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On Herrlich-Spectral Spaces

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Abstract

A T_0-space is H-spectral if its Herrlich compactification $\beta_\omega X$ is spectral. In this paper we give a necessary and sufficient conditions on a T_0-space X in order to get it H-spectral.

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Introduction

The spectrum of a commutative ring with identity R, denoted $Spec(R)$, is the collection of proper prime ideals of R. A subset C of $Spec(R)$ is a closed set of the Zariski topology $\mathcal{Z}$ on $Spec(R)$ if there exists a prime ideal I of R such that $C = \{P \in Spec(R) \mid I \subseteq P\}$. In [6] Hochster characterized the class of topological spaces $(Spec(R), \mathcal{Z})$ and called them spectral spaces. He proved that a topological space X is spectral if and only if the following statements hold:

(a) X is sober.
(b) X is compact.
(c) The compact open sets form a basis of X.
(d) The family of compact open sets of X is closed under finite intersections.

It is natural to ask which conditions should check a topological space to have its compactification spectral. In [4] authors characterize spaces such that their one point compactification is a spectral space, such spaces was called

A-spectral spaces. A complete characterization of W-spectral spaces (that is, spaces such that their Wallman compactification is a spectral space) and spaces such that their Stone Čech compactification is a spectral space has been settled in [2] and [1].

A minimal compactification of a T_0-space X, denoted by $\beta_\omega X$, has been constructed by Herrlich in [5]. In honor to Herrlich this T_0-compactification is called the Herrlich compactification. For a T_1-space, the Wallman compactification and the Herrlich compactification coincide.

A topological space X is called a H-spectral space if its Herrlich compactification $\beta_\omega X$ is a spectral space [2]. The propose of this paper is to give a characterization of H-spectral spaces.

The first section of this paper is devoted to give some remarks about the Herrlich compactification which there will be needed throughout this work.

Section 2 contains a characterization of compact open sets of the Herrlich compactification. A necessary and sufficient conditions on a topological space X are given to have the compact open sets form a basis of $\beta_\omega X$.

Section 3 deal with spaces such that their Herrlich compactification is sober.

In section 4, necessary and sufficient conditions are given on a T_0-space in order to get its Herrlich compactification spectral.

Throughout this paper we consider spaces on which no separation axioms are assumed unless explicitly stated.

1. Some remarks about Herrlich compactification

In 1993, Herrlich introduce a compactification of T_0-space as follows [5]: Let X be a T_0-space and $\Gamma(X)$ be the collection of all filters $\mathcal{F}$ on X satisfying the following two conditions:

(a) $\mathcal{F}$ does not converge in X.
(b) Every finite open cover of X contains some member of $\mathcal{F}$.

Let $\Omega(X)$ be the set of minimal elements of $\Gamma(X)$ and define:

(i) $X^*_\omega = X \cup \Omega(X)$.
(ii) $A^*_\omega = A \cup \{\mathcal{F} \mid \mathcal{F} \in \Omega(X) \text{ and } A \in \mathcal{F}\}$ for $A \subseteq X$.

Let $\beta_\omega = \{A^*_\omega \mid A \text{ open in } X\}$. Hence β_ω is a base for a topology $\mathcal{T}^*_\omega$ on X^*_ω. Then X is a dense set of the T_0-space X^*_ω. Herrlich showed also that X^*_ω is a compact space. Hence X^*_ω is a compactification of X. This compactification is called the Herrlich compactification (or the T_0-compactification) of X and it is generally denoted by $\beta_\omega X$. The following properties of $\beta_\omega X$ are frequently useful:

Proposition 1.1. *Let X be a T_0-space. Then the following properties hold:*

(1) *For A and B two subsets of X, $(A \cap B)^*_\omega = A^*_\omega \cap B^*_\omega$.*
(2) *For each $\mathcal{F} \in \Omega(X)$, $\{\mathcal{F}\}$ is a closed set of $\beta_\omega X$.*

Throughout this paper, given a T_0-space X and $\mathcal{O}$ a collection of open sets of X, we will denote $O(\mathcal{O})$ the set $\cup(O^*_\omega : O \in \mathcal{O})$ and $C(\mathcal{O})$ the set $\beta_\omega X \setminus \cup(O^*_\omega : O \in \mathcal{O})$.

The following concept has been introduced by Belaid et al [3]:

Definition 1.2. Let X be a topological space. An open cover $\mathcal{U}$ of X is said to be a good-covering (g-covering, for short) of X if it has a finite subcover, and $\mathcal{U}$ is called a bad-covering (b-covering, for short) of X if it is not a g-covering.

For a T_0-space X, two collections $\mathcal{O}_1$ and $\mathcal{O}_2$ of proper open sets of X are called H-equivalent ($\mathcal{O}_1 \sim_H \mathcal{O}_2$, for short) if for each b-covering $\mathcal{U}$ of X, $\mathcal{U}\cup\mathcal{O}_1$ is a g-covering of X if and only if $\mathcal{U} \cup \mathcal{O}_2$ is a g-covering of X.

Example 1.3.
(1) Let $\mathbb{N}$ be the set of positive integers equipped with the left topology (that is, $\mathbb{N}$ is equipped with the topology generated by $\{(\downarrow, x) \mid x \in \mathbb{N}\}$ with $(\downarrow, x) = \{y \in \mathbb{N} \mid y \leq x\}$). For $x \in \mathbb{N}$, the equivalence class of $\{(\downarrow, x)\}$ by $\sim_H$ is $\{(\downarrow, x)\}$, since every collection of proper open sets of $\mathbb{N}$ is a b-covering of $\mathbb{N}$.
(2) Let $\mathbb{Z}$ be the digital line (that is, the set of integers equipped with Khalimsky topology $\mathcal{K}$ generated by $\mathcal{K}_\mathbb{Z} = \{\{2n-1, 2n, 2n+1\} \mid n \in \mathbb{Z}\}$). For $n \in \mathbb{N}$, the equivalence class of $\{\{2n-1, 2n, 2n+1\}\}$ by $\sim_H$ is $\{O \in \mathcal{K} \mid O \text{ is finite}\}$.

The following lemma was proved in [2]:

Lemma 1.4. *Let X be a non compact T_0-space and $\mathcal{U}$ be an open cover of X. Then $\mathcal{U}$ is a g-covering of X if and only if for each $\mathcal{F} \in \beta_\omega X \setminus X$, there exists $U \in \mathcal{U}$ such that $U \in \mathcal{F}$.*

Corollary 1.5. *Let X be a non compact T_0-space. A collection of open sets $\mathcal{O}$ of X is a g-covering of X if and only if $\beta_\omega X = O(\mathcal{O})$.*

Proof. *Necessary condition.* Straightforward.

Sufficient condition. Let $\mathcal{O}$ be a collection of open sets of X such that $\beta_\omega X = \cup(O^*_\omega : O \in \mathcal{O})$. Since $\beta_\omega X$ is a compact space, there exists a finite subset $\mathcal{O}'$ of $\mathcal{O}$ such that $\beta_\omega X = \cup(O^*_\omega : O \in \mathcal{O}')$. Hence $\mathcal{O}'$ is a finite open cover of X. So $\mathcal{O}$ is a g-covering of X. □

Proposition 1.6. *Let X be a non compact T_0-space. Then following statements hold:*

(1) *$\beta_\omega X \setminus X$ is a singleton.*
(2) *For each b-covering $\mathcal{U}$ of X and each two open sets U_1, U_2 of X such that $\mathcal{U} \cup \{U_1, U_2\}$ is a g-covering of X, either $\mathcal{U} \cup \{U_1\}$ or $\mathcal{U} \cup \{U_2\}$ is a g-covering of X.*

Proof. (1) $\Longrightarrow$ (2) Let $\mathcal{U}$ be a b-covering of X and U_1, U_2 be two open sets of X such that $\mathcal{U} \cup \{U_1, U_2\}$ is a g-covering of X. Suppose that $\mathcal{U} \cup \{U_1\}$ and $\mathcal{U} \cup \{U_2\}$ are both a b-covering of X. Then there exist $\mathcal{F}_1$ and $\mathcal{F}_2$ in $\beta_\omega X \setminus X$

such that $\mathcal{F}_1 \in U^*_{1\omega} \setminus U^*_{2\omega}$ and $\mathcal{F}_2 \in U^*_{2\omega} \setminus U^*_{1\omega}$ contradicting with the fact that $\beta_\omega X \setminus X$ is a singleton. Hence either $\mathcal{O} \cup \{U_1\}$ or $\mathcal{O} \cup \{U_2\}$ is a g-covering of X.

(2) $\Longrightarrow$ (1) Suppose that there exists two distinct elements $\mathcal{F}_1$ and $\mathcal{F}_2$ in $\beta_\omega X \setminus X$. Since elements of $\beta_\omega X \setminus X$ are closed sets of $\beta_\omega X$, there exists two open sets U_1, U_2 of X such that $\mathcal{F}_1 \in U^*_{1\omega} \setminus U^*_{2\omega}$ and $\mathcal{F}_2 \in U^*_{2\omega} \setminus U^*_{1\omega}$. Let $\mathcal{U}$ be the collection of open sets $\mathcal{U}$ of X such that $U^* \subseteq \beta_\omega X \setminus \{\mathcal{F}_1, \mathcal{F}_2\}$. Hence $\mathcal{U}$, $\mathcal{U} \cup \{U_1\}$ and $\mathcal{U} \cup \{U_2\}$ are b-covering of X. That contradict hypothesis, since $\mathcal{U} \cup \{U_1, U_2\}$ is a g-covering of X. Thus $\beta_\omega X \setminus X$ is a singleton. □

Definition 1.7. A non compact T_0-space X is called a $H1$-space if one of the following statements hold:

1. (1) Every proper open cover of X is a b-covering of X.
2. (2) For each b-covering $\mathcal{U}$ of X and each two open sets U_1, U_2 of X such that $\mathcal{U} \cup \{U_1, U_2\}$ is a g-covering of X, either $\mathcal{U} \cup \{U_1\}$ or $\mathcal{U} \cup \{U_2\}$ is a g-covering of X.

Example 1.8. (1) The set of positive integers $\mathbb{N}$ equipped with the left topology is a $H1$-space.
(2) Every infinite discrete topological space is a $H1$-space.

The following proposition is an immediate consequence of the Herrlich compactification construction.

Proposition 1.9. *Let X be a T_0-space such that every proper open cover of X is a b-covering of X. Then $\beta_\omega X \setminus X$ is a singleton and for each proper open set O of X, $O^*_\omega = O$.*

Lemma 1.10. *Let X be a $H1$-space and O be an open set of X. Set $\{\infty\} = \beta_\omega X \setminus X$. Then the following statements are equivalent:*

(1) $O^*_\omega = O \cup \{\infty\}$.
(2) *For each b-covering $\mathcal{U}$ of X, $\mathcal{U} \cup \{O\}$ is a g-covering of X.*
(3) *$X \setminus O$ is a compact closed set of X.*

Proof. (1) $\Longrightarrow$ (2) Let $\mathcal{U}$ be a b-covering $\mathcal{U}$ of X. Since $\{O^*_\omega\} \cup (\cup(U^*_\omega : U \in \mathcal{U})) = \beta_\omega X$ and $\beta_\omega X$ is compact, $\mathcal{U} \cup \{O\}$ is a g-covering of X.

(2) $\Longrightarrow$ (3) Let $\mathcal{V}$ be an open cover of $X \setminus O$. Then $\mathcal{V} \cup \{O\}$ is a g-covering of X. Hence there exists a finite subset $\mathcal{V}'$ of $\mathcal{V}$ such that $\mathcal{V}' \cup \{O\}$ is an open cover of X. Thus $\mathcal{V}'$ is a finite open cover of $X \setminus O$. So $X \setminus O$ is compact.

(3) $\Longrightarrow$ (1) Since ∞ does not converge in X , for each $x \in X \setminus O$ there exists an open set U_x of X such that $\infty \notin U^*_{x\omega}$. Hence $\mathcal{U} = \{U_x \mid x \in X \setminus O\}$ is an open cover of $X \setminus O$. Since $X \setminus O$ is compact, there exists a finite subset $\mathcal{U}'$ of $\mathcal{U}$ such that $\mathcal{U}'$ is a cover of $X \setminus O$. Thus $\mathcal{U}' \cup \{O\}$ is a finite cover of X. Then $\infty \in O^*_\omega$, since $\infty \notin U^*_{x\omega}$, for each $U \in \mathcal{U}'$. □

Let X be a topological space and $\infty \notin X$. Recall that the one point compactification of X is $\widetilde{X} = X \cup \{\infty\}$ equipped with the topology whose elements are open sets of X and subsets U of $\widetilde{X}$ such that $\widetilde{X} \setminus U$ is compact closed set of X.

Proposition 1.11. *Let X be a non compact T_0-space. If X is a $H1$-space then the Herrlich compactification coincide with the one point compactification.*

Definition 1.12. A non compact T_0-space X is called a B-space if it is not a $H1$-space.

The following proposition help in understanding the importance of the notion of H-equivalence.

Proposition 1.13. *Let X be a B-space, $\mathcal{O}_1$ and $\mathcal{O}_2$ be two collections of proper open sets of a T_0-space X. Then the following statements are equivalent:*

(1) $O(\mathcal{O}_1) = O(\mathcal{O}_2)$.
(2) $\cup(O : O \in \mathcal{O}_1) = \cup(O : O \in \mathcal{O}_2)$ *and* $\mathcal{O}_1 \sim_H \mathcal{O}_2$.

Proof. (1) $\Longrightarrow$ (2) That $\cup(O : O \in \mathcal{O}_1) = \cup(O : O \in \mathcal{O}_2)$ is immediate since $O(\mathcal{O}_1) = O(\mathcal{O}_2)$. Let $\mathcal{U}$ be a b-covering of X such that $\mathcal{U} \cup \mathcal{O}_1$ is a g-covering of X. Suppose that $\mathcal{U} \cup \mathcal{O}_2$ is a b-covering of X. Then there exists $\mathcal{F} \in \beta_\omega X \setminus X$ such that $\mathcal{F} \notin V^*_\omega$, for each $V \in \mathcal{U} \cup \mathcal{O}_2$. Hence $\mathcal{F} \notin O^*_\omega$, for each $O \in \mathcal{O}_2$. Since $O(\mathcal{O}_1) = O(\mathcal{O}_2)$, $\mathcal{F} \notin O^*_\omega$, for each $O \in \mathcal{O}_1$. Contradicting the fact that $\mathcal{U} \cup \mathcal{O}_1$ is a g-covering of X. Thus $\mathcal{U} \cup \mathcal{O}_2$ is a g-covering of X. So $\mathcal{O}_1 \sim_H \mathcal{O}_2$.

(2) $\Longrightarrow$ (1) Let $\mathcal{F} \in O^*_\omega \cap (\beta_\omega X \setminus X)$ with $O \in \mathcal{O}_1$. Set $\mathcal{U}$ be the collection of open sets U of X such that $\mathcal{F} \notin U^*_\omega$. Since $\{\mathcal{F}\}$ is a closed set of $\beta_\omega X$, $\beta_\omega X \setminus \{\mathcal{F}\} = \cup(U : U \in \mathcal{U})$. Then $\mathcal{U}$ is a b-covering and $\mathcal{U} \cup \{O\}$ is a g-covering of X so that $\mathcal{U} \cup \mathcal{O}_1$ is a g-covering of X. Since $\mathcal{O}_1 \sim_H \mathcal{O}_2$, $\mathcal{U} \cup \mathcal{O}_2$ is a g-covering of X. Hence there exists $O' \in \mathcal{O}_2$ such that $\mathcal{F} \in O'^*_\omega$, since $\mathcal{F} \notin U^*_\omega$, for each $U \in \mathcal{U}$. Thus $\mathcal{F} \in O(\mathcal{O}_2)$. Therefore $O(\mathcal{O}_1) = O(\mathcal{O}_2)$. □

Example 1.14. Let $\mathbb{Z}$ be the digital line. For $n \in \mathbb{Z}$, $(\{2n-1, 2n, 2n+1\} \cup \{2n+1, 2n+2, 2n+3\})^* = (\{2n-1, 2n, 2n+1\})^* \cup (\{2n+1, 2n+2, 2n+3\})^*$, and $(\cup\{2k-1, 2k, 2k+1\} : k \leq n)^* \neq (\cup(\{2k-1, 2k, 2k+1\})^* : k \leq n)$.

2. Compact open sets of $\beta_\omega X$

Our goal in the present section is to characterize compact open sets of $\beta_\omega X$ and to give a necessary and sufficient conditions on X to have $\beta_\omega X$ with compact open base. First let us introduce the following notion:

Definition 2.1. Let X be a B-space and $\mathcal{O}$ be a collection of open sets of X. A collection $\mathcal{V}$ of open sets of X is said to be a H-cover of $\mathcal{O}$ if the following properties hold:

(1) $\cup(O : O \in \mathcal{O}) \subseteq \cup(V : V \in \mathcal{V})$.
(2) For every b-covering $\mathcal{U}$ of open sets of X, if $\mathcal{U} \cup \mathcal{O}$ is a g-covering of X then $\mathcal{U} \cup \mathcal{V}$ is a g-covering of X.

Proposition 2.2. *Let X be a B-space and $\mathcal{O}$ be a collection of open sets of X. A collection $\mathcal{V}$ of open sets of X is a H-cover of $\mathcal{O}$ if and only if $O(\mathcal{O}) \subseteq O(\mathcal{V})$.*

Proof. Necessary condition. Let $\mathcal{V}$ be a H-cover of $\mathcal{O}$. It is immediate that $\cup(O : O \in \mathcal{O}) \subseteq \cup(V : V \in \mathcal{V})$. Let $\mathcal{F} \in O(\mathcal{O}) \cap (\beta_\omega X \setminus X)$. Then there exists $O \in \mathcal{O}$ such that $\mathcal{F} \in O^*_\omega$. Set $\mathcal{U}$ the collection of open sets of X such that $\mathcal{F} \notin U^*_\omega$, for each $U \in \mathcal{U}$. Hence $\mathcal{U}$ is a b-covering of X and $\mathcal{U} \cup \{O\}$ is a g-covering of X, so $\mathcal{U} \cup \mathcal{O}$ is a g-covering of X. Since $\mathcal{V}$ is a H-cover of $\mathcal{O}$, $\mathcal{U} \cup \mathcal{V}$ is a g-covering of X. Thus there exists $V \in \mathcal{V}$ such that $\mathcal{F} \in V^*_\omega$. Then $\cup(O^*_\omega : O \in \mathcal{O}) \subseteq \cup(V^*_\omega : V \in \mathcal{V})$.

Sufficient condition. Let $\mathcal{V}$ be a collection of open sets of X such that $O(\mathcal{O}) \subseteq O(\mathcal{V})$. That $\cup(O : O \in \mathcal{O}) \subseteq \cup(V : V \in \mathcal{V})$ is straightforward. Now, let $\mathcal{U}$ be a b-covering of X such that $\mathcal{U} \cup \mathcal{O}$ is a g-covering of X. Set $A = \{\mathcal{F} \in \beta_\omega X \setminus X \mid \mathcal{F} \notin U^*_\omega, \forall U \in \mathcal{U}\}$. Then $A \subseteq O(\mathcal{O})$. Hence $\mathcal{U} \cup \mathcal{V}$ is a g-covering of X, since $\cup(O^*_\omega : O \in \mathcal{O}) \subseteq \cup(V^*_\omega : V \in \mathcal{U})$. Thus $\mathcal{V}$ is a H-cover of $\mathcal{O}$. □

We need that following definition.

Definition 2.3. Let X be a B-space and $\mathcal{O}$ be a collection of open sets of X. We say that $\mathcal{O}$ is H-compact if for each H-cover $\mathcal{V}$ of $\mathcal{O}$ there exists a finite subcollection $\mathcal{V}'$ of $\mathcal{V}$ such that $\mathcal{V}'$ is a H-cover of $\mathcal{O}$.

Example 2.4. Let $\mathbb{Z}$ be the digital line.

(1) Let $\mathcal{O} = \{\{2m-1, 2m, 2m+1\}, \{2n-1, 2n, 2n+1\}\}$ is a H-compact set of $\mathbb{Z}$ with $m, n \in \mathbb{Z}$. In fact, let $\mathcal{V}$ be H-cover $\mathcal{O}$, then there exist $V_1, V_2 \in \mathcal{V}$ such that $2m \in V_1$ and $2n \in V_2$. Hence $\{2m-1, 2m, 2m+1\} \subseteq V_1$ and $\{2n-1, 2n, 2n+1\} \subseteq V_2$. Thus for every b-covering $\mathcal{U}$ of X, if $\mathcal{U} \cup \mathcal{O}$ is a g-covering of X then $\mathcal{U} \cup \mathcal{V}$ is a g-covering of X. Then $\mathcal{O}$ is H-compact set of $\mathbb{Z}$.

(2) Let A be an infinite subset of $\mathbb{Z}$. It is immediate that $\mathcal{O} = \{\{2n-1, 2n, 2n+1\} \mid n \in A\}\}$ is not a H-compact collection of $\mathbb{Z}$, and for $p \in \mathbb{Z}$, $\mathcal{O} = \{\{2k-1, 2k, 2k+1\} \mid k \geq 2p-1\}\}$ is not a H-compact collection of $\mathbb{Z}$.

(3) Let $n \in \mathbb{Z}$ and $\mathcal{O} = \{\{k \in \mathbb{Z} \mid k \geq 2n-1\}\}$. Let $\mathcal{V}$ be a H-cover of $\mathcal{O}$. Then $\{k \in \mathbb{Z} \mid k \geq 2n-1\} \subseteq \cup(V : V \in \mathcal{V})$ and for every b-covering $\mathcal{U}$ of open sets of X, if $\mathcal{U} \cup \{\{k \in \mathbb{Z} \mid k \geq 2n-1\}\}$ is a g-covering of X then $\mathcal{U} \cup \mathcal{V}$ is a g-covering of X. Hence there exists a finite subset $\mathcal{V}'$ of $\mathcal{V}$ such that $\mathcal{V}$ is finite cover of $\{k \in \mathbb{Z} \mid k \geq 2n-1\}$. Thus $\mathcal{O}$ is H-compact.

Proposition 2.5. *Let X be a B-space. The collection of open sets $\mathcal{O}$ of X is H-compact if and only if $O(\mathcal{O})$ is a compact set of $\beta_\omega X$.*

Proof. Necessary condition. Let $\mathcal{O}$ be a collection of open sets of X such that $\mathcal{O}$ is H-compact and $\mathcal{V}$ a collection of open sets of X such that $O(\mathcal{O}) \subseteq \cup(V^*_\omega : V \in \mathcal{V})$. By Proposition 2.2 $\mathcal{V}$ is a H-cover of $\mathcal{O}$. Since $\mathcal{O}$ is H-compact, there exists a finite subset $\mathcal{V}'$ of $\mathcal{V}$ such that $\mathcal{V}'$ is a H-cover of $\mathcal{O}$. Hence $O(\mathcal{O}) \subseteq \cup(V^*_\omega : V \in \mathcal{V}')$. Thus $O(\mathcal{O})$ is a compact set of $\beta_\omega X$.

Sufficient condition. Let $\mathcal{O}$ be a collection of open sets of X such that $O(\mathcal{O})$ is a compact set of $\beta_\omega X$ and $\mathcal{V}$ be a collection of open sets of X such that $\mathcal{V}$ is a H-cover of $\mathcal{O}$. Then $O(\mathcal{O}) \subseteq \cup(V^*_\omega : V \in \mathcal{V})$. Since $O(\mathcal{O})$ is compact, there exists a finite subset $\mathcal{V}'$ of $\mathcal{V}$ such that $O(\mathcal{O}) \subseteq \cup(V^*_\omega : V \in \mathcal{V}')$. Hence $\mathcal{V}'$ is a finite subset of $\mathcal{V}$ such that $\mathcal{V}'$ is a H-cover of $\mathcal{O}$. Thus $\mathcal{O}$ is H-compact. □

Now, we are in position to give the principal result of this section.

Proposition 2.6. *Let X be a B-space. The following statements are equivalent:*

(1) *The compact open sets form a basis of $\beta_\omega X$.*
(2) *For each open set U of X there exists a family $\{\mathcal{O}_k \mid k \in K\}$ of collections of open sets of X such that:*
  (*i*) *$\mathcal{O}_k$ is H-compact, for each $k \in K$;*
  (*ii*) *$U = \cup(O : O \in \cup(\mathcal{O}_k : k \in K))$;*
  (*iii*) *$\{U\}$ and $\cup(\mathcal{O}_k : k \in K)$ are H-equivalent.*

Proof. (1) $\Longrightarrow$ (2) Let U be an open set of X. Then U^*_ω is an open set of $\beta_\omega X$. Since the compact open sets form a basis of $\beta_\omega X$., there exists a collection $\{V_k \mid k \in K\}$ of compact open sets of $\beta_\omega X$ such that $U^*_\omega = \cup(V_k : k \in K)$. For each $k \in K$, V_k is a compact open set of $\beta_\omega X$, then there exists a collection $\mathcal{O}_k$ of open sets of X such that $V_k = O(\mathcal{O}_k)$. By Proposition 2.5 $\mathcal{O}_k$ is H-compact. Since $U^*_\omega = \cup(O^*_\omega : O \in \cup(\mathcal{O}_k : k \in K))$, $U = \cup(O : O \in \cup(\mathcal{O}_k : k \in K))$ and $\{U\}$ is H-equivalent to $\cup(\mathcal{O}_k : k \in K)$ (by Proposition 1.13).

(2) $\Longrightarrow$ (1) Let U be an open set of X. Then there exists a family $\{\mathcal{O}_k \mid k \in K\}$ of collections of open sets of X such that:
(*i*) $\mathcal{O}_k$ is H-compact, for each $k \in K$;
(*ii*) $U = \cup(O : O \in \cup(\mathcal{O}_k : k \in K))$;
(*iii*) $\{U\}$ and $\cup(\mathcal{O}_k : k \in K)$ are H-equivalent.

Since $U = \cup(O : O \in \cup(\mathcal{O}_k : k \in K))$, $\cup(O^*_\omega : O \in \cup(\mathcal{O}_k : k \in K)) \subseteq U^*_\omega$. That $\cup(O^*_\omega : O \in \cup\mathcal{O}_k) = U^*_\omega$ is immediate from the fact that $\{U\}$ and $\cup(\mathcal{O}_k : k \in K)$ are H-equivalent. Hence $U^*_\omega = \cup(O(\mathcal{O}_k) : k \in K)$. On the other hand, for each $k \in K$, $\mathcal{O}_k$ is H-compact. Hence $O(\mathcal{O}_k)$ is a compact open set of $\beta_\omega X$. It follows that he compact open sets form a basis of $\beta_\omega X$. □

Example 2.7. Let $\mathbb{Z}$ be the digital line. By Example 2.4 the collection of compact open sets is not a base of the topology of $\beta_\omega \mathbb{Z}$.

3. Irreducible closed sets of $\beta_\omega X$

In this section we give necessary and sufficient conditions on a B-space to have its Herrlich compactification sober. First, let us introduce the following definition:

Definition 3.1. Let X be a B-space, $\mathcal{O}$ be a collection of open sets of X. We say that an open set V of X c-H-meet $\mathcal{O}$ ($V \cap^c_H \mathcal{O} \neq \emptyset$, for short) if at least one of the following properties hold:

(1) $V \cap (X \setminus \cup(O : O \in \mathcal{O})) \neq \emptyset$.
(2) There exists a b-covering $\mathcal{U}$ of X such that $\mathcal{U} \cup \{V\}$ is a g-covering and $\mathcal{U} \cup \mathcal{O}$ is a b-covering.

Example 3.2. Let $\mathbb{Z}$ be the digital line and $n \in \mathbb{Z}$. Let $\mathcal{O} = \{ [2k-1, +\infty[\cap \mathbb{Z} \mid k \geq -1\}$ and $V = [5, +\infty[\cap \mathbb{Z}$. Since for $\mathcal{U} = \{]-\infty, 2k+1] \cap \mathbb{Z} \mid k \in \mathbb{Z}\}$ is a b-covering of $\mathbb{Z}$ such that $\mathcal{U} \cup \{V\}$ is a g-covering and $\mathcal{U} \cup \mathcal{O}$ is a b-covering, $V \cap_H^c \mathcal{O} \neq \emptyset$.

Proposition 3.3. *Let X be a B-space, $\mathcal{O} \cup \{V\}$ be a collection of open sets of X. Then the following statements are equivalent:*
(1) $V_\omega^* \cap C(\mathcal{O}) \neq \emptyset$.
(2) $V \cap_H^c \mathcal{O} \neq \emptyset$.

Proof. (1) $\Longrightarrow$ (2) Let $\mathcal{O} \cup \{V\}$ be collection of open sets of X such that $V_\omega^* \cap C(\mathcal{O}) \neq \emptyset$ and suppose that $V \cap (X \setminus \cup(O : O \in \mathcal{H})) = \emptyset$. Then there exists $\mathcal{F} \in \beta_\omega X \setminus X$ such that $\mathcal{F} \in V_\omega^*$ and $\mathcal{F} \notin \cup(O_\omega^* : O \in \mathcal{O})$. Set $\mathcal{U}$ the collection of open sets of X such that $\mathcal{F} \notin U_\omega^*$, for each $U \in \mathcal{U}$. Hence $\mathcal{U}$ is a b-covering of X and $\mathcal{U} \cup \{V\}$ is a g-covering of X. That $\mathcal{U} \cup \mathcal{O}$ is a b-covering is immediate since $\mathcal{F} \notin O_\omega^*$, for each $O \in \mathcal{O}$. Therefore $V \cap_H^c \mathcal{O} \neq \emptyset$.

(2) $\Longrightarrow$ (1) Let $\mathcal{O} \cup \{V\}$ be collection of open sets of X such that $V \cap_H^c \mathcal{O} \neq \emptyset$. It is immediate that if $V \cap (X \setminus \cup(O : O \in \mathcal{O})) \neq \emptyset$ then $V_\omega^* \cap C(\mathcal{O}) \neq \emptyset$. Suppose, now, that $V \cap (X \setminus \cup(O : O \in \mathcal{O})) = \emptyset$. Then there exists a b-covering $\mathcal{U}$ of X such that $\mathcal{U} \cup \{V\}$ is a g-covering and $\mathcal{U} \cup \mathcal{O}$ is a b-covering. Hence there exists $\mathcal{F} \in \beta_\omega X \setminus X$ such that $\mathcal{F} \in V_\omega^*$ and $\mathcal{F} \notin \cup(O_\omega^* : O \in \mathcal{O})$. Thus $\mathcal{F} \in V_\omega^* \cap C(\mathcal{O})$ proving that $V_\omega^* \cap C(\mathcal{O}) \neq \emptyset$. □

Definition 3.4. Let X be a T_0-space and $\mathcal{O}$ be a collection of open sets of X. The collection $\mathcal{O}$ is called H-irreducible if, for each two open sets V_1 and V_2 of X such that $V_1 \cap_H^c \mathcal{O} \neq \emptyset$ and $V_2 \cap_H^c \mathcal{O} \neq \emptyset$, $(V_1 \cap V_2) \cap_H^c \mathcal{O} \neq \emptyset$.

Example 3.5. Let $\mathbb{Z}$ be the digital line and $n \in \mathbb{Z}$. Let $\mathcal{O} = \{\{k \in \mathbb{Z} \mid k \leq 2n-1\}, \{k \in \mathbb{Z} \mid k \geq 2n+2\}\}$. Then $\mathcal{O}$ is H-irreducible.

Proposition 3.6. *Let X be a B-space and $\mathcal{O}$ be a collection of open sets of X. The following statements are equivalent:*
(1) *$\mathcal{O}$ is H-irreducible.*
(2) *$C(\mathcal{O})$ is an irreducible closed set of $\beta_\omega X$.*

Proof. (1) $\Longrightarrow$ (2) Let $\mathcal{O}$ be a H-irreducible collection open sets of X, V_1 and V_2 be two open sets of X such that $V_1^* \cap C(\mathcal{O}) \neq \emptyset$ and $V_2^* \cap C(\mathcal{O}) \neq \emptyset$. Then $V_1 \cap_H^c \mathcal{O} \neq \emptyset$ and $V_2 \cap_H^c \mathcal{O} \neq \emptyset$. Since $\mathcal{O}$ is a H-irreducible collection open sets of X , $(V_1 \cap V_2) \cap_H^c \mathcal{O} \neq \emptyset$. Hence $(V_1 \cap V_2)_\omega^* \cap C(\mathcal{O}) \neq \emptyset$. Thus $C(\mathcal{O})$ is an irreducible closed set of $\beta_\omega X$.

(2) $\Longrightarrow$ (1) Let $\mathcal{O}$ be a collection of open sets of X such that $C(\mathcal{O})$ is an irreducible closed set of $\beta_\omega X$, V_1 and V_2 be two open sets of X such that $V_1 \cap_H^c \mathcal{O} \neq \emptyset$ and $V_2 \cap_H^c \mathcal{O} \neq \emptyset$. So $V_1^* \cap C(\mathcal{O}) \neq \emptyset$ and $V_2^* \cap C(\mathcal{O}) \neq \emptyset$.

Since $C(\mathcal{O})$ is an irreducible closed set of $\beta_\omega X$, $(V_1 \cap V_2)^*_\omega \cap C(\mathcal{O}) \neq \emptyset$. Hence $(V_1 \cap V_2) \cap^c_H \mathcal{O} \neq \emptyset$. Therefore $\mathcal{O}$ is H-irreducible. □

Lemma 3.7. *Let X be a non compact B-space and $\mathcal{O}$ be a b-covering of X. Then following statements hold:*

(1) *$\beta_\omega X \cap C(\mathcal{O})$ is a singleton.*
(2) *For each two open sets U_1, U_2 of X such that $\mathcal{O} \cup \{U_1, U_2\}$ is a g-covering of X, either $\mathcal{O} \cup \{U_1\}$ or $\mathcal{O} \cup \{U_2\}$ is a g-covering of X.*

Proof. (1) $\Longrightarrow$ (2) Let $\mathcal{O}$ be a b-covering of X such that $\beta_\omega X \cap C(\mathcal{O})$ is a singleton and U_1, U_2 two open sets of X such that $\mathcal{O} \cup \{U_1, U_2\}$ is a g-covering of X. Since $\mathcal{O}$ is an open cover of X, $C(\mathcal{O}) \subseteq \beta_\omega X \setminus X$. Suppose that $\mathcal{O} \cup \{U_1\}$ and $\mathcal{O} \cup \{U_2\}$ are both a b-covering of X. Then there exist $\mathcal{F}_1$ and $\mathcal{F}_2$ in $\beta_\omega X \cap C(\mathcal{O})$ such that $\mathcal{F}_1 \in U^*_{1\omega} \setminus U^*_{2\omega}$ and $\mathcal{F}_2 \in U^*_{2\omega} \setminus U^*_{1\omega}$ contradicting hypothesis. Hence either $\mathcal{O} \cup \{U_1\}$ or $\mathcal{O} \cup \{U_2\}$ is a g-covering of X.

(2) $\Longrightarrow$ (1) Let $\mathcal{O}$ be a b-covering of X such that for each two open sets U_1, U_2 of X such that $\mathcal{O} \cup \{U_1, U_2\}$ is a g-covering of X, either $\mathcal{O} \cup \{U_1\}$ or $\mathcal{O} \cup \{U_2\}$ is a g-covering of X. Since $\mathcal{O}$ is an open cover of X, $C(\mathcal{O}) \subseteq \beta_\omega X \setminus X$. Suppose that $\beta_\omega X \cap C(\mathcal{O})$ contains two distinct elements $\mathcal{F}_1$ and $\mathcal{F}_2$. That $\{\mathcal{F}_1\}$ and $\{\mathcal{F}_2\}$ are two disjoint closed set of $\beta_\omega X$ is immediate. Set $\mathcal{V}_1$ the collection of open sets V of X such that $\mathcal{F}_1 \notin V^*_\omega$ and $\mathcal{V}_2$ the collection of open sets V of X such that $\mathcal{F}_2 \notin V^*_\omega$. Then $V_1 = \cup(V : V \in \mathcal{V}_1)$ and $U_2 = \cup(U : U \in \mathcal{U}_2)$ are two distinct open sets of X such that $\mathcal{O} \cup \{U_1, U_2\}$ is a g-covering of X but neither $\mathcal{O} \cup \{V_1\}$ nor $\mathcal{O} \cup \{V_2\}$ are g-covering of X contradicting hypotheses. Thus $\beta_\omega X \cap C(\mathcal{O})$ is a singleton. □

Proposition 3.8. *Let X be a non compact B-space. The following statements are equivalent:*

(1) *$\beta_\omega X$ is sober.*
(2) *For each H-irreducible collection $\mathcal{O}$ of open sets of X , one of the following statements hold:*
   (*i*) *$X \neq \cup(O : O \in \mathcal{O})$ and there exists a unique $x \in X \setminus \cup(O : O \in \mathcal{O})$ such that $x \in V$, for each open set V such that $V \cap^c_H \mathcal{O} \neq \emptyset$.*
   (*ii*) *$X = \cup(O : O \in \mathcal{O})$ and for each U_1, U_2 two open sets such that $\mathcal{O} \cup \{U_1, U_2\}$ is a g-covering of X, either $\mathcal{O} \cup \{U_1\}$ or $\mathcal{O} \cup \{U_2\}$ is a g-covering.*

Proof. (1) $\Longrightarrow$ (2) Let $\mathcal{O}$ be a H-irreducible collection of open sets of X. By Proposition 3.6 $C(\mathcal{O})$ is an irreducible closed set of $\beta_\omega X$. Since $\beta_\omega X$ is sober, there exists a unique $x \in \beta_\omega X$ such that $C(\mathcal{O}) = \overline{\{x\}}^{\beta_\omega X}$. We discuss the following two cases:

Case 1: $X \neq \cup(O : O \in \mathcal{O})$. Then $C(\mathcal{O}) \cap X \neq \emptyset$. Since singletons of $\beta_\omega X \setminus X$ are closed sets of $\beta_\omega X$, $x \in X \setminus \cup(O : O \in \mathcal{O})$. Let V be an open set of X such that $V \cap^c_H \mathcal{O} \neq \emptyset$. Then $V^*_\omega \cap C(\mathcal{O}) \neq \emptyset$. Thus $x \in V$.

Now, let $y \in X \setminus \cup(O : O \in \mathcal{O})$ such that $y \in V$, for each open set V such that $V \cap_H^c \mathcal{O} \neq \emptyset$. Then $y \in C(\mathcal{O})$ so that $y \in \overline{\{x\}}^{\beta_\omega X}$. Since each open neighborhood V of x meet $C(\mathcal{O})$, $x \in \overline{\{y\}}^X$. Hence $x = y$.

Case 2: $X = \cup(O : O \in \mathcal{O})$. Then $C(\mathcal{O}) \subseteq \beta_\omega X \setminus X$. Since $C(\mathcal{O}) = \overline{\{x\}}^{\beta_\omega X}$, $C(\mathcal{O}) = \{x\}$ with $x \in \beta_\omega X \setminus X$. By Lemma 3.7, for each U_1, U_2 two open sets such that $\mathcal{O} \cup \{U_1, U_2\}$ is a g-covering of X, either $\mathcal{O} \cup \{U_1\}$ or $\mathcal{O} \cup \{U_2\}$ is a g-covering of X.

(2) $\Longrightarrow$ (1) Let C be an irreducible closed set of $\beta_\omega X$. Then there exists a collection $\mathcal{O}$ of open sets of X such that $C = C(\mathcal{O})$. Hence $\mathcal{O}$ is a H-irreducible of X. Thus one of the following statements hold:

(*i*) $\cup(O : O \in \mathcal{O}) \neq X$ and there exists a unique $x \in X \setminus \cup(O : O \in \mathcal{O})$ such that $x \in V$, for each open set V such that $V \cap_H^c \mathcal{O} \neq \emptyset$. In this case $C(\mathcal{O}) = \overline{\{x\}}^{\beta_\omega X}$. In fact, let $y \in C(\mathcal{O})$ and V be an open neighborhood of y. Hence $V \cap_H^c \mathcal{O} \neq \emptyset$ so $x \in V$. Thus $y \in \overline{\{x\}}^{\beta_\omega X}$ and $C(\mathcal{O}) \subseteq \overline{\{x\}}^{\beta_\omega X}$. Since $x \in C(\mathcal{O})$, $C(\mathcal{O}) = \overline{\{x\}}^{\beta_\omega X}$.

(*ii*) $\cup(O : O \in \mathcal{O}) = X$ and for each U_1, U_2 two open sets such that $\mathcal{O} \cup \{U_1, U_2\}$ is a g-covering of X, either $\mathcal{O} \cup \{U_1\}$ or $\mathcal{O} \cup \{U_2\}$ is a g-covering. By Lemma 2 3.7 there exists a unique $\mathcal{F} \in \beta_\omega X \setminus X$ such that $C(\mathcal{O}) = \{\mathcal{F}\}$.

Therefore $\beta_\omega X$ is sober. □

4. **H-spectral spaces**

The goal of the present section is to give a necessary and sufficient conditions on a T_0-space to be a H-spectral space. First, we need the following lemma:

Lemma 4.1. *Let X be a B-space. The following statements are equivalent:*

1. *The family of compact open sets of $\beta_\omega X$ is closed under finite intersections.*
2. *For each two collections $\mathcal{O}_1$ and $\mathcal{O}_2$ of open sets of X, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are H-compact then there exists a collection $\mathcal{O}_3$ of open sets of X such that $\mathcal{O}_3$ is H-compact and:*
   (*i*) $\cup(O_1 \cap O_2 : O_1 \in \mathcal{O}_1$ *and* $O_2 \in \mathcal{O}_2) = \cup(O : O \in \mathcal{O}_3)$.
   (*ii*) $\{O_1 \cap O_2 \mid O_2 \in \mathcal{O}_1$ *and* $O_2 \in \mathcal{O}_2\}$ *and* $\mathcal{O}_3$ *are* H*-equivalent.*

Proof. (1) $\Longrightarrow$ (2) Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two collections of open sets of X such that $\mathcal{O}_1$ and $\mathcal{O}_2$ are H-compact. By Proposition 2.5 $O(\mathcal{O}_1)$ and $O(\mathcal{O}_2)$ are two compact open sets of $\beta_\omega X$. Then $O(\mathcal{O}_1) \cap O(\mathcal{O}_2)$ is a compact open set of $\beta_\omega X$. Hence there exists a collection $\mathcal{O}_3$ of open sets of X such that $O(\mathcal{O}_1)$ is a compact set of $\beta_\omega X$ and $O(\mathcal{O}_1) \cap O(\mathcal{O}_2) = O(\mathcal{O}_3)$. Since $O(\mathcal{O}_1) \cap O(\mathcal{O}_2) = \cup(O^*_{1\omega} \cap O^*_{2\omega} : O_1 \in \mathcal{O}_1$ and $O_2 \in \mathcal{O}_2)$ and $O^*_{1\omega} \cap O^*_{2\omega} = (O_1 \cap O_2)^*_\omega$, for each $O_1 \in \mathcal{O}_1$ and $O_2 \in \mathcal{O}_2$, $O(\mathcal{O}_3) = \cup((O_1 \cap O_2)^*_\omega : O_1 \in \mathcal{O}_1$ and $O_2 \in \mathcal{O}_2)$. Then $\{O_1 \cap O_2 \mid O_2 \in \mathcal{O}_1$ and $O_2 \in \mathcal{O}_2\}$ and $\mathcal{O}_3$ are H-equivalent, and $\cup(O_1 \cap O_2 : O_1 \in \mathcal{O}_1$ and $O_2 \in \mathcal{O}_2) = \cup(O : O \in \mathcal{O}_3)$.

(2) $\Longrightarrow$ (1) Let U_1 and U_2 be two compact open sets of $\beta_\omega X$. Then there exists two collections $\mathcal{O}_1$ and $\mathcal{O}_2$ of open sets of X such that $O(\mathcal{O}_1) = U_1$ and $O(\mathcal{O}_2) = U_2$. So $\mathcal{O}_1$ and $\mathcal{O}_2$ are H-compact. Hence there exists a collection $\mathcal{O}_3$ of open sets of X such that $\mathcal{O}_3$ is H-compact and:

(i) $\cup(O_1 \cap O_2 : O_1 \in \mathcal{O}_1$ and $O_2 \in \mathcal{O}_2) = \cup(O : O \in \mathcal{O}_3)$.

(ii) $\{O_1 \cap O_2 \mid O_2 \in \mathcal{O}_1$ and $O_2 \in \mathcal{O}_2\}$ and $\mathcal{O}_3$ are H-equivalent.

The condition (i) and (ii) induce that $\{(O_1 \cap O_2)^*_\omega \mid O_2 \in \mathcal{O}_1$ and $O_2 \in \mathcal{O}_2\} = O(\mathcal{O}_3)$. Since, for each $O_1 \in \mathcal{O}_1$ and $O_2 \in \mathcal{O}_2$, $O^*_{1\omega} \cap O^*_{2\omega} = (O_1 \cap O_2)^*_\omega$, $\{O^*_{1\omega} \cap O^*_{2\omega} \mid O_2 \in \mathcal{O}_1$ and $O_2 \in \mathcal{O}_2\} = O(\mathcal{O}_3)$. Hence $\cup(O^*_{1\omega} \cap O^*_{2\omega} : O_1 \in \mathcal{O}_1$ and $O_2 \in \mathcal{H}_2) = O(\mathcal{O}_3)$. Thus $O(\mathcal{O}_3) = O(\mathcal{O}_1) \cap O(\mathcal{O}_2)$ is a compact open set of $\beta_\omega X$. Therefore the family of compact open sets of $\beta_\omega X$ is closed under finite intersections. □

Now, we are in position to give a characterization of H-spectral space of T_0-space. By Proposition 1.11, if X is a $H1$-space then $\beta_\omega X$ coincide with the one point compactification of X. Spaces such their one point compactification is a spectral space are called A-spectral spaces. A complete characterization of A-spectral spaces was given in [4]. Since the collection of T_0-spaces is a disjoint union of $H1$-spaces and B-spaces, to complete characterization of H-spectral spaces of T_0-spaces we give the following proposition:

Proposition 4.2. *Let X be a non compact B-space. The following statements are equivalent:*

(1) *X is a H-spectral space.*
(2) *X has the following properties:*
  (a) *For each open set U of X there exists a family $\{\mathcal{O}_k \mid k \in K\}$ of collections of open sets of X such that:*
    (i) *$\mathcal{O}_k$ is H-compact, for each $k \in K$;*
    (ii) *$U = \cup(O : O \in \cup(\mathcal{O}_k : k \in K))$;*
    (iii) *$\{U\}$ and $\cup(\mathcal{O}_k : k \in K)$ are H-equivalent.*
  (b) *For each two collections $\mathcal{O}_1$ and $\mathcal{O}_2$ of open sets of X, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are H-compact then there exists a collection $\mathcal{O}_3$ of open sets of X such that $\mathcal{O}_3$ is H-compact and:*
    (i) *$\cup(O_1 \cap O_2 : O_1 \in \mathcal{O}_1$ and $O_2 \in \mathcal{O}_2) = \cup(O : O \in \mathcal{O}_3)$.*
    (ii) *$\{O_1 \cap O_2 \mid O_2 \in \mathcal{O}_1$ and $O_2 \in \mathcal{O}_2\}$ and $\mathcal{O}_3$ are H-equivalent.*
  (c) *For each H-irreducible collection $\mathcal{O}$ of open sets of X, one of the following statements hold:*
    (i) *$X \neq \cup(O : O \in \mathcal{O})$ and there exists a unique $x \in X \setminus \cup(O : O \in \mathcal{O})$ such that $x \in V$, for each open set V such that $V \cap^c_H \mathcal{O} \neq \emptyset$.*
    (ii) *$X = \cup(O : O \in \mathcal{O})$ and for each U_1, U_2 two open sets such that $\mathcal{O} \cup \{U_1, U_2\}$ is a g-covering of X, either $\mathcal{O} \cup \{U_1\}$ or $\mathcal{O} \cup \{U_2\}$ is a g-covering.*

Example 4.3. Let $\mathbb{N}$ be the set of positive integers and $\leq$ be the order on X defined by $2n \leq 2n+2$ and $2n+1 \leq 2n+3$, for each $n \in X$.

$$\begin{array}{ccccccccccccc} 0 & \longleftarrow & 2 & \longleftarrow & 4 & \longleftarrow & \cdots & 2n & \longleftarrow & 2n+2 & \longleftarrow & \cdots & \cdots \\ & 1 & \longleftarrow & 3 & \longleftarrow & 5 & \longleftarrow & \cdots & 2n+1 & \longleftarrow & 2n+3 & \longleftarrow & \cdots \end{array}$$

Equip $\mathbb{N}$ with the left topology T_L generated by $\{(\downarrow, n) \mid n \in \mathbb{N}\}$. Then $\mathbb{N}$ is a B-space and $\beta_\omega \mathbb{N}$ is a spectral space.

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