^n$, $p>1$ and $\tau\in[-1,1]$, if $\phi\in SL(n)$, then*

$$\nabla_p^{\tau}\phi K=\phi\nabla_p^{\tau}K. \tag{3.3}$$

Proof. From $V_p(K,Q)=V_p(\phi K,\phi Q)$ for any $K,Q\in\mathcal{K}_o^n$ and $\phi\in SL(n)$, we have

$$\begin{aligned}
V_p(\phi\nabla_p^{\tau}K,Q) &= V_p(\nabla_p^{\tau}K,\phi^{-1}Q)\\
&= V_p(f_1(\tau)\cdot K\sharp_p f_2(\tau)\cdot(-K),\phi^{-1}Q)\\
&= f_1(\tau)V_p(K,\phi^{-1}Q)+f_2(\tau)V_p(-K,\phi^{-1}Q)\\
&= f_1(\tau)V_p(\phi K,Q)+f_2(\tau)V_p(-\phi K,Q)\\
&= V_p(f_1(\tau)\cdot\phi K\sharp_p f_2(\tau)\cdot(-\phi K),\phi^{-1}Q)\\
&= V_p(\nabla_p^{\tau}\phi K,Q).
\end{aligned}$$

So (3.3) is proved.

4 Proofs of Theorems

In this section, we complete the proofs of Theorems 1.1-1.4.

Proof of Theorem 1.1. By (1.10) and (1.12), we get for any $\tau\in[-1,1]$ and $Q\in\mathcal{K}_o^n$,

$$\begin{aligned}
V_p(\nabla_p^{\tau}K,Q) &= V_p(f_1(\tau)\cdot K\sharp_p f_2(\tau)\cdot(-K),Q)\\
&= f_1(\tau)V_p(K,Q)+f_2(\tau)V_p(-K,Q)\\
&\geq V(Q)^{\frac{p}{n}}[f_1(\tau)V(K)^{\frac{n-p}{n}}+f_2(\tau)V(-K)^{\frac{n-p}{n}}]\\
&= V(Q)^{\frac{p}{n}}V(K)^{\frac{n-p}{n}}.
\end{aligned}$$

Let $Q = \nabla_p^{\tau} K$

$$V(\nabla_p^{\tau} K)^{\frac{n-p}{n}} \geq V(K)^{\frac{n-p}{n}}$$

Therefore, we obtain for $n > p > 1$,

$$V(\nabla_p^{\tau} K) \geq V(K). \tag{4.1}$$

This gives the right inequality of (1.13).

Clearly, equality holds in (4.1) if $\tau = \pm 1$. Besides, if $\tau \neq \pm 1$, then by the condition of equality in (2.1), we see that equality holds in (4.1) if and only if K and $-K$ are dilates, this yields $K = -K$, i.e., K is an origin-symmetric convex body. This means that if $\tau \neq \pm 1$, then equality holds in the right inequality of (1.13) if and only if K is origin-symmetric.

Now, we prove the left inequality of (1.13). From (1.5) and (1.7), we know that for any $u \in S^{n-1}$,

$$\begin{aligned} S_p(\nabla_p^{\tau} K, u) + S_p(\nabla_p^{-\tau} K, u) &= f_1(\tau) S_p(K, u) + f_2(\tau) S_p(-K, u) \\ &\quad + f_2(\tau) S_p(K, u) + f_1(\tau) S_p(-K, u) \\ &= S_p(K, u) + S_p(-K, u) \\ &= 2 S_p(\nabla_p K, u) \\ &= S_p(2 \cdot \nabla_p K, u). \end{aligned}$$

So

$$\nabla_p^{\tau} K \sharp_p \nabla_p^{-\tau} K = 2 \cdot \nabla_p K.$$

By $2 \cdot \nabla_p K = 2^{\frac{1}{n-p}} \nabla_p K$, we get

$$V(2 \cdot \nabla_p K)^{\frac{n-p}{n}} = 2 V(\nabla_p K)^{\frac{n-p}{n}}.$$

For any $\tau \in [-1, 1]$ and $Q \in \mathcal{K}_o^n$, we have

$$\begin{aligned} V_p(2 \cdot \nabla_p K, Q) &= V_p(\nabla_p^{\tau} K \sharp_p \nabla_p^{-\tau} K, Q) \\ &= V_p(\nabla_p^{\tau} K, Q) + V_p(\nabla_p^{-\tau} K, Q) \\ &\geq V(Q)^{\frac{p}{n}} [V(\nabla_p^{\tau} K)^{\frac{n-p}{n}} + V(\nabla_p^{-\tau} K)^{\frac{n-p}{n}}] \\ &= V(Q)^{\frac{p}{n}} [V(\nabla_p^{\tau} K)^{\frac{n-p}{n}} + V(-\nabla_p^{\tau} K)^{\frac{n-p}{n}}]. \end{aligned}$$

Let $Q = 2 \cdot \nabla_p K$, This gives that for $n > p > 1$,

$$V(\nabla_p K) \geq V(\nabla_p^{\tau} K). \tag{4.2}$$

This just the left inequality of (1.13).

Obviously, if $\tau = 0$, then equality holds in (4.2). If $\tau \neq 0$, according to the equality condition of (2.1), we see that equality holds in (4.2) if and only if $\nabla_p^{\tau} K$ and $\nabla_p^{-\tau} K$ are dilates, this implies $\nabla_p^{\tau} K = \nabla_p^{-\tau} K$. Therefore, using Corollary 3.1, we obtain that if K is not origin-symmetric body, then equality holds in (4.2) if and only if $\tau = 0$. This shows that if $\tau \neq 0$, then equality holds in the left inequality of (1.13) if and only if K is origin-symmetric.

Proof of Theorem 1.2. From definition (1.7) and (2.3), we have that

$$f_p(\nabla_p^{\tau} K, u) = f_1(\tau) f_p(K, u) + f_2(\tau) f_p(-K, u).$$

By (2.6), we get

$$\begin{aligned}\Omega_{-p}(\nabla_p^{\tau} K, Q) &= \int_{S^{n-1}} f_p(\nabla_p^{\tau} K, u) f_p(Q, u)^{-\frac{p}{n+p}} dS(u) \\ &= f_1(\tau) \int_{S^{n-1}} f_p(K, u) f_p(Q, u)^{-\frac{p}{n+p}} dS(u) \\ &\quad + f_2(\tau) \int_{S^{n-1}} f_p(-K, u) f_p(Q, u)^{-\frac{p}{n+p}} dS(u) \\ &= f_1(\tau) \Omega_{-p}(K, Q) + f_2(\tau) \Omega_{-p}(-K, Q).\end{aligned}$$

By Lemma 2.1, we know $\Omega_p(K) = \Omega_p(-K)$. Let $Q = \nabla_p^{\tau} K$ and combine with (2.9), we know

$$\Omega_p(\nabla_p^{\tau} K) \geq \Omega_p(K), \tag{4.3}$$

i.e., the right inequality of (1.14) is obtained.

Clearly, equality holds in (4.3) if $\tau = \pm 1$. If $\tau \neq \pm 1$, equality of (4.3) holds if and only if K and $-K$ are dilates. This yields $K = -K$, thus K is an origin-symmetric star body. Therefore, if $\tau \neq \pm 1$, equality holds in the right inequality of (1.14) if and only if K is origin-symmetric.

Further, we complete proof of the left inequality of (1.14). From Theorem 3.1, we know that

$$\nabla_p^{-\tau} K = -\nabla_p^{\tau} K.$$

Thus, (3.1) and (3.2) can be written as

$$\nabla_p K = \frac{1}{2} \cdot \nabla_p^{\tau} K \sharp_p \frac{1}{2} \cdot \nabla_p^{-\tau} K.$$

Similar to the proof of inequality (4.3), we have

$$\begin{aligned}\Omega_{-p}(2 \cdot \nabla_p K, Q) &= \Omega_{-p}(\nabla_p^{\tau} K \sharp_p \nabla_p^{-\tau} K, Q) \\ &= \Omega_{-p}(\nabla_p^{\tau} K, Q) + \Omega_{-p}(\nabla_p^{-\tau} K, Q) \\ &= \Omega_{-p}(\nabla_p^{\tau} K, Q) + \Omega_{-p}(-\nabla_p^{\tau} K, Q)\end{aligned}$$

Let $Q = 2 \cdot \nabla_p K$ and combine with (2.9), we know

$$\Omega_p(\nabla_p K) \geq \Omega_p(\nabla_p^{\tau} K). \tag{4.4}$$

This yields the left inequality of (1.14).

Similar to the proof of equality in inequality (4.2), we easily know that equality holds in (4.4) if and only if $\nabla_p^\tau K = \nabla_p^{-\tau} K$ when $\tau \neq 0$. Hence, if $\tau \neq 0$, using Theorem 3.2 to get equality holds in the left inequality of (1.14) if and only if K is origin-symmetric.

In order to prove Theorems 1.3-1.4, the following lemma is required.

Lemma 4.1. *If $K \in K_o^n$, $p > 1$ and $\tau \in [-1, 1]$, then*

$$\Pi_p(\nabla_p^\tau K) = \Pi_p K.$$

Proof. From definition (1.10) and by (1.3), we have that

$$\begin{aligned}
h^p_{\Pi_p(\nabla_p^\tau K)} &= \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} | u \cdot v |^p \, dS_p(\nabla_p^\tau K, v) \\
&= f_1(\tau) \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} | u \cdot v |^p \, dS_p(K, v) \\
&\quad + f_2(\tau) \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} | u \cdot v |^p \, dS_p(-K, v) \\
&= f_1(\tau) \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} | u \cdot v |^p \, dS_p(K, v) \\
&\quad + f_2(\tau) \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} | u \cdot v |^p \, dS_p(K, -v) \\
&= \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} | u \cdot v |^p \, dS_p(K, v) = h^p_{\Pi_p K}(u)
\end{aligned}$$

i.e.

$$\Pi_p(\nabla_p^\tau K) = \Pi_p K.$$

Proof of Theorem 1.3. Since K is not origin-symmetric convex body, thus from Theorem 1.1, we know that if $\tau \neq \pm 1$, then

$$V(\nabla_p^\tau K) > V(K).$$

Choose $\varepsilon > 0$, such that $V\left((1-\varepsilon)\nabla_p^\tau K\right) > V(K)$. Therefore, let $L = (1-\varepsilon)\nabla_p^\tau K$ (for $\tau = 0$, $L \in K_{os}^n$; for $\tau \neq 0$, $L \in K_o^n$), then

$$V(L) > V(K).$$

But from Lemma 4.1, and notice that

$$\Pi_p\left((1-\varepsilon)K\right) = (1-\varepsilon)^{\frac{n-p}{p}} \Pi_p K,$$

we can get

$$\Pi_p L = \Pi_p\left((1-\varepsilon)\nabla_p^\tau L\right) = (1-\varepsilon)^{\frac{n-p}{p}} \Pi_p(\nabla_p^\tau K) = (1-\varepsilon)^{\frac{n-p}{p}} \Pi_p K \subset \Pi_p K.$$

Proof of Theorem 1.4. Since K is not origin-symmetric convex body, thus by Theorem 1.2, we know that for $\tau \neq \pm 1$,

$$\Omega_p(\nabla_p^\tau K) > \Omega_p(K).$$

Choose $\varepsilon > 0$, such that $\Omega_p\Big((1-\varepsilon)\nabla_p^\tau K\Big) > \Omega_p(K)$. Therefore, let $L = (1-\varepsilon)\nabla_p^\tau K$, then $L \in K_o^n$ and

$$\Omega_p(L) > \Omega_p(K).$$

But, similar to the proof of Theorem 1.3, we may obtain $\Pi_p L \subset \Pi_p K$.

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