Pure Mathematical Sciences, Vol. 14, 2025, no. 1, 5 - 15
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/pms.2025.91604

Study of Some Asymptotic Properties of the Axis-quasi-graduations of the Same Ring, Particular Case of Quasi-graduations

Eugène Deval Béché

Université Polytechnique de Man
UFR des Sciences et Technologies
Département de Mathématiques et Informatique
BP 20 Man, Cote d'Ivoire

Abstract

This article generally studies a new class of families not necessarily nested with subgroups obeying a certain compatibility, or from two axis-quasi-graduations of an arbitrarily chosen ring we give properties of binary relations and classical operations obtained from two rings axis-quasi-graduations. The main results obtained will be used to study and characterize certain concepts of commutative algebra.

Keywords: axis-quasi-graduation, good, Noetherian, strongly Noetherian, approximately subgroup power, finer, axial sum, axially adic

1. Introduction

All rings are assumed to be commutative and unitary. The following notions: **good, Noetherian, strongly Noetherian, approximately power** etc.... have been the subject of several research works in the asymptotic theory of ideals and filtrations through several results obtained in relation to other notions of commutative algebra. Hence the richness of these notions which have had a great impact in the theory of filtrations through the inspiration of certain researchers. As part of our research we will consider two axis-quasi-graduations of the same ring, presented some properties taken from binary relations in [2] and important properties on the notions **good, Noetherian, strongly Noetherian, approximately power** etc... presented in [3] for the ring quasi-graduations which have been introduced and studied since 2002 by Youssouf DIAGANA in [6] and author of several publications, other researchers such as Deval Béché and Brou Kouadjo have also worked on this notion which today makes it possible to give several extensions of the very important concepts of commutative algebra. These different properties obtained will allow us to establish several characterizations of the aforementioned notions.

2. Generality on the axis-quasi-graduations of ring

2.1. Definitions.

Definition 2.1.1. *Let $\mathcal{R}$ be a ring and $g = (G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ a family of subgroups of $\mathcal{R}$. We say that g is a quasi-graduation of $\mathcal{R}$ if :*

(1) $\mathcal{A} = G_0$ *un subring of* $\mathcal{R}$
(2) $G_\infty = (0)$
(3) $G_pG_q \subseteq G_{p+q} \ \ \forall\, p, q \in \mathbb{N}$.

Example 2.1.1. (i) *Let $f = (F_n)_{n\in\mathbb{Z}}$ be a ring filtration $\mathcal{A}$. By posing $F_\infty = (0)$ and $f_\infty = (F_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$, f_∞ is a quasi-graduation of the ring $\mathcal{A}$.*

(ii) *Let $\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathrm{A}_n$ a graduated ring. So by posing $\mathrm{A}_\infty = (0)$ and $g = (\mathrm{A}_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$, it's clear that g is a quasi-graduation of the ring $\mathcal{A}$.*

(iii) *Let p a prime number. Let's pose $\mathcal{R} = \mathbb{R}\,;\ \mathcal{A} = \mathbb{Z}\,;\ I = \sqrt{p}\mathbb{Z}$ and $g = (G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ the family defined by: $G_n = \sqrt{p}^n\mathbb{Z}\,,\ \forall\, n \geq 1\,;\ G_n = (0)\,,\ \forall\, n < 0\,;\ G_0 = \mathbb{Z}$ et $G_\infty = (0)$. it's clear that g is a family of subgroups of $\mathcal{R}$ and g is a quasi-graduation of the ring $\mathcal{R}$.*

Definition 2.1.2. *Let $\mathcal{R}$ be a ring and $g = (G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ a family of subgroups of $\mathcal{R}$. We say that g is a axis-quasi-graduation of $\mathcal{R}$ if :*

(1) $\mathcal{A} = G_0$ *a subring of* $\mathcal{R}$
(2) $G_\infty = (0)$
(3) $G_pG_q \subseteq G_0G_{p+q} \ \ \forall\, p, q \in \mathbb{N}$.

Example 2.1.2. *Let's pose $\mathcal{R} = \mathbb{R}$ and $g = (G_n)$ a sequence defined by :*

$$\begin{cases} G_n = \mathbb{Z}\left[\sqrt{p}\right]\ ,\ \text{if } n \leq 0\,; \\ G_\infty = (0)\,; \\ G_n = \left(\dfrac{q}{r}\right)^n \sqrt{p}^{n+1}\mathbb{Z}\,,\ \text{if } n \geq 1 \end{cases}$$

Or p, q and r are pairwise distinct prime numbers. We rigorously show that g is a axis-quasi graduation of $\mathcal{R}$

Example 2.1.3. *Put $\mathcal{R} = \mathbb{C}$, Let p be a prime number . Let $g = (G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ the family defined by :*

$$G_{+\infty} = (0)\,,\ G_0 = \mathbb{Z}[i]\ \text{ and } G_n = \begin{cases} (0)\ ,\ \text{if } n < 0\,; \\ i^n\sqrt{p}^n\mathbb{Z}\,,\ \text{if } n \geq 1\,. \end{cases} \qquad \text{with } i^2 = -1$$

It is easily shown that , g is an axis-quasi-graduation of $\mathbb{C}$ which isn't a quasi-graduation of $\mathbb{C}$.

Notation 2.1.1. *We denote respectively $\mathbb{A}_{QG}(\mathcal{R})$ and $\mathbb{Q}_G(\mathcal{R})$ $\mathbb{A}_{QG}(\mathcal{R})$ the set of axis-quasi-graduations and the quasi-graduations of the ring $\mathcal{R}$. these sets are ordered by the relation ($\leq$), defined by $g = (G_n) \leq f = (F_n)$ if and only if $G_n \subseteq F_n, \forall\, n \in \mathbb{Z}$.*

2.2. **Algebraic structure of $\mathbb{A}_{\mathbf{QG}}(\mathcal{R})$ and $\mathbb{Q}_{\mathbf{G}}(\mathcal{R})$.**

Lemma 2.2.1. *The sets $\Big(\mathbb{A}_{QG}(\mathcal{R}), +\Big)$, $\Big(\mathbb{A}_{QG}(\mathcal{R}), \times\Big)$, $\Big(\mathbb{Q}_G(\mathcal{R}), +\Big)$ and $\Big(\mathbb{Q}_G(\mathcal{R}), \times\Big)$ are abelian semi-groups.*

Proof. Let us show that $\Big(\mathbb{A}_{\mathrm{QG}}(\mathcal{R}), +\Big)$, $\Big(\mathbb{A}_{\mathrm{QG}}(\mathcal{R}), \times\Big)$ are abelian semigroups.
Let $f = (F_n), g = (G_n), h = (H_n) \in \mathbb{A}_{\mathrm{QG}}(\mathcal{R})$.

i) $f + g = \left(\sum\limits_{i=0}^{n} F_i G_{n-i}\right)_n = \left(\sum\limits_{j=0}^{n} G_j F_{n-j}\right)_n = g + f \in \mathbb{A}_{\mathrm{QG}}(\mathcal{R})$. Therefore $(\mathbb{A}_{\mathrm{QG}}(\mathcal{R}), +)$ is an abelian semigroup.
Likewise we check that $(\mathbb{A}_{\mathrm{QG}}(\mathcal{R}), \times)$, $(\mathbb{Q}_{\mathrm{G}}(\ mathcalB), +)$ and $(\mathbb{Q}_{\mathrm{G}}(\mathcal{R}), \times)$ are also abelian semigroups. □

Remark 2.2.1. *1. The sets $\Big(\mathbb{A}_{QG}(\mathcal{R}), +\Big)$, $\Big(\mathbb{A}_{QG}(\mathcal{B}), \times\Big)$, $\Big(\mathbb{Q}_G(\mathcal{R}), +\Big)$ and $\Big(\mathbb{Q}_G(\mathcal{R}), \times\Big)$ are abelian monoids. Indeed : In $\Big(\mathbb{A}_{QG}(\mathcal{R}), +\Big)$ and $\Big(\mathbb{Q}_G(\mathcal{R}), +\Big)$ we have, $g + g_\infty = g$ (where $g_\infty = (0)$) is the element neutral for addition.*

In $\Big(\mathbb{Q}_G(\mathcal{R}), \times\Big)$ we have, $g.g^* = g$ (where $g^* = G_0$) is the neutral element for multiplication.

2. $\Big(\mathbb{Q}_G(\mathcal{R}), +\Big)$ is a submonoid of $\Big(\mathbb{A}_{QG}(\mathcal{R}), +\Big)$.

Remark 2.2.2. *By definition, any quasi-graduation of a ring $\mathcal{R}$ is an axis-quasi-graduation of this ring and through* ***previous examples*** *and we see that all axis-quasi-graduations aren't necessarily quasi-graduations. We have :*

(1) *If g is an axis-quasi-graduation of $\mathcal{R}$ and $\mathcal{A}G_m = G_m$ for all $m \in \mathbb{Z}$, then g is called a* ***quasi-graduation*** *of $\mathcal{R}$.*
(2) *If g is an axis-quasi-graduation of $\mathcal{R}$, we say that g is an* ***axis-filtration*** *of $\mathcal{R}$ if for all $n \in \mathbb{Z}$, $\mathcal{A}G_{n+1} \subseteq \mathcal{A}G_n$.*

Definition 2.2.1. *Let $\mathcal{R}$ be a ring and $g = (G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ an axis-quasi-graduation of $\mathcal{R}$. We pose $\mathcal{A} = G_0$ et $R(\mathcal{A}, g) = \bigoplus_{n\in\mathbb{N}} \mathcal{A}G_nX^n$; $\mathfrak{R}(\mathcal{A}, g) = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}G_nX^n$. $R(\mathcal{A}, g)$ et $\mathfrak{R}(\mathcal{A}, g)$ are respectively called Rees ring and generalized Rees ring of g.*

Definition 2.2.2. *Let $g = (G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ be an axis-quasi-graduation of $\mathcal{R}$ and I a subgroup of $(\mathcal{R}, +)$ such as $IG_0 = I$. We say that g is $I-$good if :*
i) $\forall\, n \in \mathbb{N} : \ IG_n \subseteq G_{n+1}$.
ii) $\exists\, n_0 \in \mathbb{N} : \ \forall\, n \geq n_0\,,\ IG_n = G_{n+1}$.

Consequence 2.2.1. (1) $I^nG_{n_0} = G_{n+n_0}, \forall\, n \in \mathbb{N}$,
(2) *g is $I-$good $\Rightarrow$ g is G_1-good* ,
(3) $I^n \subseteq G_0G_n, \forall\, n \in \mathbb{N}$

Definition 2.2.3. *Let $g = (G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ be an axis-quasi-graduation of $\mathcal{R}$.*
1) we say that g is Approximately Power of subgroups of $(\mathcal{R}, +)$ (abbreviated AP) if there exets a sequence $(k_n)_{n\in\mathbb{N}}$ of naturals integers non-zero such that :
i) $\displaystyle\lim_{n\to+\infty} \frac{k_n}{n} = 1$
ii) $\forall\, m, n \in \mathbb{N}, G_{k_nm} \subseteq G_0G_n^m$
2) We say that g is strongly AP if there exets $m \geq 1$ checking :
$G_m^{n+1} \subseteq G_0G_{mn+j} \subseteq G_0G_m^n, \forall\, n \geq 0$ *an* $\forall\, j \in \{0, 1, ..., m-1\}$

Definition 2.2.4. *Let $g = (G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ be an axis-quasi-graduation of $\mathcal{R}$. We say that g is $k-$regular if there exets a non-zero integer k such as $G_0g^{(k)} = g_{(G_0G_k)}$ that's to say $\widetilde{g^{(k)}} = \widetilde{g_{G_k}}$.*

Remark 2.2.3. *If g is strongly AP, so there exets $k \geq 1$ such as g is $k-$regular*

Proposition 2.2.1. *Let $g = (G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ be an axis-quasi-graduation of $\mathcal{R}$.*
1) *g $I-$adic $\Rightarrow$ g $I-$good $\Rightarrow$ g $k-$regular* .
2) *g strongly AP $\Rightarrow$ g AP*

Proof 2.2.1. *See* [3] .

Definition 2.2.5. *We say g is an axis-quasi-filtration of $\mathcal{R}$ if the sequence $(G_n)_{n\in\mathbb{N}^*}$ is decreasing .*

For this guy of the axis-quasi-graduation, we defined the class of the axis-quasi-filtrations noetherian :

Definition 2.2.6. *Let $g = (G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ be an axis-quasi-filtration of $\mathcal{R}$.*
1) We say that g is noetherian if :
i) *G_0 is noetherian* ,
ii) *$R(G_0, G_0g)$ is noetherian* .
2) We say that g is strongly Noetherian if :
i) *G_0 is noetherian* ,
ii) $\exists\, k \geq 1$ *such as* $\forall\, m, n \geq k,\ G_0G_mG_n = G_0G_{m+n}$.
In other words g is strongly Noetherian if only if the quasi-graduation from g is strongly Noetherian .

3. Classical operations on ring axis-quasi-graduations

Definition 3.0.7. *Let $f=(F_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ and $g=(G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ are two axis-quasi-graduations of the same ring $\mathcal{R}$. We have the following relations:*
1) ***The total sum*** *$f+g=(T_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$, defined by:*

$$T_\infty=(0),\, T_m=(0)\,,\ \forall\, m<0 \ et\ T_m=F_0G_0\sum_{r=0}^{m}F_rG_{m-r},\,\forall\ m\geq 0\,.$$

2) ***The axial sum*** *$f\top g=(S_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$, defined by:*

$$S_{+\infty}=(0),\, S_n=(0)\,,\forall\, n<0\ et\ S_n=F_0G_0(F_n+G_n),\,\forall\, n\geq 0\,.$$

3) We say that f ***is less than or equal to*** *g and we note $f\leq g$ if, $\forall\, n\in\mathbb{N},\, F_n\subseteq G_n$.*
4) We say that f is ***thinner*** *than g and we note $f\precsim g$ if:*

$$\forall\, m,n\in\mathbb{N},\, F_mG_n\subseteq F_0G_{m+n}\,.$$

5) We say that f est ***axially*** *$g-$**adic** and we note $f\sim g$ if,*

$$\forall\, m,n\in\mathbb{N},\, F_mG_n=F_0G_{m+n}.$$

Remark 3.0.4. *Let $\mathcal{R}$ be a ring and $f=(F_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ and $g=(G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ two axis-quasi-graduations (resp. two quasi-graduations) of the ring $\mathcal{R}$. So :*

(1) *The following families are the axis-quasi-graduations (resp. the quasi-graduations) of the ring $\mathcal{R}$: fg and $f+g$.*
(2) *Si $f\precsim g$, so the sum axial $f\top g=F_0g$ and is an axis-quasi-graduation (resp. une quasi-graduation) of the ring $\mathcal{B}$, $F_0f\top g$ and $f\top G_0g$ are also axis-quasi-graduations (resp. the quasi-graduations) of the ring $\mathcal{R}$.*

Proposition 3.0.2. *Let $f=(F_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ and $g=(G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ are two axis-quasi-graduations (resp. two quasi-graduations) of the ring $\mathcal{R}$ such as $f\sim g$. We have,*

(1) *$F_0G_{n+1}=F_1^nG_1,\,\forall\, n\in\mathbb{N}^*$;*
(2) *If more $F_0=G_0$, $\widetilde{f}=\widetilde{g}$ (resp. $f=g$) .*

4. Some properties of Characterization of the notion of good axis-quasi-graduations

Theorem 4.0.1. *Let $\mathcal{R}$ be a ring and $f=(F_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ and $g=(G_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$ are two axis-quasi-graduations of $\mathcal{R}$. Let I et J of subgroups of $(\mathcal{R},+)$ such as $IF_0=I$ and $JG_0=J$.*

Suppose that f and g are respectively $I-$good et $J-$good .

$$So,\ f+g\ is\ (I+J)-good\ and\ fg\ is\ (IJ)-good$$

Proof 4.0.2. *Suppose that f is $I-$good and that g is $J-$good . Let's show that $f+g$ is $(I+J)-$ good.*
f and g being respectively $J-$good and $I-$good, so we have :

i) $\forall\, n \in \mathbb{N}$ $IF_n \subseteq F_{n+1}$ *and* $JG_n \subseteq G_{n+1}$.

$$\begin{aligned}(I+J)T_n &= F_0G_0\left(\sum_{i=0}^{n} IF_iG_{n-i} + \sum_{i=0}^{n} F_iJG_{n-i}\right)\\ &\subseteq F_0G_0\sum_{i=0}^{n} F_{i+1}G_{n-i} + F_0G_0\sum_{i=0}^{n} F_iG_{n-i+1}\\ &= F_0G_0\left(\sum_{i=0}^{n} F_{i+1}G_{n-i} + \sum_{i=0}^{n} F_iG_{n-i+1}\right)\\ &\subseteq F_0G_0\sum_{i=0}^{n+1} F_iG_{n+1-i}\\ &\subseteq T_{n+1}\end{aligned}$$

ii) $\exists\, n_0 \in \mathbb{N}\, :\, \forall\, n \geq n_0\,,\ IF_n = F_{n+1}$ *and* $JG_n = G_{n+1}$ *and we have,*

$$\begin{aligned}T_{n+1} &= F_0G_0\sum_{i=0}^{n+1} F_iG_{n+1-i}\\ &= F_0G_0\,(F_0G_{n+1} + F_1G_n + F_2G_{n-1} + ... + F_{n-1}G_2 + F_nG_1 + F_{n+1}G_0)\\ &= F_0G_0\left(\sum_{i=0}^{n} F_{i+1}G_{n-i} + \sum_{i=0}^{n} F_iG_{n-i+1}\right)\\ &= (I+J)T_n\end{aligned}$$

As a result $(f+g)$ *is* $(I+J)-$*good* .

Corollary 4.0.1. *Let* $\mathcal{R}$ *be a ring and* $f = (F_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ *and* $g = (G_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$ *are two axis-quasi-graduations of* $\mathcal{R}$*. Let* I *and* J *the subgroups of* $(\mathcal{R},+)$ *such as* $IF_0 = I = IG_0$*.*

So, f *and* g $I-$*good* $\Rightarrow f+g$ *is* $I-$*good* $\Rightarrow f+g$ *is* $k-$*regular.*

Proof 4.0.3. *See* ***Theorem*** *4.0.1*

Proposition 4.0.3. *Let* $\mathcal{R}$ *be a ring and* $f = (F_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ *and* $g = (G_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ *are two axis-quasi-graduations of* $\mathcal{R}$ *such as* $f \precsim g$ *and* $F_0 \subseteq G_0$*. Let* I *be a subgroup of* $(\mathcal{R},+)$ *such as* $IG_0 = I$ *. Then*

$$i)\ \tilde{g}\ \ I-good \iff f+g\ \ I-good \iff f\top g\ \ I-good$$

$$ii)\ g\ \ I-good \Rightarrow f+g\ \ I-good \Rightarrow f\top g\ \ I-good$$

Proof 4.0.4. *Suppose that* $f \precsim g$ *so* $F_mG_n \subseteq F_0G_{m+n}\,,\ \forall\, m,n \geq 0$
i) Let's show that g $I-$*good* $\iff$ $f+g$ $I-$*good* $\iff$ $f\top g$ $I-$*good . Suppose that* g *est* $I-$*good , so* $IG_n \subseteq G_{n+1}\,,\ \forall\, n \in \mathbb{N}$ *and there* $\exists\, n_0 \in \mathbb{N}$ *such as* $IG_n = G_{n+1}\,,\ \forall\, n \geq n_0$ *.*
$IT_n = I(F_0G_n) = F_0(IG_n) \subseteq F_0G_{n+1} = T_{n+1},\ \forall\, n \in \mathbb{N}\ \ (1)$.

$\forall\, n \geq n_0$, $T_{n+1} = F_0G_{n+1} = F_0(IG_n) = I(F_0G_n) = IT_n$ (2) . From (1) and (2) it is clear that $f+g$ is $I-$good . Suppose that $f+g$ is $I-$good, so $IT_n \subseteq T_{n+1}$, $\forall\, n \in \mathbb{N}$ and there $\exists\, n_0 \in \mathbb{N}$ such as $IT_n = T_{n+1}$, $\forall\, n \geq n_0$.

$$\begin{aligned} \forall\, n \in \mathbb{N},\ IG_n &\subseteq I(F_0G_n) \\ &\subseteq IT_n \\ &\subseteq T_{n+1} \\ &\subseteq F_0G_{n+1} \\ &\subseteq G_0G_{n+1} \end{aligned}$$

$$\begin{aligned} \textit{Let } n \geq n_0,\ G_{n+1} &\subseteq F_0G_{n+1} \\ &\subseteq T_{n+1} \\ &\subseteq IT_n \\ &\subseteq F_0(IG_n) \\ &\subseteq IG_0G_n \end{aligned}$$

as a result $I(G_0G_n) = G_0G_{n+1}$, $\forall\, n \geq n_0$. Then $\widetilde{g}$ is $I-$good . Therefore $\widetilde{g}$ $I-good$ $\iff$ $f+g$ $I-good$ $\iff$ $f \top g$ $I-good$.

$ii)$ Immediate

Proposition 4.0.4. *Let $\mathcal{R}$ be a ring and $f = (I_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ and $g = (G_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$ two axis-quasi-graduations of $\mathcal{R}$ such as $f \sim g$ and $G_0 = I_0$. Let I be a subgroup of $(\mathcal{R},+)$ such us $IG_0 = I$. We have*

$i)$ $\widetilde{g}$ $I-$good $\Leftrightarrow$ $\widetilde{f}$ $I-$good $\Leftrightarrow$ $f+g$ $I-$good .

$ii)$ $\widetilde{g}$ AP $\Leftrightarrow$ $\widetilde{f}$ AP $\Leftrightarrow$ $f+g$ is AP.

$iii)$ $\widetilde{g}$ $k-$ regular$\Leftrightarrow$ $\widetilde{f}$ $k-$ regular $\Leftrightarrow$ $f+g$ is regular.

Proof 4.0.5. ***Trivial***

5. Around a Characterization of the Noetherian axis-quasi-filtrations

5.1. Characterization of an axis-quasi-Noetherian filtration.

Theorem 5.1.1. *Let $\mathcal{R}$ be a ring and $g = (G_n)_{n\in\mathbb{N}^*}$ be an axis-quasi-filtration of $\mathcal{R}$ such as G_0 is a ring noetherian, then the following assertions are equivalent :*

$i)$ g is noetherian

$ii)$ $\mathcal{R}(G_0,g)$ is noetherian

$iii)$ $\mathfrak{R}(G_0,g)$ is noetherian

$iv)$ $\mathcal{R}(G_0,g)$ is an G_0-algebra finished type

$v)$ $\mathfrak{R}(G_0,g)$ is an G_0-algebra finished type

$vi)$ $\exists\, k \geq 1$ such as $g^{(k)} = g_{G_k}$

$vii)$ $\exists\, k \geq 1$ such as $(\mathcal{R}\,(G_0,g))^{(k)} \simeq \mathcal{R}(G_0,G_k)$

$viii)$ $\exists\, k \geq 1$ such as $(\mathcal{R}\,(G_0,g))^{(k)}$ is noetherian

ix) $\exists\, k \geq 1$ *such as* $\forall\, j \geq k\,,\ G_0G_{k+j} = G_0G_kG_j$

Proof 5.1.1. *See* [3] .

Remark 5.1.1. *Let* $\mathcal{R}$ *be a ring and* $g = (G_n)_{n\in\mathbb{N}^*}$ *be an axis-quasi-filtration of* $\mathcal{R}$ *such as* G_0 *is a noetherian ring and* I *a subgroup of* $(\mathcal{R},+)$ *such as* $IG_0 = I$. *So we have :* g $I-$*good* $\Rightarrow$ g *is strongly noetherian* $\Rightarrow$ g *is noetherian .*

5.2. Characterization of Noetherian axis-quasi-Filtrations.

Theorem 5.2.1. *Let* $\mathcal{R}$ *be a ring and* $f = (F_n)_{n\in\mathbb{N}^*}$ *and* $g = (G_n)_{n\in\mathbb{N}^*}$ *are two axis-quasi-filtrations of* $\mathcal{R}$ *such as* $f \sim g$ *is* $F_0 = G_0$. *If* F_0 *is noetherian, then the following assertions are equivalent :*

i) g *is noetherian*
ii) f *is noetherian*
iii) $f+g$ *is noetherian*
iv) $\mathcal{R}(F_0G_0, f+g)$ *is noetherian*
v) $\mathfrak{R}(F_0G_0, f+g)$ *eis noetherian*
vi) $\mathcal{R}(F_0G_0, f+g)$ *is an* F_0G_0-*finite type algebra*
vii) $\mathfrak{R}(F_0G_0, f+g)$ *is an* F_0G_0-*finite type algebra*
viii) $\exists\, k \geq 1$ *such as* $(f+g)^{(k)} = (f+g)_{T_k}$
ix) $\exists\, k \geq 1$ *such as* $(\mathcal{R}\,(F_0G_0, f+g))^{(k)} \simeq \mathcal{R}(F_0G_0, T_k)$
x) $\exists\, k \geq 1$ *such as* $\forall\, j \geq k\,,\ T_{k+j} = T_kT_j$
xi) $\exists\, k \geq 1$ *such as* $\forall\, j \geq k\,,\ G_0G_{k+j} = G_0G_kG_j$
xii) $\exists\, k \geq 1$ *such as* $\forall\, j \geq k\,,\ F_0F_{k+j} = F_0F_kF_j$
xiii) $\exists\, k \geq 1$ *such as* $(\mathcal{R}\,(F_0G_0, f+g))^{(k)}$ *is noetherian*

Proof 5.2.1. *Under the hypotheses of* ***Theorem 5.2.1*** , *we have:* (i) g *is noetherian* $\Leftrightarrow$ (xi) $\exists\, k \geq 1$ *such as* $\forall\, j \geq k\,,\ G_{k+j} = G_0G_kG_j$. *As* $f \sim g$ *so for everything* $n \in \mathbb{N}, G_0F_n = G_0G_n$,*So we have* $(xi) \Leftrightarrow\ F_0F_{k+j} = G_0G_{k+j} = G_0G_kG_j = (F_0F_k)(F_0F_j) = F_0F_{k+j} \Leftrightarrow (xiii) \Leftrightarrow (ii)$ f *is noetherian* $\Leftrightarrow (x)$ $\exists\, k \geq 1$ *such as* $\forall\, j \geq k\,,\ T_{k+j} = T_0F_{k+j} = T_0F_kF_j = T_kT_j \Leftrightarrow (iii)$ $f+g$ *is noetherian* $\Leftrightarrow$ *iv)* $\mathcal{R}(F_0G_0, f+g)$ *is noetherian* $\Leftrightarrow$ (v) $\mathfrak{R}(F_0G_0, f+g)$ *is noetherian* $\Leftrightarrow$ (vi) $\mathcal{R}(F_0G_0, f+g)$ *is an* F_0G_0-*finite type algebra* $\Leftrightarrow$ (vii) $\mathfrak{R}(F_0G_0, f+g)$ *is a* F_0G_0-*finite type algebra.*

$$\begin{aligned}
(i)\ g\text{is noetherian} &\Leftrightarrow\ \exists\, k \geq 1 \text{ such as } G_{nk} = G_k^n, \forall\, n \in \mathbb{N} \\
&\Leftrightarrow\ F_0G_0G_{nk} = (F_0G_0G_k)^n \\
&\Leftrightarrow\ T_{nk} = T_k^n \\
&\Leftrightarrow\ (f+g)^{(k)} = (f+g)_{T_k} \\
&\Leftrightarrow\ (viii) \\
&\Leftrightarrow\ (ix), \text{ according to the } \textbf{Theorem 5.1.1}
\end{aligned}$$

Proposition 5.2.1. *Let* $\mathcal{R}$ *be a ring and* $f = (F_n)_{n\in\mathbb{N}^*}$ *et* $g = (G_n)_{n\in\mathbb{N}^*}$ *are two axis-quasi-filtrations of* $\mathcal{R}$ *such as* $f \precsim g$. *If* F_0 *et* G_0 *are noetherians, then the following assertions are equivalent:*

i) g is noetherian
ii) $f+g$ is noetherian
iii) $\mathcal{R}(F_0G_0, f+g)$ is noetherian
iv) $\mathfrak{R}(F_0G_0, f+g)$ is noetherian
v) $\mathcal{R}(F_0G_0, f+g)$ is a F_0G_0-finite type algebra
vi) $\mathfrak{R}(F_0G_0, f+g)$ is a F_0G_0-finite type algebra
vii) $\exists\, k \geq 1$ such as $(f+g)^{(k)} = (f+g)_{T_k}$
viii) $\exists\, k \geq 1$ such as $(\mathcal{R}\,(F_0G_0, f+g))^{(k)} \simeq \mathcal{R}(F_0G_0, T_k)$
ix) $\exists\, k \geq 1$ such as $\forall\, j \geq k\,,\ T_{k+j} = T_kT_j$
x) $\exists\, k \geq 1$ such as $\forall\, j \geq k\,,\ G_0G_{k+j} = G_0G_kG_j$

Proof 5.2.2. *See* ***Theorem*** *5.2.1*

Proposition 5.2.2. *Let $\mathcal{R}$ be a ring and $f = (F_n)_{n\in\mathbb{N}^*}$ and $g = (G_n)_{n\in\mathbb{N}^*}$ are two axis-quasi-filtrations dof $\mathcal{R}$ such as $f \sim g$ and $F_0 = G_0$. If F_0 is noetherian, then we have the following assertions :*

$\widetilde{g}$ is $I-$good	$\Rightarrow$	g is strongly noetherian	$\Rightarrow$	g is noetherian
$\Updownarrow$		$\Updownarrow$		$\Updownarrow$
$\widetilde{f}$ is $I-$good	$\Rightarrow$	f is strongly noetherian	$\Rightarrow$	f is noetherian
$\Updownarrow$		$\Updownarrow$		$\Updownarrow$
$f+g$ is I-good	$\Rightarrow$	$f+g$ is strongly noetherian	$\Rightarrow$	$f+g$ is noetherian

Proof 5.2.3. *Indications: Assume that $f \sim g$ and $F_0 = G_0$ so we have, $\widetilde{f} = \widetilde{g}$ and $f+g = \widetilde{f+g} = F_0g = G_0f$.*

5.3. **Case of ring quasi-graduations.**

Corollary 5.3.1. *let $\mathcal{R}$ be a ring, $f = (F_n)_{n\in\mathbb{N}^*}$ and $g = (G_m)_{m\in\mathbb{N}^*}$ are two quasi-filtrations of $\mathcal{R}$ such as $f \sim g$ and $G_0 \subseteq I_0$. Let I a subgroup of $(\mathcal{R}, +)$ such as $IG_0 \subseteq I$. We have :*

(1) *g $I-$ good $\Longrightarrow$ f strongly noetherian .*
(2) *g noetherian $\Longrightarrow$ f noetherian .*
(3) *g strongly noetherian $\Longrightarrow$ f strongly noetherian .*
(4) *g $k-$ regular $\Longrightarrow$ f $k-$regular .*

Proof 5.3.1. *Let us verify the following assertions:*
1) Let's show that g $I-$ good $\Longrightarrow$ f strongly noetherian .

Assume that g is $I-$ good , so $\exists\, n_0 \geq 1$: $I^nG_{n_0} = G_{n+n_0}$ and $I^n \subseteq G_n\,, \forall\, n \geq 0$. Let $m, n \geq k$, let's choose $k = n_0$, so $\exists\, r, s \in \mathbb{N}$: $m = r+k$ and $n = s+k$.
$F_{m+n} = F_{r+k+s+k} \subseteq G_0F_{r+k+s+k} \subseteq F_0G_{r+k+s+k} = F_0I^{r+s+k}G_k = F_0I^rG_kI^{s+k}$, *since* $F_0I^rG_kI^{s+k} \subseteq F_0G_mF_0G_n = G_0F_mF_n \subseteq F_mF_n$, *because* $G_0 \subseteq F_0$.

Then, $F_{m+n} = F_m F_n$ *. Thus* f *is strongly Noetherian .*
2. Let's show that g *noetherian* $\Longrightarrow$ f *noetherian .*
Assume that g *is noetherian , then there exets* $k \geq 1$ *such that* $\forall\, j \geq k$ *we have,* $G_{k+j} = G_k G_j$ *.* $I_{k+j} \subseteq G_0 I_{k+j} = I_0 G_{k+j} = I_0 G_k G_j = G_0 I_k I_j \subseteq I_k I_j$ *. Thus,* $I_{k+j} \subseteq I_k I_j \subseteq I_{k+j} \Longrightarrow I_{k+j} = I_k I_j$ *. Therefore* f *is noetherian .*
3. Obvious .
4. Let's show that g $k-$ *regular* $\Longrightarrow$ f $k-$*regular*
Assume that g *is* $k-$ *regular , then there exets an integer* $k \geq 1$ *such that* $G_{nk} = G_k^n$*, and we have* $I_{nk} \subseteq G_0 I_{nk} = I_0 G_{nk} = I_0 G_k^n = (I_0 G_k)^n = (G_0 I_k)^n \subseteq I_k^n$ *, thus* $I_{nk} \subseteq I_k^n \subseteq I_{nk} \Longrightarrow I_{nk} = I_k^n$*. So* f *is* $k-$*regular .*

Corollary 5.3.2. *Let* $\mathcal{R}$ *be a ring and* $f = (I_n)_{n \in \mathbb{Z} \cup \{+\infty\}}$ *and* $g = (G_m)_{m \in \mathbb{Z} \cup \{+\infty\}}$ *are two quasi-graduations of the ring* $\mathcal{R}$ *such that* $f \sim g$ *and* $G_0 \subseteq I_0$ *. Let* I *be a subgroup of* $(\mathcal{R}, +)$ *such that* $IG_0 \subseteq I$*. We have*

(1) g $I-$*good* $\Longrightarrow$ f $I-$*good .*
(2) g $AP \Longrightarrow$ f AP *.*
(3) g *strongly* $AP \Longrightarrow$ f *strongly* AP *.*
(4) g $k-$*regular* $\Longrightarrow$ f $k-$*regular .*

Proof 5.3.2. *Let's check the assertions above:*
1. Let us show that g $I-$*good* $\Longrightarrow$ f $I-$*good . Suppose* g *is* $I-$*good, then*

(1) $IG_n \subseteq G_{n+1}\,,\ \forall\, n \in \mathbb{N}$ *.*
(2) $\exists\, n_0 \in \mathbb{N}$ *tel que* $IG_n = G_{n+1}\,,\ \forall\, n \geq n_0$ *.*

$II_n \subseteq I(G_0 I_n) = I(I_0 G_n) = I_0(IG_n) \subseteq I_0 G_{n+1} = I_{n+1}$ *.*
Let $n \geq n_0$*, we have* $I_{n+1} \subseteq G_0 I_{n+1} = I_0 G_{n+1} = I_0(IG_n) \subseteq II_n$ *.*
Hence f *is* $I-$*good. Likewise we obtain the other assertions.*

Corollary 5.3.3. *Let* $\mathcal{R}$ *be a ring and* $f = (I_n)_{n \in \mathbb{Z} \cup \{+\infty\}}$ *and* $g = (G_m)_{m \in \mathbb{Z} \cup \{+\infty\}}$ *are two quasi-graduations of* $\mathcal{R}$ *such that* $f \precsim g$ *and* $I_0 \subseteq G_0$ *. Let* I *be a subgroup of* $(\mathcal{R}, +)$ *such that* $IG_0 \subseteq I$*. We have*

(1) g $AP \Longleftrightarrow$ $f+g$ AP *.*
(2) g $k-$*regular* $\Longleftrightarrow$ $f+g$ $k-$*regular .*

Proof 5.3.3. *Let's check the following assertions:*
1. Let us show that g $AP \Longleftrightarrow$ $f+g$ AP *.*
Suppose that g *est* AP *, then if there exets a sequence* $(w_n)_{n \in \mathbb{N}}$ *of non-zero natural integers such that :* $\lim\limits_{n \longrightarrow +\infty} \dfrac{w_n}{n} = 1$ *et* $\forall\, m, n \in \mathbb{N}\,,\ G_{w_n m} \subseteq G_n^m$
$T_{w_n m} = I_0 G_{w_n m} \subseteq I_0 G_n^m = (I_0 G_n)^m = T_n^m$ *. Therefore* $f+g$ *is AP. The converse is trivial.*
2. Let us show that g $k-$ *regular* $\Longleftrightarrow$ $f+g$ $k-$ *regular*
Suppose g *is* $k-$ *regular, then there exets an integer* $k \geq 1$ *such that* $g^{(k)} = g_{G_k}$*. Let* $n \in \mathbb{N}$, $T_{kn} = I_0 G_{kn} = I_0 G_k^n = (I_0 G_k)^n = T_k^n$ *. The converse is trivial.*

References

[1] E. D. Béché, *axis-quasi-graduations of ring : operations, generalized analytic independence, and analytic spread comparison,* Thesis sustained in 2022 at the University Nangui Abrogoua of Abidjan (Ivory Coast).

[2] E. D. Béché, Y. M. Diagana and Pierre K. Brou, Independence on quasi-bigraduation of rings and analytic spread, *African Mathematics Annals*, **7** (2018), 129-136.

[3] P. K. Brou and Y. M. Diagana, Independences weakened analytics and extensions of the analytic spread of a quasi-graduation, *African Mathematics Annals*, **4** (2013), 139-150.

[4] W. Bruns, On the number of elements independent with respect to an ideal, *J. London Math. Soc.*, **22** (1980), no. 2, 57-62. https://doi.org/10.1112/jlms/s2-22.1.57

[5] Y. M. Diagana, H. Dichi and D. Sangaré, Filtrations, generalized analytic indepedence, analytic spread, *Afrika Mathematika*, **4** (1994), no. 3, 101-1016.

[6] Y. M. Diagana, Quasi-graduations of rins : regular analytic independence and extensions of analytic spread, *Communications in Algebra*, **30** (2002), no 6, 2745-2761.

[7] Y. M. Diagana, Quasi-graduations of rings, generalized analytic independence, extensions of analytic spread, *Afrika Mathematika*, **15** (2003), no. 3, 93-108.

[8] D. G. Northcott and D. Rees, Reductions of ideals in locals rings, *Proc. Cambridge Philos. Soc.*, **50** (1954), 145-158. https://doi.org/10.1017/s0305004100029194

Received: March 1, 2025; Published: June 11, 2025