Pure Mathematical Sciences, Vol. 14, 2025, no. 1, 33 - 41
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/pms.2025.91611

# On the Weakened Properties of Analytic Independence and Extensions of the Analytic Spread to an Axis-Quasi-Graduation of Ring

**Kouadjo Pierre Brou**

Universite Nangui Abrogoua, UFR-SFA
Laboratoire de Mathematiques et Informatique
02 BP 801 Abidjan 02, Cote d'Ivoire

**Eugène Deval Béché**

Université Polythecnique de Man, UFR-ST
Laboratoire de Mathématiques et Informatique
BP 20 Man, Cote d'Ivoire



**Abstract**

In this paper, we define a family of subgroups in an arbitrarily chosen commutative and unitary ring. These families obey a certain compatibility and are called axis-quasi-graduations of ring which are in general not quasi-graduations and filtrations. We will extend the concept of analytic width to this notion of axis-quasi-graduation of ring through the notion of analytically independent elements.This leads us to the study of three weakened notions of $J-$independence that we will use in this document to introduce new extensions of the analytic spread with respect to axis-quasi-graduations. The last part of our manuscript will be devoted to studying these new extensions in order to give some properties that characterize these new notions and those that already exist.

**Keywords:** Axis-quasi-graduation , regular analytic spread, additive analytic spread, quasi-analytic spread, weak analytic spread

## 1. Introduction

All rings used in our work are assumed to be commutative and unitary, and an ideal of a ring is different from the ring itself. The study of weakened properties of analytic independence and extensions of analytic spread is a subject of much thought in commutative algebra due to its richness. This concept represents a rich and complex area in filtration theory since its introduction by Northcott and Rees in [7] to ideals and then extended to filtrations a few years later by J. S. Okon in [8] . In 2002 Youssouf M. Diagana became the first African author to introduce this notion to quasi-graduations in [5] . Many other authors have made important extensions of the analytic spread following several approaches and have obtained very good results which are related to certain characteristics of commutative algebra and to applications in algebraic geometry.

In [2] Brou and Diagana used the concepts of weakened analytic independence of quasi-graduations which are a generalization of filtrations and this allowed to obtain important results on the extensions of the analytic spread. In this work, we focus on axis-quasi-graduations of ring defined as families which also respect three compatibility properties like quasi-graduations, but the big difference between these two notions (quasi-graduation and axis-quasi-graduation) is at the level of the third property, which makes axis-quasi-graduation a superb generalization of quasi-graduations.

We also highlight the implications of these concepts on the characteristics of commutative algebra such as Krull dimension, transcendence degree which are essential elements in the analysis of algebraic structures. This paper aims to establish new results on the analytic spread of order $k$, as well as its extensions with emphasis on the arithmetic and algebraic aspects of axis-quasi-graduations. Through illustrative examples and rigorous demonstrations, we hope to contribute to a better understanding of the dynamics underlying these analytical properties established in this manuscript.

## 2. Generality on axis-quasi-graduations

### 2.1. Definitions.

**Definition 2.1.1.** *Let $\mathcal{B}$ be a ring and $g = (G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ be a family of subgroups of $\mathcal{B}$. We say that $g$ is an axis-quasi-graduation of $\mathcal{B}$ if:*

(1) $G_0$ *is a subring of* $\mathcal{B}$
(2) $G_\infty = (0)$
(3) $G_pG_q \subseteq G_0G_{p+q} \;\; \forall\, p, q \in \mathbb{N}$.

**Example 2.1.1.** *Let $\mathcal{B}=\mathbb{R}$ and $g=(G_n)$ be a family defined by:*

$$\begin{cases} G_n=\mathbb{Z}\left[\sqrt{p}\right],\ \text{if } n\leq 0;\\ G_\infty=(0);\\ G_n=\left(\frac{q}{r}\right)^n\sqrt{p}^{n+1}\mathbb{Z},\ \text{if } n\geq 1\end{cases}$$

*where $p$, $q$ and $r$ are pairwise distinct prime numbers. We can clearly see that $g=(G_n)$ is a family of subgroups of $\mathcal{B}$. And we show that $g$ is an axis-quasi-graduation of $\mathcal{B}$*

**Remark 2.1.1.** *By definition, any quasi-graduation of ring $\mathcal{B}$ is an axis-quasi-graduation of this ring and through **the example 2.1.1** we have:*

$$\forall\, n\geq 1,\ G_0G_n=\left(\mathbb{Z}+\sqrt{p}\mathbb{Z}\right)\left(\frac{q}{r}\right)^n\sqrt{p}^{n+1}\mathbb{Z}=G_n+G_0G_n\neq G_n$$

*this shows that the axis-quasi-graduation are not necessarily quasi-graduations because for a quasi-graduation $\forall\, n,\ G_0G_n=G_n$.*

2.2. **Other types of axis-quasi-ring graduation.**

**Definition 2.2.1.** *Let $\mathcal{B}$ be a ring, $k\in\mathbb{N}^*\cup\{+\infty\}$. We have the following notions:*

***1-2-1)*** *We say that $g$ is an $k$**-axis-quasi-filtration** if $g$ is $k-$ decreasing, that is: $\forall\, i\geq 0$, $(G_0G_{n+ik})\cap(G_0G_n)\subseteq(G_0G_{n+k})\cap(G_0G_n)$*

***1-2-2)*** *We say that $g$ is an **axe-quasi-filtration** if the sequence $(G_0G_n)_{n\in\mathbb{N}^*}$ is decreasing.*

**Remark 2.2.1.** *Any decreasing axis-pregraduation is an axis-quasi-filtration.*

## 3. Generalized Analytical Independence and Extensions of Analytic Spread

3.1. **Concept of $J-$independence.**

**Definition 3.1.1.** *Let $\mathcal{B}$ be a ring, $g=(G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ an axis-quasi-graduation of $\mathcal{B}$, $k\in\mathbb{N}^*\cup\{+\infty\}$, $J$ an ideal of $G_0$ and $a_1,...,a_r$ elements of $\mathcal{B}$. We have the following definitions:*

***2-0-1)*** *Suppose that $J+(G_0G_k)\cap G_0\neq G_0$, the elements $a_1,...,a_r$ are $J-$independent of order $k$ with respect to $g$ if the following assertions are verified:*

*i)* $a_1,...,a_r\in\mathcal{A}G_1$,

*ii) For any homogeneous polynomial $\mathcal{P}\in G_0[X_1,...,X_r]$ of degree $s$, the relation $\mathcal{P}(a_1,...,a_r)\in JG_s+(G_0G_{s+k})$ implies that $\mathcal{P}\in(J+G_0G_k)$.*

***2-0-2)*** *The elements $a_1,...,a_r$ are regularly $J-$ independent of order $k$ with respect to $g$ if there exists a natural number $p\geq 1$ such that $J+(G_0G_{pk})\cap G_0\neq G_0$ and $a_1,...,a_r$ are $J-$independent of order $k$ with* *respect to $g^{(p)}$. From the previous definitions, we pose:*

- •) *We call $J-$**analytic spread** of order $k$ of $g$ the number:*

$$\ell_J(g,k) = \sup\left\{ \begin{array}{c} r \in \mathbb{N} \,:\, \exists\, a_1, ...a_r \in J\,,\, J-\textit{independnt} \\ \textit{of order} \quad k \quad \textit{with respect} \quad g \end{array} \right\}$$

- •) *We call $J-$**regular analytic spread** of order $k$ of $g$ the number:*

$$\ell_J^a(g,k) = \sup\left\{ \begin{array}{c} r \in \mathbb{N} \,:\, \exists\, a_1, ...a_r \in J \textit{ regularly } J-\textit{independent} \\ \textit{of order} \quad k \quad \textit{with respect to} \quad g \end{array} \right\}$$

- •) $\sup J = \sup\left\{ \; r \in \mathbb{N} \,:\, \exists\, a_1, ...a_r \in J\,,\, J-\textit{independents} \; \right\}$

### 3.2. Weakened notions of $J-$independence.

**Definition 3.2.1.** *Let $\mathcal{B}$ be a ring, $g = (G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ an axis-quasi-graduation of $\mathcal{B}$, $k \in \mathbb{N}^* \cup \{+\infty\}$, $J$ an ideal of $G_0$ and $a_1, ..., a_r$ the elements of $\mathcal{B}$. We have the following definitions :*

***2-2-1)*** *The elements $a_1, ..., a_r$ are **additively** $J-$**independent** of order $k$ with respect to $g$ if there exists a strictly increasing sequence of integers $(\mu_n)_{n\in\mathbb{Z}\cup\{\infty\}}$ such as $\mu_0 = 0$, $\mu_\infty = +\infty$, $J + (G_0G_{\mu_k}) \cap G_0 \neq G_0$ and $G_{\mu_{p_1}+\mu_{p_2}+...+\mu_{p_m}} = G_{\mu_{p_1+p_2+...+p_m}}, \forall\, m, p_1, ..., p_m \in \mathbb{N}^*$, with $a_1, ..., a_r$ $J-$independent of order $k$ with respect $(G_{\mu_n})$.*

***2-2-2)*** *The elements $a_1, ..., a_r$ are **quasi** $J-$**independent** of order $k$ with respect to $g$ if there exists a strictly increasing sequence of integers $(\mu_n)_{n\in\mathbb{Z}\cup\{\infty\}}$ such as $\mu_0 = 0$, $\mu_\infty = +\infty$, $J + (G_0G_{\mu_k}) \cap G_0 \neq G_0$ and $G_{\mu_{p_1}+\mu_{p_2}+...+\mu_{p_m}} \subseteq G_0G_{\mu_{p_1+p_2+...+p_m}}, \forall\, m, p_1, ..., p_m \in \mathbb{N}^*$, with $a_1, ..., a_r$ $J-$independent of order $k$ with respect to $(G_{\mu_n})$. Under these conditions, we have:*

$G^m_{\mu_p} \subseteq G_0G_{m\mu_p} \subseteq G_0G_{\mu_{mp}}, \forall\, p \in \mathbb{Z}$ *et* $\forall\, m \in \mathbb{N}$.

***2-2-3)*** *The elements $a_1, ..., a_r$ are **weakly** $J-$**independent** of order $k$ with respect to $g$ if there exists a strictly increasing sequence of integers $(\mu_n)_{n\in\mathbb{Z}\cup\{\infty\}}$ such that $\mu_0 = 0$, $\mu_\infty = +\infty$, $J + (G_0G_{\mu_k}) \cap G_0 \neq G_0$ and $G_{\mu_p}G_{\mu_p} \subseteq G_0G_{\mu_{p+q}}, \forall\, p, q \in \mathbb{Z}$, with $a_1, ..., a_r$ $J-$independent of order $k$ with respect to $(G_{\mu_n})$. Note that: $G^m_{\mu_p} \subseteq G_0G_{\mu_{mp}}, \forall\, p \in \mathbb{Z}$ et $\forall\, m \in \mathbb{N}$.*

*These defined notions lead to the following extensions of the analytic spread of an axis-quasi-graduation :*

- •) *We call $J-$**additive analytic spread** of order $k$ of $g$ the number:*

$$\ell_J^+(g,k) = \sup\left\{ \begin{array}{c} r \in \mathbb{N} \,:\, \exists\, a_1, ...a_r \in J \textit{ additively } J-\textit{independent} \\ textoforder \quad k \quad \textit{with respect to} \quad g \end{array} \right\}$$

- •) *We call **quasi** $J-$ **analytic spread** of order $k$ of $g$ the number:*

$$\ell_J^q(g,k) = \sup\left\{ \begin{array}{c} r \in \mathbb{N} \,:\, \exists\, a_1, ...a_r \in J \textit{ quasi } J-\textit{independent} \\ \textit{of order} \quad k \quad \textit{with respect to} \quad g \end{array} \right\}$$

•) *We call* $J-$ ***weak analytic spread*** *of order* $k$ *of* $g$ *the number:*

$$\ell_J^*(g,k) = \sup \left\{ \begin{array}{c} r \in \mathbb{N} \,:\, \exists\, a_1, ...a_r \in J \textit{ weakly } J - \textit{independent} \\ \textit{of order} \quad k \quad \textit{with respect to} \quad g \end{array} \right\}$$

**Proposition 3.2.1.** *Let* $\mathcal{B}$ *be a ring and* $g = (G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ *be an axis-quasi-graduation of* $\mathcal{B}$*. Let* $k \in \overline{\mathbb{N}^*}$ *and* $J$ *be an ideal of* $G_0$*. We have the following inequalities :*

*1)* $\ell_J\,(g,k) \leq \ell_J^a\,(g,k) \leq \ell_J^+\,(g,k) \leq \ell_J^q\,(g,k) \leq \ell_J^*\,(g,k)$

*2) Suppose that* $\forall\, i \geq 0\,,\, J \supseteq (G_0G_{i+k}) \cap G_0$*, we have :*
*i)* $\ell_J\,(g,k) \leq \ell_J^a\,(g,k) \leq \ell_J^+\,(g,k) \leq \ell_J^q\,(g,k) \leq \sup J$
*If moreover* $G_0$ *is Noetherian and if* $P \in Spec\,(G_0)$ *on* $J$*, then :*
$\ell_J\,(g,k) \leq \ell_J^a\,(g,k) \leq \ell_J^+\,(g,k) \leq \ell_J^q\,(g,k) \leq \sup J \leq ht\,(J) \leq \dim\,(G_0)_P < +\infty$
*ii)* $\ell_J\,(g,k) \leq \ell_J^a\,(g,k) \leq \ell_J^+\,(g,k) \leq \ell_J^q\,(g,k) \leq \ell_J^*\,(g,k) \leq \sup J$
*If moreover* $G_0$ *is Noetherian and if* $P \in Spec\,(G_0)$ *on* $J$*, then :*
$\ell_J\,(g,k) \leq \ell_J^a\,(g,k) \leq \ell_J^+\,(g,k) \leq \ell_J^q\,(g,k) \leq \ell_J^*\,(g,k) \leq \sup J \leq ht\,(J) \leq \dim\,(G_0)_P < +\infty$

**Proof 3.2.1.** *The first inequality follows from the previous definitions. 2) In this section, we will assume that* $\forall\, i \geq 0,\ J \supseteq (G_0G_{i+k}) \cap G_0$.

*i) Let us show that* $\ell_J^q\,(g,k) \leq \sup J$.

*Let us assume that the elements* $a_1, ..., a_r$ *are quasi* $J-$ *independent of order* $k$ *with respect to* $g$*, then:*

•) *There exists a strictly increasing sequence of integers* $(s_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ *such that* $s_0 = 0,\ s_\infty = +\infty,\ J + (G_0G_{s_k}) \cap G_0 \neq G_0$,

••) $G_{s_{p_1}+s_{p_2}+...+s_{p_m}} \subseteq G_0G_{s_{p_1+p_2+...+p_m}}, \forall\, m, p_1, p_2, ..., p_m \in \mathbb{N}^*$*, with* $a_1, ..., a_r$ $J-$*independent of order* $k$ *with respect to* $(G_{s_n})$*. For any homogeneous polynomial* $\mathcal{F} \in G_0\,[X_1, ..., X_r]$*, the relation* $\mathcal{F}(a_1, ..., a_r) = 0$ *(resp.* $\mathcal{F}(a_1, ..., a_r) \in JI^d + I^{d+\lambda+k}$*) with* $d = \deg \mathcal{F}$ *and* $I = (a_1, ..., a_r)_{G_0}$ *the sub*$-G_0-$*module of* $\mathcal{B}$ *(and of* $G_0G_{s_1}$*) generated by the elements* $a_1, ..., a_r$ *. Let us set* $t = \mathcal{F}(a_1, ..., a_r)$*, so we have ,*

$t \in \left[JG_{s_1}^d + G_0G_{s_1}^{d+\lambda+k}\right] \cap G_{s_1}^d \Rightarrow t \in JG_{s_d} + G_0\left[G_{s(d+\lambda+k)} \cap G_{s_d}\right]$*. Since the elements* $a_1, ..., a_r$ *are* $J-$*independent of order* $k$ *relatively to* $(G_{s_n})$ *then all the coefficients of* $\mathcal{F}$ *are in* $J + \left(G_0G_{s_{k+\lambda}}\right) \cap G_0$*, by hypothesis it results that:*
$J + \left(G_0G_{s_{k+\lambda}}\right) \cap G_0 = J = J + (G_0G_{k+\lambda}) \cap G_0 = J + I^{k+\lambda} \cap G_0 \subseteq J + I^{k+\lambda}$ *for all* $\lambda \in \{(n-1)k, ..., +\infty\}$*. By following* $\ell_J^q\,(g,k) \leq \sup J$ *and using **1)** we obtain **i)** .*

*If in addition the ring* $G_0$ *is Noetherian and if* $P$ *is an ideal of* $G_0$ *prime over* $J$ *then,* $\sup J \leq htJ \leq \dim(G_0)_P = htP \leq +\infty$.

*ii) Let us show that* $\ell_J^*\,(g,k) \leq \sup J$.

*For this we will assume that the elements* $a_1, ..., a_r$ *are weakly* $J-$*independent*

*of order $k$ relative to $g$ if there exists a strictly increasing sequence of integers $(s_n)_{n\in\mathbb{Z}\cup\{\infty\}}$ such that $s_0 = 0$, $s_\infty = +\infty$, $J + (G_0G_{s_k}) \cap G_0 \neq G_0$ and $G_{s_p}G_{s_p} \subseteq G_0G_{s_{p+q}}$, $\forall\, p, q \in \mathbb{Z}$, with $a_1, ..., a_r$ $J-$independent of order $k$ relative to $(G_{\mu_n})$. For any homogeneous polynomial $\mathcal{F} \in G_0\,[X_1, ..., X_r]$, the relation $\mathcal{F}(a_1, ..., a_r) = 0$ (resp. $\mathcal{F}(a_1, ..., a_r) \in JI^d + I^{d+k}$) with $d = \deg \mathcal{F}$ and $I = (a_1, ..., a_r)_{G_0}$ the sub$-G_0-$module of $\mathcal{B}$ (and of $G_0G_{s_1}$) generated by the elements $a_1, ..., a_r$. Let us set $t = \mathcal{F}(a_1, ..., a_r)$, so we have ,*
*$t \in \left[JG_{s_1}^d + G_0G_{s_1}^{d+k}\right] \cap G_{s_1}^d \Rightarrow t \in JG_{s_d} + G_0\left[G_{s(d+k)} \cap G_{s_d}\right]$ therefore,*
*$\mathcal{F} \in \Big(J + (G_0G_{s_k}) \cap G_0\Big)\,[X_1, ..., X_r]$ or $J + (G_0G_{s_k}) \cap G_0 = J = J + I^k \cap G_0$. Which proves that we actually have $r$ elements $J-$independent (resp. of order $k$). From where $\ell_J^*(g, k) \leq \sup J$ and according to inequality 1) we have:*

$$\ell_J(g, k) \leq \ell_J^a(g, k) \leq \ell_J^+(g, k) \leq \ell_J^q(g, k) \leq \ell_J^*(g, k) \leq \sup J.$$

*If furthermore the ring $G_0$ is Noetherian and if $P$ is an ideal of $G_0$ prime over $J$ then, $\sup J \leq htJ \leq \dim(G_0)_P = htP \leq +\infty$. Hence the desired result.*

## 4. Weakened Properties of $J$-Independence

### 4.1. Some Arithmetic Properties on Axis-Quasi-Graduation.

**Proposition 4.1.1.** *Let $\mathcal{B}$ be a ring, $k \in \overline{\mathbb{N}^*}$, $g = (G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ be a quasi-graduation axis of $\mathcal{B}$. Let $u, v$ be two natural numbers such that $u$ divides $v$ and $P$ be a prime ideal of $G_0$*

*(3.1.1) $P \supseteq G_v \cap G_0 \Longrightarrow P \supseteq (G_0G_u) \cap G_0$ .*

*(3.1.2) $P \supseteq G_{kn} \cap G_0, \forall\, n \geq 1 \Longrightarrow P \supseteq (G_0G_k) \cap G_0$, so if there $\exists\, n \geq 1$ such that $P \supseteq G_{kn} \cap G_0$ then $P \supseteq (G_0G_k) \cap G_0$ .*

*(3.1.3) If $k \neq \infty$ and $\forall i \geq 1$, we have: $P \supseteq G_{ik} \cap G_0 \Longrightarrow P \supseteq (G_0G_k) \cap G_0$.*

*(3.1.4) If we assume that $G_{kn} \cap G_0 \subseteq G_k \cap G_0$, then we have the following equivalences:*
*i) $\exists\, n \geq 1 \,:\, P \supseteq G_{kn} \cap G_0 \Rightarrow P \supseteq (G_0G_k) \cap G_0 \Leftrightarrow P \supseteq (G_0G_{ik}) \cap G_0, \forall\, i \geq 0$.*
*ii) Si $k \neq \infty\,, \exists\, n \geq 1 \,:\, P \supseteq G_{kn} \cap G_0 \Rightarrow P \supseteq (G_0G_k) \cap G_0 \Leftrightarrow P \supseteq (G_0G_i) \cap G_0, \forall\, i \geq 0$.*

### 4.2. Weakening and comparison.

**Theorem 4.2.1.** *Let $\mathcal{B}$ be a ring, $k \in \overline{\mathbb{N}^*}$, $g = (G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ be a axis-quasi-graduation of $\mathcal{B}$ and $J$ be a strict ideal of $G_0$.*
*1- We have :*

*i) If $g$ is an $k-$axis-quasi-graduation and if $J \supseteq (G_0G_k) \cap G_0$ ( in particular if $k$ is infinite), then:*
*$\ell_J\left(g^{(n)}, k\right) \leq \ell_J\left(g^{(pn)}, k\right) \leq \ell_J^a(g, k) \leq \ell_J^+(g, k) \leq \ell_J^q(g, k) \leq \ell_J^*(g, k)\,, \forall\, n, p \in \mathbb{N}^*$.*

*ii) Let $P$ be an ideal of $G_0$ prime over $J$. The relation $J \supseteq (G_0G_{i+k}) \cap G_0$ (in particular if $k$ is infinite) implies the following inequalities:*

*$\ell_J^*(g,k) \le \sup J \le \sup P$. If moreover $G_0$ is Noetherian then:*
$\ell_J\left(g^{(n)},k\right) \le \ell_J^a(g,k) \le \ell_J^+(g,k) \le \ell_J^q(g,k) \le \ell_J^*(g,k) \le \sup J \le \sup P \le htP < +\infty$, $\forall\, n,p \in \mathbb{N}^*$.

*iii) We assume that for all $i \ge 1$, $G_{ik} \cap G_0 \subseteq G_k \cap G_0$, let $P$ be an ideal of $G_0$ prime over $J + (G_0G_k) \cap G_0$ (in particular if $k$ is infinite) and $P$ be an ideal of $G_0$ prime over $J$, then we have: $\ell_J^*(g,k) \le \sup J$. Moreover if $G_0$ is Noetherian then for all $\forall\, n \in \mathbb{N}^*$, then:*
$\ell_J\left(g^{(n)},k\right) \le \ell_J^a(g,k) \le \ell_J^+(g,k) \le \ell_J^q(g,k) \le \ell_J^*(g,k) \le \sup J \le \sup P \le htP < +\infty$.

*iv) If $G_0$ is Noetherian and if $P$ is a maximal ideal of $G_0$, then:*
$\ell_J^*(g,k) \le \sup P \le htP < +\infty$, *thus* $\forall\, n \in \mathbb{N}^*$ *we have:*
$\ell_J\left(g^{(n)},k\right) \le \ell_J^a(g,k) \le \ell_J^+(g,k) \le \ell_J^q(g,k) \le \ell_J^*(g,k) < +\infty$.
*2- We assume that the following conditions are met:*
$C_1 : J \supseteq (G_0G_{i+k}) \cap G_0, \forall\, i \ge 0$
$C_2 : (G_{n+ik} \cap G_n)_{i\ge 0}$ *is decreasing* $\forall\, n \ge 0$. *So we have:*

*i)* $\exists\, k_0 \in \mathbb{N}^*$ *such that* $\ell_J^q(g,k) = \ell_J\left(g^{(nk_0)},k\right), \forall\, n \in \mathbb{N}^*$.

*ii)* $\ell_J^q(g,k) = \sup\left\{\ell_J(g^{(n)},k),\, n \in \mathbb{N}^*\right\} = \ell_J^a(g,k) = \ell_J^+(g,k)$.

*iii)* $\ell_J^q(g,k) = \ell_J^q(g^{(p)},k), \forall\, p \in \mathbb{N}^*$

**Proof 4.2.1.** *(1)-*
*i-(a)* $\ell_J(g,k) \le \ell_J\left(g^{(p)},k\right), \forall\, p \ge 1$, *indeed if $r$ elements $a_1,...,a_r$ of $J$ are $J-$ independent of order $k$ with respect to $g$, $\forall\, \mathcal{P} \in G_0[X_1,...,X_r]$ homogeneous polynomial of degree $d$, the relation:*

$$\mathcal{P}(a_1^p,...,a_r^p) \in JG_{dp} + G_0G_{dp+pk} \implies \mathcal{P}(a_1^p,...,a_r^p) \in \Big[JG_{dp} + G_0\left(G_{(dp+pk)} \cap G_{dp}\right)\Big] \subseteq JG_{dp} + G_0\left(G_{(dp+k)} \cap G_{dp}\right)$$ *as a result, we have:*

$\mathcal{P} \in \Big(J + (G_0G_{pk} \cap G_0)\Big)\Big[X_1,...,X_r\Big]$. *So the $r$ elements $a_1^p,...,a_r^p$ are $J-$ independent of order $k$ with respect to $g^{(p)}$.*
*i-(b) Let $n \ge 1$, suppose that elements $a_1,...,a_r$ of $J$ are $J-$ independent of order $k$ with respect to $g^{(n)}$ then it is clear that they are regularly $J-$ independent of order $k$ with respect to $g$ and we have: $\ell_J^a(g,k) \ge r$, consequently* $\ell_J\left(g^{(n)},k\right) \le \ell_J^a(g,k)$.

*So* $\ell_J\left(g^{(n)},k\right) \le \ell_J\left(\left[g^{(n)}\right]^{(p)},k\right) = \ell_J\left(g^{(pn)},k\right) \le \ell_J^a(g,k), \forall\, p \ge 1$. *According to the* ***Proposition 3.2.1*** *we have:*
$\ell_J\left(g^{(n)},k\right) \le \ell_J\left(g^{(pn)},k\right) \le \ell_J^a(g,k) \le \ell_J^+(g,k) \le \ell_J^q(g,k) \le \ell_J^*(g,k), \forall\, n,p \in \mathbb{N}^*$.
*(ii) If elements $a_1,...,a_r$ of $J$ are $J-$ independent of order $k$ with respect*

*to* $(G_{s_k})$ *with* $G_{s_p}G_{s_q} \subseteq G_0G_{s_{p+q}}$ *for all integers* $p$ *and* $q$*, then they are* $J$-*independent.*
*If* $\mathcal{P} \in G_0[X_1,...,X_r]$ *is a homogeneous polynomial with* $\mathcal{P}(a_1,...,a_r) = 0$*, then* $P$ *has its coefficients in* $J + (G_0G_{s_k}) \cap G_0 = J$ *because the sequence* $(s_n)$ *is increasing. Thus* $\ell_J^*(g,k) \leq \sup J$ *and we conclude with the* ***Proposition 3.2.1*** *and the fact that if* $G_0$ *is Noetherian then* $\sup P \leq htP < +\infty$*.*
*(iii) In the same way as before, if elements* $a_1,...,a_r$ *of* $J$ *are* $J-$*independent of order* $k$ *with respect to* $(G_{s_k})$ *with* $G_{s_p}G_{s_q} \subseteq G_0G_{s_{p+q}}$ *for all integers* $p$ *and* $q$*, then they are* $P-$*independent. If* $\mathcal{P}$ *is a homogeneous polynomial with coefficients in* $G_0$ *and with* $r$ *indeterminate with* $\mathcal{P}(a_1,...,a_r) = 0$ *which implies that* $\mathcal{P}$ *has its coefficients in* $J + (G_0G_{s_k}) \cap G_0 \subseteq P + (G_0G_{s_k}) \cap G_0$*.*

- *If* $k$ *is infinite then* $J + (G_0G_{s_k}) \cap G_0 = J \subseteq P$*,*
- *Otherwise like* $P \supseteq (G_0G_k) \cap G_0$*, according to* ***Proposition 4.1.1*** *it results that* $P \supseteq (G_0G_{s_k}) \cap G_0 \Rightarrow P + (G_0G_{s_k}) \cap G_0 = P$*.*

*Then* $\ell_J^*(g,k) \leq \sup P$ *and if* $G_0$ *is Noetherian* $\ell_J^*(g,k) \leq \sup P \leq htP < +\infty$*.*

*iv) If* $P$ *is a maximal ideal of* $G_0$ *and if* $\ell_P^*(g,k) \neq 0$*, then there exist elementsa*$_1,...,a_r$ *of* $P$ *;* $P-$ *independent of order* $k$ *relative to* $(G_{s_n})_n$ *with the relation* $\left(G_{s_p}\right)\left(G_{s_q}\right) \subseteq G_0G_{s_{p+q}}$ *for all* $p,q$ *. The elements* $a_1,...,a_r$ *are also* $P-$ *independent: if* $\mathcal{P}$ *is a homogeneous polynomial, with coefficients in* $G_0$ *and with* $r$ *indeterminate with* $\mathcal{P}(a_1,...,a_r) = 0$*,*
*then* $\mathcal{P} \in \left(P + (G_0G_{s_k}) \cap G_0\right)[X_1,...,X_r]$*. Since* $P + (G_0G_{s_k}) \cap G_0 \neq G_0$ *and in addition* $P$ *is maximal, it follows that* $P + (G_0G_{s_k}) \cap G_0 = P$*. Consequently* $\ell_J^*(g,k) \leq \sup P$ *and if* $G_0$ *is Noetherian, we have:* $\ell_J^*(g,k) \leq \sup P \leq htP < +\infty$ *2- Let us prove assertions i) and ii). According to the assertions* $1-(i)$ *and* $1-(ii)$ *we have:* $\ell_J^+(g,k) \leq \ell_J^q(g,k) \leq \ell_J^*(g,k) \leq \sup(J)$*. Let us assume that* $\ell_J^q(g,k) \neq 0$*, or an integer* $r \leq \ell_J^q(g,k)$*, then there exists a sequence of integers* $(s_n)_n$ *such that* $G_{s_{p_1}+...+s_{p_n}} \subseteq G_{s_{p_1+...+p_n}}, \forall\, m, p_1,...,p_m \in \mathbb{N}^*$ *and elements* $b_1,...,b_r \in (G_0G_{s_1}) \cap J$*,* $J-$ *independent of order* $k$ *with respect to* $(G_{s_n})$*. Do we have* $\forall\, p \geq 1$*,* $b_1^p,...,b_r^p$ $J-$*independent of order* $k$ *with respect to* $g^{(ps_1)} = (G_{nps_1})_{n\in\mathbb{N}}$*? If* $\mathcal{F}$ *is a homogeneous polynomial of degree* $d$ *with coefficients in* $G_0$ *with* $r$ *indeterminate, we have:* $\mathcal{F}\left(b_1^p,...,b_r^p\right) = \sum_{i_1+i_2+...+i_r=d} \beta_{i_1...i_r}(b_r^p)^{i_1}...(b_r^p)^{i_r}$*, let* $x = \mathcal{F}\left(b_1^p,...,b_r^p\right)$*.*

$x \in JG_{ps_1d} + \left(G_0G_{ps_1(d+k)}\right) \Rightarrow x \in JG_{ps_1d} + \left(G_{ps_1d+ps_1k} \cap G_{ps_1d}\right)$*, or*

$JG_{ps_1d} + \left(G_{ps_1d+ps_1k} \cap G_{ps_1d}\right) \subseteq JG_{ps_1d} + G_0\left(G_{ps_1d+s_1k} \cap G_{ps_1d}\right)$ *therefore*

$x \in JG_{s_{pd}} + G_0\left(G_{s_{pd+k}} \cap G_{s_{pd}}\right) \Rightarrow \beta_{i_1...i_r} \in J + G_0G_{s_k} \cap G_0 = J = J + G_0G_{ps_1k} \cap G_0$*, indeed* $k \leq s_k$ *and* $k \leq ps_1k$*.Consequently* $\ell_J^q(g,k) \leq \ell_J(g^{(ps_1)},k)$ *and* $1-(i)$

*entail the equality of* $\ell_J^q(g,k)$ *with all* $\ell_J(g^{(ps_1)},k)$ *as well as with* $\ell_J^a(g,k)$ *and* $\ell_J^+(g,k)$ *.*
*Let us prove* $(iii)$*. Referring to assertions* $1-(i)$ *and* $2-(ii)$*, we have:*

$$\ell_J^q(g,k) = \sup\left\{\ell_J(g^{(n)},k)\,;\, n\in\mathbb{N}^*\right\} \le \sup\left\{\ell_J(g^{(pn)},k)\,;\, n\in\mathbb{N}^*\right\} \le \ell_J^q(g,k),$$

$\ell_J^q(g,k) = \sup\left\{\ell_J(g^{(pn)},k)\,;\, n\in\mathbb{N}^*\right\} = \ell_J^q(g^{(n)},k)$ *et* $\ell_J^q(g,k) = \ell_J^a(g,k) = \ell_J^+(g,k)$

**Corollary 4.2.1.** *Let* $\mathcal{B}$ *be a ring,* $m\in\mathbb{N}^*$, $k\in\overline{\mathbb{N}^*}$ *and* $g=(G_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ *a axis-quasi-graduation of* $\mathcal{B}$*. Assume that* $G_0$ *is Noetherian and that* $g$ *is* $m-$*regular. Let* $J$ *be a proper ideal of* $G_0$ *such that* $J\supseteq (G_0G_{i+k})\cap G_0$, $\forall\, i\ge 0$*. If* $g$ *is* $k-$*decreasing then we have:*

*i)* $\ell_J^a(g,k) = \ell_J^a(G_{mn},k) = \ell_J^q(g,k)\,,\ \forall\, n\in\mathbb{N}^*$ *.*

*ii)* $\exists\, p\in\mathbb{N}^*$ *multiple of* $m$ *such that ,* $\ell_J^a(g,k) = \ell_J^a(G_{pn},k)\,,\ \forall\, n\in\mathbb{N}^*$ *.*

*iii)* $\ell_J^a(g,k) = \sup\{\ell_J(G_{mn},k)\,,\ \forall\, n\in\mathbb{N}^*\}$

*iv) The sequence* $n\mapsto \ell_J(G_{mn},k)$ *is increasing and converges to the integer* $\ell_J^a(g,k)$*.*

*v) The sequence* $n\mapsto \ell_J(G_n,k)$ *converges to the integer* $\ell_J^a(g,k)$*.*

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**Received: March 23, 2025; Published: July 3, 2025**