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Nearly $\Omega-$Boundedness in $L-$Topological Space

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Abstract

In this paper, we introduce and study the notion of nearly $\Omega-$boundedness on arbitrary $L-$sets in $L-$topological spaces by using the notion of $\delta-$upper limit of $\Omega-$nets. Several characterizations of nearly $\Omega-$boundedness in terms of convergence theory of constant α-nets, α-ideals are obtained. We prove that the notion is good extension, productive and topologically invariant.

Keywords: Molecules, $R-$neighborhoods, $L-$topological space, $\delta-$limit and $\delta-$cluster points, $\Omega-$nets, constant $\alpha-$nets, $\alpha-$filters, $\alpha-$ideals, nearly $\Omega-$compact and nearly $\Omega-$bounded sets

1. Introduction

Boundedness, as a natural generalization of relative compactness was considered by several authors (see [12] and [13]). In depth analysis of boundedness and its various weaker forms was done by Lamprinos in [13] and [14]. A subset A of a space X is said to be bounded if every open cover of X has a finite subfamily which covers A. In 1997 Georgiou and Papadopoulos [7] introduced the notion of nearly $\Omega-$compact, nearly (α,β)-compact topological and fuzzy topological spaces, nearly $\Omega-$bounded, nearly (α,β)-bounded sets and fuzzy sets. Then he give the characterizations of nearly compact topological and fuzzy topological spaces of weakly $\theta-$upper limit and fuzzy weakly $\theta-$upper limit of nets and fuzzy nets. Finally he give the characterizations of the nearly bounded sets and fuzzy sets of weakly $\theta-$upper limit and fuzzy weakly $\theta-$upper limit of nets and fuzzy nets. In 1997 Georgiou and Papadopoulos [8] gave characterizations of fuzzy nearly compactness by used the notion of fuzzy weakly $\theta-$upper limit of fuzzy nets. Also, he studied new fuzzy compactness and fuzzy boundedness in fuzzy topological spaces. In 2000 Georgiou and Papadopoulos

[10] introduced and studied fuzzy boundedness by used the notion of fuzzy upper limit of fuzzy nets.

Recently, Georgiou and Papadopoulos in [9] and [10] extended be the concept of bounded set to fuzzy topology and introduced the notion of fuzzy boundedness using the fuzzy compactness given by Change [2], which is not good extension of ordinary compactness. Hence the notion of fuzzy boundedness in [9] is not good extension of ordinary bounded and so it is unsatisfactory.

In This paper, we introduce and study the concept of nearly $\Omega-$boundedness on arbitrary $L-$sets in $L-$topological spaces along the line of nearly $\Omega-$ compactness defined by Georgiou and Papadopoulos [9] and remoted neighborhood due to Wang [18]. Then we give new characterizations and properties of nearly $\Omega-$boundedness in terms of convergence theory of constant α-nets, α-filters and α-ideals. We prove that the notion is good extension, productive and topologically invariant.

2. Preliminaries

Through this paper $L = L(\leq,\vee,\wedge,')$ denotes a completely distributive complete lattice with a smallest element 0 and a largest element 1 $(0 \neq 1)$ and with an order reversing involution on it. An $\alpha \in L$ is called a molecule of L if $\alpha \neq 0$ and $\alpha \leq \nu \vee \gamma$ implies $\alpha \leq \nu$ or $\alpha \leq \gamma$ for all $\nu, \gamma \in L$. The set of all molecules of L is denoted by $M(L)$. Let X be a nonempty set. L^X denotes the family of all mappings from X to L. The elements of L^X are called L–subsets on X. L^X can be made into a lattice by inducing the order and involution from L. We denote the smallest element and the largest element of L^X by 0_X and 1_X, respectively. If $\alpha \in L$, then the constant mapping $\underline{\alpha} : X \to \{\alpha\}$ is L–subset [11]. An L-point (or molecule on L^X), denoted by x_α, $\alpha \in M(L)$ is a L-subset which is defined by $x_\alpha(y) = \begin{cases} \alpha \ : \ x = y \\ 0 \ : \ x \neq y \end{cases}$.

The family of all molecules of L^X is denoted by $M(L^X)$ [19]. For $\mu \in L^X$ and $\alpha \in L$ we defined the set $\mu_{w\alpha} = \{x \in X : \mu(x) \geq \alpha\}$, which it is called weak α-cut of μ. The set $\mu_{s\alpha} = \{x \in X : \mu(x) \not\leq \alpha\}$, it is called strong α-cut of μ and $Supp(\mu) = \{x \in X : \mu(x) > 0\}$ is called support of μ [15]. For any $\lambda \in L^X$ and $\alpha \in M(L)$ with $\alpha' \geq \alpha$, we have $(\lambda_{w\alpha})' \subseteq (\lambda')_{w\alpha}$. For $\Psi \subset L^X$, we define $2^{(\Psi)}$ bythe set $\{\omega \subset \Psi : \omega \textit{ is finite subfamily of } \Psi\}$. An L-topology on X is a subfamily τ of L^X closed under arbitrary unions and finite intersections. The pair (L^X, τ) is called an L–topological space (or L–ts, for short) [20]. If (L^X, τ) is an L–ts, then for each $\eta \in L^X$, $cl(\eta)$, $\mathrm{int}(\eta)$ and η' will denote the closure, interior and complement of η. A mapping $f : L^X \to L^Y$ is said to be an L-valued

Zadeh function induced by a mapping $f: X \to Y$, iff $f(\mu)(y) = \vee\{\mu(x): f(x) = y\}$ for every $\mu \in L^X$ and every $y \in Y$ [19]. An L-ts (L^X, τ) is called fully stratified if for each $\alpha \in L$, $\underline{\alpha} \in \tau$ [15]. If (L^X, τ) is an L-ts, then the family of all crisp open sets in τ is denoted by $[\tau]$ i.e., $(X, [\tau])$ is a crisp topological space [16].

Definition 2.1 [17]: If (L^X, τ) is L-ts, then $\mu \in L^X$ is called regular open set iff $\mu = \text{int}(cl(\mu))$. The family of all regular open sets is denoted by $RO(L^X, \tau)$. The complement of the regular open set is called regular closed set and satisfy $\mu = cl(\text{int}(\mu))$. The family of all regular closed sets is denoted by $RC(L^X, \tau)$.

Definition 2.2 [21]: Let (L^X, τ) be an L-ts and $x_\alpha \in M(L^X)$. Then $\lambda \in \tau'$ is called an remoted neighborhood (R-nbd, for short) of x_α if $x_\alpha \notin \lambda$. The set of all R-nbds of x_α is called remoted neighborhood system and is denoted by R_{x_α}.

Definition 2.3 [21]: Let (L^X, τ) be an L-ts, $\mu \in L^X$ and $\alpha \in M(L)$. Then $\Psi \subset \tau'$ is called an:

(i) α-remoted neighborhood family of μ, briefly α-RF of μ, if for each L-point $x_\alpha \in \mu$ there is $\lambda \in \Psi$ such that $\lambda \in R_{x_\alpha}$.

(ii) $\overline{\alpha}$-remoted neighborhood family of μ, briefly $\overline{\alpha}$-RF of μ, if there exists $\gamma \in \beta^*(\alpha)$ such that Ψ is a γ-RF of μ, where $\beta^*(\alpha) = \beta(\alpha) \cap M(L)$, and $\beta(\alpha)$ denotes the union of all the minimal sets relative to α.

Definition 2.4 [5]: Let (L^X, τ) be an L-ts, $\mu \in L^X$ and $\alpha \in M(L)$. Then $\Psi \subset RC(L^X, \tau)$ is called an α-regular closed remoted neighborhood family of μ, briefly α-RCRF of μ, if for each L-point $x_\alpha \in \mu$ there is $\lambda \in \Psi$ such that $\lambda \in R_{x_\alpha}$.

Definition 2.5 [18]: Let (L^X, τ) be an L-ts, $\mu \in L^X$ and $\alpha \in M(L)$. An α-RF $\Psi = \{\eta_j : j \in J\}$ of μ is called:

(i) Directed if $\eta_1, \eta_2 \in \Psi$ there is $\eta_3 \in \Psi$ such that $\eta_3 \le \eta_1 \wedge \eta_2$.

(ii) Regular if :

(a) For each $j \in J$ there is $\lambda_j \in RO(L^X, \tau) \setminus \{1_X\}$ such that $\eta_j \le \lambda_j$.

(b) The family $\{cl(\lambda_j) : j \in J\}$ is α-RF of μ.

Definition 2.6 [3]: Let $(D, \leq)$ be a directed set. Then the mapping $S : D \to L^X$ and denoted by $S = \{\mu_n : n \in D\}$ is called a net of L-subsets in X. Specially, the mapping $S : D \to M(L^X)$ is said to be a molecular net in L^X. If $\mu \in L^X$ and for each $n \in D, S \in \mu$ then S is called a net in μ.

Remark 2.7 [10]: We denote by Ω a class of directed sets. Let $S = \{\mu_n : n \in D\}$ be a net of L-subsets in L^X. If $D \in \Omega$, then this net is called $\Omega-$net.

Definition 2.8 [21]: Let (L^X, τ) be an L-ts and $S = \{S(n) : n \in D\}$ be a molecular net in L^X. S is called an molecular α-net ($\alpha \in M(L)$), if for each $\gamma \in \beta^*(\alpha)$ there exists $n \in D$ such that $\vee(S(m)) \geq \gamma$ whenever $m \geq n$, where $\vee(S(m))$ is the height of the molecular $S(m)$. If $\vee(S(m)) = \alpha$ for each $m \in D$, then $\{S(m) : m \in D\}$ is called constant molecular α-net.

Definition 2.9 [21]: Let $S = \{S(n) : n \in D\}$ and $T = \{T(m) : m \in E\}$ be a be molecular nets in (L^X, τ). Then T is said to be is a molecular subnet of S if there is a mapping $f : E \to D$ satisfies the following conditions:

(i) $T = S \circ f$

(ii) For each $n \in D$ there is $m \in E$ such that $f(l) \geq n$ for each $l \in E$, $l \geq m$.

Definition 2.10 [3]: Let (L^X, τ) be an L-ts and $\Delta = \{\mu_n : n \in D\}$ be a net of L-subsets in (L^X, τ) and $x_\alpha \in M(L^X)$.

Then:

(i) x_α is called a $\delta-$limit point of Δ, in symbols $\Delta \xrightarrow{\delta} x_\alpha$ if for each $\eta \in R_{x_\alpha}$ there is an $m \in D$ such that $\mu_n \notin cl(\text{int}(\eta))$ for all $n \in D$, $n \geq m$. The union of all $\delta-$limit points of Δ are denoted by $\delta.\underline{\lim}(\Delta)$.

(ii) x_α is called a $\delta-$cluster ($\delta-$adherent) point of Δ, in symbols $\Delta \overset{\delta}{\propto} x_\alpha$ if for each $\eta \in R_{x_\alpha}$ and for each $n \in D$ there is an $m \in D$ such that $m \geq n$ and $\mu_n \notin cl(\text{int}(\eta))$. The union of all $\delta-$cluster points of Δ are denoted by $\delta.\overline{\lim}(\Delta)$.

If $\delta.\underline{\lim}(\Delta) = \delta.\overline{\lim}(\Delta) = \mu$, then we say that μ is $\delta-$limit of Δ, or we say that Δ $\delta-$converges to μ, in symbol $\delta.\lim(\Delta) = \mu$.

The $\delta-$limit and $\delta-$cluster points of a molecular net are defined similarly in [21].

Definition 2.11 [10]: Let (L^X, τ) be an L-ts, $\mu \in L^X$. Then μ is called nearly $\Omega-$compact (or $N..\Omega-$compact) set in (L^X, τ) if every Ω-net $\{\eta_n : n \in D\}$ of closed L-subsets in L^X such that $\delta.\overline{\lim}(\eta_n)(x) < \alpha$ for each $x \in X$ there exists $n_\circ \in D$ for which $\eta_n = 0_X$, for every $n \in D$, $n \geq n_\circ$.

Definition 2.12 [6]: An L-ts (L^X, τ) is said to be :

(i) LR_2-space (regular space) iff for all $\alpha \in M(L)$, $x \in X$ and for each $\lambda \in R_{x_\alpha}$ there is $\eta \in R_{x_\alpha}$, $\rho \in \tau'$ such that $\eta \vee \rho = 1_X$ and $\lambda \wedge \rho = 0_X$.

(ii) LSR_2-space (semi regular space) iff for all $x_\alpha \in M(L^X)$ and for each $\lambda \in R_{x_\alpha}$ there is $\eta \in R_{x_\alpha}$ such that $\lambda \leq cl(\text{int}(\eta))$.

Theorem 2.13 [17]: If (L^X, τ) is LR_2-space, then it is LSR_2-space

Definition 2.14 [23]: The nonempty family $\mathcal{F} \subset L^X$ is called an L-filter if the following conditions satisfies, for each $\mu_1, \mu_2 \in L^X$

(i) $0_X \notin \mathcal{F}$

(ii) If $\mu_1 \leq \mu_2$ and $\mu_1 \in \mathcal{F}$, then $\mu_2 \in \mathcal{F}$.

(iii) If $\mu_1, \mu_2 \in \mathcal{F}$, then $\mu_1 \wedge \mu_2 \in \mathcal{F}$.

Definition 2.15 [23]: A filter $\mathcal{F}$ in L^X is called an α-filter ($\alpha \in M(L)$), if for every $\lambda \in \mathcal{F}$, $\bigvee_{x \in X} \lambda(x) \geq \alpha$.

Definition 2.16 [23]: Let (L^X, τ) be an L-ts and $\mathcal{F}$ be an L- filter in L^X. Then $x_\alpha \in M(L^X)$ is called $\delta-$cluster point of $\mathcal{F}$, in symbol $\mathcal{F} \propto^\delta x_\alpha$ if for each $\lambda \in \mathcal{F}$ and each $\mu \in R_{x_\alpha}$, $\lambda \not\leq cl(\text{int}(\mu))$. The union of all $\delta-$cluster points of $\mathcal{F}$ are denoted by $\delta.adh(\mathcal{F})$

Definition 2.17 [22]: The nonempty family $I \subset L^X$ is called an ideal if the following conditions satisfies, for each $\mu_1, \mu_2 \in L^X$

(i) $1_X \notin I$

(ii) If $\mu_1 \leq \mu_2$ and $\mu_2 \in I$, then $\mu_1 \in I$.

(iii) If $\mu_1, \mu_2 \in I$, then $\mu_1 \vee \mu_2 \in I$.

Definition 2.18 [22]: Let I be an ideal in an L-ts (L^X, τ) and $\alpha \in M(L)$. Then I is said to be an α-ideal, if $\vee_{x \in X} \eta(x) < \alpha$ for each $\eta \in I$.

Definition 2.19 [22] : Let I be an ideal in an L-ts (L^X, τ) and $\alpha \in M(L)$. Then $x_\alpha \in M(L^X)$ is called $\delta-$cluster point of I, in symbol $I \propto^\delta x_\alpha$ if for each $\lambda \in I$ and each $\mu \in R_{x_\alpha}$, $\lambda \vee cl(\text{int}(\mu)) \neq 1_X$. The union of all $\delta-$cluster points of I are denoted by $\delta.adh(I)$.

Theorem 2.20 [4]: Let $\mathcal{F}$ be an L-filter in an L-ts (L^X, τ) and $S(\mathcal{F})$ be the L-molecular net induced by $\mathcal{F}$. Then $\delta.adh(\mathcal{F}) = \delta.adh(S(\mathcal{F}))$.

Theorem 2.21 [4]: Suppose that S is an L-net in an L-ts (L^X, τ) and $\mathcal{F}(S)$ is the L-filter induced by S. Then $\delta.adh(S) = \delta.adh(\mathcal{F}(S))$.

Theorem 2.22 [4]: Suppose that I is an ideal in an L-ts (L^X, τ) and $S(I)$ is the L-molecular net induced by I. Then $\delta.adh(I) = \delta.adh(S(I))$).

3. Nearly $\Omega-$Boundedness in L-topological spaces

In this section, we introduce the concept of nearly Ω-bounded sets in L-topological spaces. Then we obtain several characterizations of nearly Ω-bounded sets.

Definition 3.1. Let (L^X, τ) be an L-ts, $\mu \in L^X$ and $\alpha \in M(L)$, then $\mu \in L^X$ is called Nearly Ω-bounded (or, $N..\Omega-$bounded) set in (L^X, τ) if every Ω-net $\{\rho_n : n \in D\}$ of closed L-subsets in L^X such that $\delta.\overline{\lim}(\rho_n)(x) < \alpha$ for each $x \in X$ there exists $n_\circ \in D$ for which $\rho_n \wedge \mu = 0_X$, for every $n \in D$, $n \geq n_\circ$.

Remark 3.2. We note that every Ω-net of L-subsets of X is a net of L-subsets of X. But if Ω is the class of all directed sets, then Ω-net and a net of L-subsets of X are equivalent.

The following example show that the converse is not true in general.

Example 3.3. Let $L = X = [0,1]$ and let $\tau = \{0_X, 1_X, \mu\}$, where $\mu \in L^X$ such that $\mu(x) = \begin{cases} \frac{1}{3} : x = 0 \\ 0 : x \neq 0 \end{cases}$

Clearly, the pair (L^X,τ) is L-ts. Let $S=\{\mu_n : n\in N\}$ such that $\mu_n = x_{.25}$ for every $n\in N$. Then S is a net of L-subsets of X. But S is not Ω-net, since $D=N$ is not class of all directed sets.

Theorem 3.4. Let (L^X,τ) be an L-ts. If $\mu\in L^X$ is Ω-bounded [1], then μ is $N..\Omega-$bounded set.

Proof. Let $\mu\in L^X$ be an Ω-bounded set and let $\{\rho_n : n\in D\}$ be an Ω-net of closed L-subsets in L^X such that $\delta.\overline{\lim}(\rho_n)(x)<\alpha$ for each $x\in X$, then for each $x\in X$ there is $\eta\in R_{x_\alpha}$ and there is $m\in D$ such that $\rho_n\le cl(\operatorname{int}(\eta))$ for all $n\ge m$, since $cl(\operatorname{int}(\eta))\le\eta$, then $\rho_n\le\eta$. Since μ is Ω-bounded set, then there exists $n_\circ\in D$ for which $\rho_n\wedge\mu=0_X$, for every $n\in D$, $n\ge n_\circ$. Hence μ is $N..\Omega-$bounded set.

Example 3.5. Let $L=[0,1]$, $X=N$ and let $\tau=\{0_X, x_{.5}, 1_X\}$. Then (L^X,τ) is L-ts. Let $S=\{x_{.5} : x\in X\}$ be an $\Omega-$net, then 1_X is not $\Omega-$bounded set. Now, let $S=\{x_\alpha : x\in X, \alpha\in L\}$ be an $\Omega-$net, then 1_X is $N..\Omega-$bounded set.

Definition 3.6. Let (L^X,τ) be an L-ts and $x_\alpha\in M(L^X)$. If $\mu\in L^X$ is closed and $N..\Omega-$bounded set, then μ is called $N\Omega B-$remoted neighborhood of x_α ($N\Omega BR-$nbd, for short) of x_α if $x_\alpha\notin\mu$. The set of all $N\Omega BR-$nbds of x_α is denoted by $N\Omega BR_{x_\alpha}$.

We note that $\Omega BR_{x_\alpha}\subseteq N\Omega BR_{x_\alpha}\subseteq R_{x_\alpha}$.

Example 3.7. By Example 3.5., let $S=\{x_{.3} : x\in X\}$ be an $\Omega-$net, then $\mu=x_{.4}$ is not $\Omega-$bounded set. Now, let $S=\{x_\alpha : x\in X, \alpha\le .4\}$ be an $\Omega-$net, then μ is $N..\Omega-$bounded set.

Definition 3.8. Let (L^X,τ) be an L-ts and $\mu\in L^X$. Then $x_\alpha\in M(L^X)$ is called $N\Omega$-bounded adherent point of μ and write $x_\alpha\in N\Omega.cl(\mu)$ iff $\mu\not\le\lambda$ for each $\lambda\in N\Omega BR_{x_\alpha}$. If $\mu=N\Omega.cl(\mu)$, then μ is called $N\Omega$-closed $L-$subset. The family of all $N\Omega$-closed $L-$subsets is denoted by $N\Omega C(L^X,\tau)$ and its complement is called the family of all $N\Omega$-open $L-$subsets and denoted by $N\Omega O(L^X,\tau)$.

Theorem 3.9. Let (L^X,τ) be an L-ts and let $\mu\in L^X$. Then the following statements are true::

(i) $\mu\le cl(\mu)\le N\Omega.cl(\mu)$ and $N\Omega.cl(\mu)\le \Omega.cl(\mu)$.

(ii) If $\eta\in L^X$ and $\mu\le\eta$ then $N\Omega.cl(\mu)\le N\Omega.cl(\eta)$.

(iii) $N\Omega.cl(N\Omega.cl(\mu))=N\Omega.cl(\mu)$.

(iv) $N\Omega.cl(\mu)=\wedge\{\eta\in L^X : \eta\in N\Omega C.(L^X,\tau),\mu\le\eta\}$.

Proof. (i) Let $x_\alpha\in M(L^X)$ such that $x_\alpha\notin N\Omega.cl(\mu)$, then there exists $\lambda\in N\Omega BR_{x_\alpha}$ such that $\mu\le\lambda$. Since $N\Omega BR_{x_\alpha}\subseteq R_{x_\alpha}$ and so $\lambda\in R_{x_\alpha}$ and hence $x_\alpha\notin cl(\mu)$. Thus $cl(\mu)\le N\Omega.cl(\mu)$.

(ii) Let $x_\alpha\in M(L^X)$ such that $x_\alpha\notin N\Omega.cl(\eta)$, then there exists $\lambda\in N\Omega BR_{x_\alpha}$ such that $\eta\le\lambda$. Since $\mu\le\eta$, then $\mu\le\lambda$ and so $x_\alpha\notin N\Omega.cl(\mu)$. Thus $N\Omega.cl(\mu)\le N\Omega.cl(\eta)$.

(iii) Suppose $x_\alpha\in M(L^X)$ such that $x_\alpha\in N\Omega.cl(N\Omega.cl(\mu))$. According to Definition 3.8. we have $N\Omega.cl(\mu)\not\le\lambda$ for each $\lambda\in N\Omega BR_{x_\alpha}$. Hence, there exists $y_\gamma\in M(L^X)$ such that $y_\gamma\in N\Omega.cl(\mu)$ with $y_\gamma\notin\lambda$ and so $\mu\not\le\lambda$, that is, $x_\alpha\in N\Omega.cl(\mu)$. This shows that $N\Omega.cl(N\Omega.cl(\mu))\le N\Omega.cl(\mu)$. On other hand, $\mu\le N\Omega.cl(\mu)$ follows from (i) and so $N\Omega.cl(\mu)\le N\Omega.cl(N\Omega.cl(\mu))$. Therefore, $N\Omega.cl(N\Omega.cl(\mu))=N\Omega.cl(\mu)$.

(iv) On account of (i) and (iii). $N\Omega.cl(\mu)$ is an $N\Omega$-closed set containing μ, and so $N\Omega.cl(\mu)\ge\wedge\{\eta\in L^X : \eta\in N\Omega C.(L^X,\tau),\mu\le\eta\}$. Conversely, in case $x_\alpha\in M(L^X)$ sand $x_\alpha\in N\Omega.cl(\mu)$, then $\mu\not\le\lambda$ for each $\lambda\in N\Omega BR_{x_\alpha}$. Hence, if η is an $N\Omega$-closed set containing μ, then $\eta\not\le\lambda$, and then $x_\alpha\in N\Omega.cl(\eta)=\eta$. This implies that $N\Omega.cl(\mu)\le\wedge\{\eta\in L^X : \eta\in N\Omega C.(L^X,\tau),\mu\le\eta\}$. Hence

$$N\Omega.cl(\mu)=\wedge\{\eta\in L^X : \eta\in N\Omega C.(L^X,\tau),\mu\le\eta\}$$

From Theorem 3.9., one can see that every $N\Omega$-closed $L-$subset is closed $L-$ subset, but the inverse is not true since every closed $L-$subset is not $N\Omega$-bounded set in general as the following example shows.

Example 3.10. By Example 3.5., let $\mu=x_{.5}$, then μ is closed $L-$subset because $\mu\in\tau'$ where $\tau'=\{0_X,x_{.5},1_X\}$. Now, μ is not $N..\Omega-$bounded. In fact, we suppose that $\Omega-$net $S=\{x_{.5}:x\in X\}$, then μ is not $N..\Omega-$bounded set.

Definition 3.11. Let (L^X,τ) be an L-ts and S be a molecular net in L^X. Then $x_\alpha\in M(L^X)$ is called $N\Omega$-bounded limit point of S, (or S $N\Omega$-converges to x_α) in symbol $S\xrightarrow{N\Omega}x_\alpha$ if for every $\mu\in N\Omega BR_{x_\alpha}$ there is an $n\in D$ such that

$m\in D$ and $m\geq n$ we have $S(m)\not\in\mu$. The union of all $N\Omega$-bounded limit points of S is denoted by $N\Omega.\lim(S)$.

Definition 3.12. Let (L^X,τ) be an L-ts and S be a molecular net in L^X. Then $x_\alpha\in M(L^X)$ is called $N\Omega$-bounded cluster point of S, in symbol $S\propto^{N\Omega}x_\alpha$ if for every $\mu\in N\Omega BR_{x_\alpha}$ and every $n\in D$ there is an $m\in D$ such that $m\geq n$ and $S(m)\not\in\mu$. The union of all $N\Omega$-bounded cluster points of S is denoted by $N\Omega.adh(S)$.

Theorem 3.13. (The goodness of $N\Omega$-bounded**ness)** Let $(L^{X_i},\omega_L(T))$ be the induced L-ts by the ordinary space (X,T), $\alpha\in M(L)$ and $\mu\in L^X$. Then μ is $N..\Omega-$bounded in $(L^{X_i},\omega_L(T))$ iff $\mu_{\omega\alpha}=\{x\in X:\mu(x)\geq\alpha\}$ is $N..\Omega-$bounded in (X,T).

Proof. Let $\mu\in L^X$ be an $N..\Omega-$bounded and $\{k_n:n\in D\}$ be an Ω-net of closed subsets in X such that $\delta.\overline{\lim}(k_n)(x)=\phi$ for each $x\in X$. Then the family $\{1_{k_n}:n\in D\}$ is Ω-net of closed L-subsets in L^X such that $\delta.\overline{\lim}(1_{k_n})(x)=0_X<\alpha$ for each $x\in X$, since μ is $N..\Omega-$bounded, there is $n_\circ\in D$ for which $1_{k_n}\wedge\mu=0_X$ for every $n\in D, n\geq n_\circ$ and so $(1_{k_n}\wedge\mu)_{\omega\alpha}=(1_{k_n})_{\omega\alpha}\cap\mu_{\omega\alpha}=k_n\cap\mu_{\omega\alpha}=\phi$ for every $n\in D$, $n\geq n_\circ$. Thus $\mu_{\omega\alpha}$ is $N..\Omega-$bounded in (X,T).

Conversely, suppose that $\mu_{\omega\alpha}$ is $N..\Omega-$bounded for any $\alpha\in M(L)$ and $\{\rho_n:n\in D\}$ is an Ω-net of closed L-subsets in L^X such that $\delta.\overline{\lim}(\rho_n)(x)<\alpha$ for each $x\in X$. Then the family $\{(\rho_n)_{\omega\alpha}:n\in D\}$ is Ω-net of closed subsets in X such that $\delta.\overline{\lim}(\rho_n)_{\omega\alpha}=\phi$ for each $x\in X$. Since $\mu_{\omega\alpha}$ is $N\Omega$-bounded for any $\alpha\in M(L)$ there is $n_\circ\in D$ for which $(\rho_n)_{\omega\alpha}\cap\mu_{\omega\alpha}=\phi$ for every $n\in D$, $n\geq n_\circ$ therefore $\rho_n\wedge\mu=0_X$ for every $n\in D$, $n\geq n_\circ$. Thus μ is $N..\Omega-$bounded in $(L^{X_i},\omega_L(T))$.

Theorem 3.14. Let (L^X,τ) be an L-ts and let $\mu\in L^X$. Then:

(i) If μ is $N..\Omega-$compact, then μ is $N..\Omega-$bounded.

(ii) If μ is closed and $N..\Omega-$bounded, then μ is $N..\Omega-$compact.

(iii) If η is $N..\Omega-$bounded and $\mu\leq\eta$, then μ is $N..\Omega-$bounded.

(iv) If η is $N..\Omega-$compact and $\mu\leq\eta$, then μ is $N..\Omega-$bounded.

(v) If $\mu_1,\mu_2,...,\mu_m$ are $N..\Omega-$bounded sets, then $\bigvee_{i=1}^{m}\mu_i$ is $N..\Omega-$bounded.

Proof. (i) Let μ be an $N..\Omega-$compact and let $\{\rho_n : n \in D\}$ be an Ω-net of closed L-subsets such that $\delta.\overline{\lim}(\rho_n)(x) < \alpha$ for each $x \in X$. Then for every $x_\alpha \in M(L^X)$ there exists $\lambda_{x_\alpha} \in R_{x_\alpha}$ and an element $n_{x_\alpha} \in D$ such that $\rho_n \leq cl(\text{int}(\lambda_{e_\alpha}))$ for each $n \in D$, $n \geq n_{x_\alpha}$. Clearly, the family $\Psi = \{cl(\text{int}(\lambda_{x_\alpha})) : x_\alpha \in M(L^X)\}$ is an α-RF of μ. Since μ is $N..\Omega-$compact, then there exists $n_\circ \in D$ for which $\rho_n = 0_X$ for every $n \in D$, $n \geq n_\circ$. Thus $cl(\text{int}(\lambda_{e_\alpha})) \wedge \mu = 0_X$ and hence $\rho_n \wedge \mu = 0_X$ for every $n \in D$, $n \geq n_\circ$. Hence μ is $N..\Omega-$bounded.

(ii) Let μ be an $N..\Omega-$bounded and let $\{\rho_n : n \in D\}$ be an Ω-net of closed L-subsets such that $\delta.\overline{\lim}(\rho_n)(x) < \alpha$ for each $x \in X$. Since μ is $N..\Omega-$bounded, then there is $n_\circ \in D$ such that $\rho_n \wedge \mu = 0_X$ for every $n \in D$, $n \geq n_\circ$. Since μ is closed, then $\rho_n \wedge \mu = \mu = 0_X$ and so $\rho_n = 0_X$. Hence μ is $N..\Omega-$compact.

(iii) Let η be an $N..\Omega-$bounded and let $\{\rho_n : n \in D\}$ be an Ω-net of closed L-subsets such that $\delta.\overline{\lim}(\rho_n)(x) < \alpha$ for each $x \in X$. Since η is $N..\Omega-$bounded, then there is $n_\circ \in D$ such that $\rho_n \wedge \eta = 0_X$ for every $n \in D$, $n \geq n_\circ$. Since $\mu \leq \eta$,then $\rho_n \wedge \mu = 0_X$ for every $n \in D$, $n \geq n_\circ$. Hence μ is $N..\Omega-$bounded.

(iv) Let η be an $N..\Omega-$compact and let $\{\rho_n : n \in D\}$ be an Ω-net of closed L-subsets such that $\delta.\overline{\lim}(\rho_n)(x) < \alpha$ for each $x \in X$. Since η is $N..\Omega-$compact, then there is $n_\circ \in D$ such that $\rho_n = 0_X$ for every $n \in D$, $n \geq n_\circ$ and so $\rho_n \wedge \eta = 0_X$. Since $\mu \leq \eta$,then $\rho_n \wedge \mu = 0_X$ for every $n \in D$, $n \geq n_\circ$. Hence μ is $N..\Omega-$bounded.

(v) Let $\mu_1, \mu_2, ..., \mu_n$ be an $N..\Omega-$bounded sets and let $\{\rho_n : n \in D\}$ be an Ω-net of closed L-subsets such that $\delta.\overline{\lim}(\rho_n)(x) < \alpha$ for each $x \in X$. Since μ_i is $N..\Omega-$bounded for each $i = 1,2,...,m$, then there exists $n_\circ \in D$ such that $\rho_n \wedge \mu_1 = 0_X$, $\rho_n \wedge \mu_2 = 0_X, \ldots, \rho_n \wedge \mu_m = 0_X$ for every $n \in D$, $n \geq n_\circ$. Thus there is $n_\circ \in D$ such that $\rho_n \wedge (\bigvee_{i=1}^{m} \mu_i) = 0_X$ for every $n \in D$, $n \geq n_\circ$. Hence $\bigvee_{i=1}^{m} \mu_i$ is $N..\Omega-$bounded.

Example 3.15. Let $X = \{x\}$, $L = [0,1]$ and let $\tau = \{0_X, 1_X, x_{\frac{1}{4}}, x_{\frac{8}{9}}\}$. Then (L^X, τ) is L-ts , we suppose that $S = \{x_\alpha : x \in X,\ \alpha \leq \frac{1}{2}\}$ is Ω-net, then by remark 3.2, we have $\mu = x_{\frac{1}{2}} \in M(L^X)$ is $N..\Omega-$bounded set but is not $N..\Omega-$compact set.

Theorem 3.16. Let (L^X, τ) be an L-ts, $\phi \neq Y \subseteq X$ and $\mu \in L^X$. If μ is $N..\Omega-$bounded set in (L^X, τ), then μ is $N..\Omega-$bounded set in (L^Y, τ_Y).

Proof. Let μ be an $N..\Omega-$bounded set in (L^X, τ) and $\phi \neq Y \subseteq X$. Let $\{\rho_n : n \in D\}$ be an Ω-net of closed L-subsets in Y such that $\delta.\overline{\lim}(\rho_n)(y) < \alpha$, for each $y \in Y$. Then $\{\rho_n = \eta_n \wedge Y : n \in D,\ \eta_n\ is\, closed\ L-subsets\, in\, X\}$. Hence $\{\eta_n : n \in D\}$ is an Ω-net of closed L-subsets in X such that $\delta.\overline{\lim}(\eta_n)(x) < \alpha$, for each $x \in X$. Since μ is $N..\Omega-$bounded set in (L^X, τ), then there exists $n_\circ \in D$ for which $\eta_n \wedge \mu = 0_X$, for every $n \in D$, $n \geq n_\circ$ and so $(\eta_n \wedge Y) \wedge \mu = 0_Y$. Hence $\rho_n \wedge \mu = 0_Y$ and so μ is $N..\Omega-$bounded set in (L^Y, τ_Y).

Theorem 3.17. Let $\{\mu_n : n \in D\}$ be a net of closed L-subsets in L^X such that $\mu_{n_1} \leq \mu_{n_2}$ then $\delta.\overline{\lim}(\mu_n) \leq \wedge\{\mu_n : n \in D\}$ iff $n_2 \leq n_1$.

Proof. Let $x_\alpha \in \delta.\overline{\lim}(\mu_n)$ and let $x_\alpha \notin \wedge\{\mu_n : n \in D\}$. Hence there exists $n_\circ \in D$ such that $x_\alpha \notin \mu_{n_\circ}$. Let $\rho = \mu_{n_\circ}$, then $\rho \in R_{x_\alpha}$. Since $x_\alpha \in \delta.\overline{\lim}(\mu_n)$ and $\rho \in R_{x_\alpha}$, there is $n \in D$, $n \geq n_\circ$ such that $\mu_n \leq \mu_{n_\circ} = \rho$, i.e., $\mu_n \leq cl(\mathrm{int}(\rho))$. This contradicts the hypothesis that $x_\alpha \in \delta.\overline{\lim}(\mu_n)$. Thus $x_\alpha \in \wedge\{\mu_n : n \in D\}$.

Theorem 3.18. Let (L^X, τ) be an L-ts and $\mu \in L^X$. Then μ is $N..\Omega-$bounded set iff for each α-RF Ψ of 1_X, there exists $\Psi_\circ \in 2^{(\Psi)}$ such that $\Psi_\circ$ is an α-RCRF of μ.

Proof. Let $\mu \in L^X$ be an $N..\Omega-$bounded set and let $\Psi \subseteq \tau'$ be an α-RF of 1_X. Let $D = 2^{(\Psi)}$ be the set of all finite subsets of Ψ directed by inclusion and let $\{\eta_\Psi : \Psi \in D\}$ be a net of closed L-subsets in L^X such that $\eta_\Psi = \wedge\{cl(\mathrm{int}(\rho)) : \rho \in \Psi\}$ Obviously, $\eta_{\Psi_1} \leq \eta_{\Psi_2}$ iff $\Psi_2 \subseteq \Psi_1$. Hence by Theorem 3.17. it follows that $\delta.\overline{\lim}(\eta_\Psi) \leq \wedge\{\eta_\Psi : \Psi \in D\}$.
So $(\wedge\{\eta_\Psi : \Psi \in D\})(x) = \wedge(\wedge\{cl(\mathrm{int}(\rho)) : \rho \in \Psi\})(x) < \alpha$ for each $x \in X$. Thus $\delta.\overline{\lim}(\eta_\Psi) < \alpha$ for each $x \in X$. Since μ is $N..\Omega-$bounded set, then there exists an element $\Psi_\circ \in D$ for which $\eta_\Psi \wedge \mu = 0_X$ for every $\Psi \in D$, $\Psi \geq \Psi_\circ$. By the

above we have $\eta_{\Psi_\circ} \wedge \mu = 0_X$ and so for each $x_\alpha \in \mu$ there is $cl(\text{int}(\rho)) \in \Psi_\circ$ such that $cl(\text{int}(\rho)) \in R_{x_\alpha}$. Thus $\Psi_\circ \in 2^{(\Psi)}$ is an α-RCRF of μ.

Conversely, suppose that $\mu \in L^X$ satisfies the condition. We prove that μ is $N..\Omega-$bounded set. Let $\{\eta_n : n \in D\}$ be a net of closed L-subsets in L^X such that $\delta.\overline{\lim}(\eta_n)(x) < \alpha$, for each $x \in X$. Then for every molecular $x_\alpha \in M(L^X)$ there exists $\rho_x \in R_{x_\alpha}$ and $n_x \in D$ such that $\eta_n \le cl(\text{int}(\rho_x))$ for every $n \in D$, $n \ge n_x$. Since $\rho_x \in R_{x_\alpha}$ for every $x \in X$, then the family $\Psi = \{\rho_x : x \in X \text{ and } \alpha \in M(L)\}$ is an α-RF of 1_X. By assumption there exist $\Psi_\circ = \{cl(\text{int}(\rho_{x^i})) : i = 1,2,..., k\} \in 2^{(\Psi)}$ such that $\Psi_\circ$ is an α-RCRF of μ. Put $\rho = \overset{k}{\underset{i=1}{\wedge}} \rho_{x^i}$, then $cl(\text{int}(\rho)) \in R_{x_\alpha}$. Since D is a directed set, there is $n_\circ \in D$ such that $n_\circ \ge n_{x^i}$ for every $i = 1,2,..., k$. Then for every $n \in D$, $n \ge n_\circ$ we have $\eta_n \le cl(\text{int}(\rho))$, whenever $n \ge n_\circ$. Since $cl(\text{int}(\rho)) \wedge \mu = 0_X$, then $\eta_n \wedge \mu = 0_X$ for every $n \in D$, $n \ge n_\circ$. Thus μ is $N..\Omega-$bounded set.

Corollary 3.19. An L-ts (L^X, τ) is $N..\Omega-$compact iff for each α-RF Ψ of 1_X, there exists $\Psi_\circ \in 2^{(\Psi)}$ such that $\Psi_\circ$ is an α-RCRF of 1_X.

Proof. Let (L^X, τ) be an $N..\Omega-$compact. Then 1_X is $N..\Omega-$compact. let $\Psi \subseteq \tau'$ be an α-RF of 1_X. Let $D = 2^{(\Psi)}$ be the set of all finite subsets of Ψ directed by inclusion and let $\{\mu_\Psi : \Psi \in D\}$ be a net of closed L-subsets in L^X such that $\mu_\Psi = \wedge\{cl(\text{int}(\rho)) : \rho \in \Psi\}$. Obviously, $\mu_{\Psi_1} \le \mu_{\Psi_2}$ iff $\Psi_2 \subseteq \Psi_1$. Hence by Theorem 3.17. it follows that $\delta.\overline{\lim}(\mu_\Psi) \le \wedge\{\mu_\Psi : \Psi \in D\}$. Hence $(\wedge\{\mu_\Psi : \Psi \in D\})(x) = \wedge(\wedge\{cl(\text{int}(\rho)) : \rho \in \Psi\})(x) < \alpha$ for all $x \in X$. Thus $\delta.\overline{\lim}(\mu_\Psi) < \alpha$ for all $x \in X$. Since 1_X is an $N..\Omega-$compact, then there exists an element $\Psi_\circ \in D$ for which $\mu_\Psi = 0_X$ for every $\Psi \in D$, $\Psi \ge \Psi_\circ$. By the above we have $\mu_{\Psi_\circ} = 0_X$ and so $x_\alpha \notin \mu_{\Psi_\circ} = \wedge\{cl(\text{int}(\rho)) : \rho \in \Psi_\circ\}$ for each $x_\alpha \in M(L^X)$ and hence $\Psi_\circ \in 2^{(\Psi)}$ is an α-RCRF of 1_X.

Conversely, suppose that 1_X satisfies the condition. We prove that 1_X is $N..\Omega-$compact. Let $\{\mu_n : n \in D\}$ be a net of closed L-subsets in L^X such that $\delta.\overline{\lim}(\eta_n)(x) < \alpha$, for each $x \in X$. Then for every molecular $x_\alpha \in M(L^X)$ there exists $\rho_x \in R_{x_\alpha}$ and $n_x \in D$ such that $\mu_n \le cl(\text{int}(\rho_x))$ for every $n \in D$, $n \ge n_x$. Since $\rho_x \in R_{x_\alpha}$ for every $x \in X$, then the family

$\Psi=\{\rho_x : x\in X \ and\ \alpha\in M(L)\}$ is an α-RF of1_X. By assumption there exist $\Psi_\circ=\{cl(\text{int}(\rho_{x^i})):i=1,2,...,k\}\in 2^{(\Psi)}$ such that $\Psi_\circ$ is an α-RCRF of 1_X. Then $(\forall x\in X)\ (\exists cl(\text{int}(\rho_{x^i}))\in\Psi_\circ, i\le k)\ (cl(\text{int}(\rho_{x^i}))\in R_{x_\alpha})$. Put $\rho=\overset{k}{\underset{i=1}{\wedge}}\rho_{x^i}$, then $cl(\text{int}(\rho))\in R_{x_\alpha}$. Since D is a directed set, there is $n_\circ\in D$ such that $n_\circ\ge n_{x^i}$ for every $i=1,2,...,k$. Hence for every $n\in D$, $n\ge n_\circ$ we have $\mu_n\le cl(\text{int}(\overset{k}{\underset{i=1}{\wedge}}\rho_{x^i}))$ and so $\mu_n\le cl(\text{int}(\rho))$, whenever $n\ge n_\circ$. Since $cl(\text{int}(\rho))=0_X$, then $\mu_n=0_X$ for every $n\in D$, $n\ge n_\circ$.Hence 1_X is $N..\Omega-$compact.

Theorem 3.20. Let (L^X,τ) be an L-ts and $\mu\in L^X$ is $N..\Omega-$bounded set. Then μ is $\Omega-$ bounded set if (L^X,τ) is LR_2-space.

Proof. Let $\mu\in L^X$ be an $N..\Omega-$bounded and let and let $\Psi=\{\rho_j : j\in J\}\subseteq\tau'$ be an α-RF of1_X. Then for each $x_\alpha\in M(L^X)$ there is $\rho\in\Psi$ such that $\rho\in R_{x_\alpha}$. Since (L^X,τ) is LR_2-space, then by Theorem 2.13 we have $\tau'=RC(L^X,\tau)$. Since μ is $N..\Omega-$bounded set, then there is $\Psi_\circ\in 2^{(\Psi)}$ such that $\Psi_\circ$ is an α-RCRF of μ, since $\tau'=RC(L^X,\tau)$, then $\Psi_\circ$ is an α-RF of μ. This shows that μ is $\Omega-$ bounded set

4. α-Nets characterizations of $N.\Omega-$Boundedness

In this section we give several characterizations of $N.\Omega-$Boundedness in terms of both $\delta-$upper limit of Ω-nets of L-subsets and $\delta-$cluster points of constant molecular α-nets.

Theorem 4.1. Let (L^X,τ) be an L-ts , $\alpha\in M(L)$ and $\mu\in L^X$. Then μ is $N.\Omega-$bounded iff for each constant molecular α-net S contained in μ has $\delta-$cluster point in X with height α.

Proof. Suppose that μ is $N.\Omega-$bounded, $\alpha\in M(L)$ and $S=\{S(n):n\in D\}$is a constant molecular α-net in μ. If S does not have any $\delta-$cluster point in X with height α. Then for all $x_\alpha\in M(L^X)$, x_α is not $\delta-$ cluster point of $S$and so there exists $\lambda_x\in R_{x_\alpha}$ and $n_x\in D$ such that $S(m)\in cl(\text{int}(\lambda_x))$ for every $m\in D$ and$m\ge n_x$. Put $\Psi=\{cl(\text{int}(\lambda_x)):x\in X\ and\ \alpha\in M(L)\}$, then Ψ is an α-RF of1_X. Sinceμ is $N.\Omega-$bounded, then by Theorem 3.18. there exist

$\Psi_\circ = \{cl(\text{int}(\lambda_{x^i})) : i = 1,2,...,k\} \in 2^{(\Psi)}$ such that $\Psi_\circ$ is an α-RCRF of μ. Hence, for each $i \le k$ we have $n_{x^i} \in D$ when $m \ge n_{x^i}$, $S(m) \in cl(\text{int}(\lambda_{x^i}))$. Since D is a directed set, then there is $n_\circ \in D$ such that $n_\circ \ge n_{x^i}$ $(i = 1,2,...,k)$. Hence $S(m) \in cl(\text{int}(\lambda_{x^1})) \wedge cl(\text{int}(\lambda_{x^2})) \wedge ... \wedge cl(\text{int}(\lambda_{x^k}))$ whenever $m \ge n_\circ$. This means that $S(m)$ not have α-RCRF in $\Psi_\circ$ and so $\Psi_\circ$ is not α-RCRF of μ. This contradicts the hypothesis that μ is $N.\Omega-$bounded. Thus S has a $\delta-$cluster point in X with height α.

Conversely, suppose that μ is not $N.\Omega-$bounded. Then by Theorem 3.18. there exist $\alpha \in M(L)$ and a family Ψ which is an α-RF of 1_X, but for any family $\Psi_\circ \in 2^{(\Psi)}$, we have $\Psi_\circ$ is not α-RCRF of μ. Then there exists molecule $x_\alpha \in \mu$ with height α such that for each $cl(\text{int}(\lambda)) \in \Psi_\circ$, $cl(\text{int}(\lambda)) \in R_{x_\alpha}$ we have $x_\alpha \le \wedge\Psi_\circ$ and x_α is denoted by $(x(\Psi_\circ))_\alpha$. Since $2^{(\Psi)}$ is a directed set with relation $\le$, then $S = \{(x(\Psi_\circ))_\alpha : \Psi_\circ \in 2^{(\Psi)}\}$ is a constant molecular α-net in μ. Take an arbitrary point y_α in X with height α, since Ψ is an α-RF of 1_X, then there is $\lambda \in \Psi$ such that $\lambda \in R_{y_\alpha}$. Hence for each $\Psi_\circ \in 2^{(\Psi)}$ such that $cl(\text{int}(\lambda)) \in \Psi_\circ$ there is $(x(\Psi_\circ))_\alpha \le \wedge\Psi_\circ \le cl(\text{int}(\lambda))$, i.e., $S \le cl(\text{int}(\lambda))$. This shows that y_α is not a $\delta-$cluster point of S, which contradicts to the hypothesis. Thus μ is $N.\Omega-$bounded.

Theorem 4.2. Let $\{(L^{X_i}, \tau_i) : i = 1,2,...,m\}$ be an L-ts's and μ_i be an $N.\Omega-$boundedset in (L^{X_i}, τ_i) for each $i = 1,2,...,m$, then the product set $\mu = \prod_{i=1}^{m} \mu_i$ is an $N.\Omega-$bounded in the product space.

Proof. Let $\mu_i \in L^{X_i}$ be an $N.\Omega-$bounded set for each $i = 1,2,...,m$ and let $\{\rho_n : n \in D\}$ be an Ω-net of closed L-subsets in L^X such that $\delta.\overline{\lim}(\rho_n)(x) < \alpha$ for each $x \in X$. Therefore there is $n_{\circ_1} \in D$ such that $\rho_n \wedge \mu_1 = 0_{X_1}$, for every $n \in D$, $n \ge n_{\circ_1}$, there is $n_{\circ_2} \in D$ such that $\rho_n \wedge \mu_2 = 0_{X_2}$ for every $n \in D$, $n \ge n_{\circ_2}$, ..., there is $n_{\circ_m} \in D$ such that $\rho_n \wedge \mu_m = 0_{X_m}$ for every $n \in D$, $n \ge n_{\circ_m}$. Thus for each Ω-net $\{\rho_n : n \in D\}$ of closed L-subsets in L^X there is $n_{\circ_i} \in D$ such that $\rho_n \wedge (\prod_{i=1}^{m} \mu_i) = 0_X$ for every $n \in D$, $n \ge n_{\circ_h}$, $i = \{1,2,...,m\}$. Put $\mu = \prod_{i=1}^{m} \mu_i$ and therefore for each Ω-net $\{\rho_n : n \in D\}$ of closed L-subsets in L^X there is

$n_\circ \in D$ such that $\rho_n \wedge \mu = 0_X$ for every $n \in D$, $n \geq n_\circ$. Hence $\mu = \prod_{i=1}^{m} \mu_i$ is $N.\Omega -$ bounded in L^X .

Theorem 4.3. If $f_L : (L^X, \tau) \to (L^Y, \Delta)$ is almost continuous mapping and $\mu \in L^X$ is $N.\Omega -$ bounded, then $f_L(\mu)$ is $N.\Omega -$ bounded.

Proof. Suppose that $\mu \in L^X$ is $N.\Omega -$ bounded and $S = \{\rho_n : n \in D\}$ is an Ω-net of closed L-subsets in L^Y such that $\delta.\overline{\lim}(\rho_n)(y) < \alpha$ for each $y \in Y$. Then $(\forall\ y \in Y,\ \alpha \in M(L))\ (\exists\ \eta \in R_{y_\alpha})\ (\exists\ m \in D)\ (\rho_n \leq cl(\text{int}(\eta)))\ (\forall\ n \geq m)$. Since f_L is almost continuous mapping, then $f_L^{-1}(S) = \{ f_L^{-1}(\rho_n) : n \in D\}$ is an Ω-net of closed L-subsets in L^X such that $(\forall\ x \in X,\ x \in f^{-1}(y),\ \alpha \in M(L))$ $(\exists\ f_L^{-1}(\eta) \in R_{x_\alpha})\ (\exists m \in D)\ (f_L^{-1}(\rho_n) \leq f_L^{-1}(cl(\text{int}(\eta))) = cl(\text{int}(f_L^{-1}(\eta))))$ $(\forall n \geq m)$ and so $\delta.\overline{\lim}(f_L^{-1}(\rho_n))(x) < \alpha$ for each $x \in X$. Now, since μ is $N.\Omega -$ bounded, then there is $n_\circ \in D$ such that $f_L^{-1}(\rho_n) \wedge \mu = 0_X$ for each $n \in D$, $n \geq n_\circ$. Thus $f_L(f_L^{-1}(\rho_n) \wedge \mu) \leq f_L f_L^{-1}(\rho_n) \wedge f_L(\mu) = f_L(0_X) = 0_Y$ and so $\rho_n \wedge f_L(\mu) = 0_Y$. Hence $f_L(\mu)$ is $N.\Omega -$ bounded.

Theorem 4.4. If (L^X, τ) is L-ts , $\mu \in L^X$ and $x_\alpha \in M(L^X)$, $x_\alpha \in N\Omega..cl(\mu)$, then there exists a molecular net S in μ such that S is $N\Omega$-converges to x_α.

Proof. Let $x_\alpha \in M(L^X)$ such that $x_\alpha \in N\Omega.cl(\mu)$, then $\mu \not\leq \lambda$ for each $\lambda \in N\Omega BR_{x_\alpha}$. Since $\mu \not\leq \lambda$, then there exists $\alpha(\mu,\lambda) \in M(L)$ such that $x_{\alpha(\mu,\lambda)} \in \mu$ with $x_{\alpha(\mu,\lambda)} \notin \lambda$. The pair $(N\Omega BR_{x_\alpha}, \geq)$ is a directed set and so we can define a molecular net $S : N\Omega BR_{x_\alpha} \to M(L^X)$ as follows $S(\lambda) = x_{\alpha(\mu,\lambda)}$ for each $\lambda \in N\Omega BR_{x_\alpha}$ Hence S is a molecular net in μ. Now let $\eta \in N\Omega BR_{x_\alpha}$ such that $\lambda \leq \eta$, so we have there exists $S(\eta) = x_{\alpha(\mu,\eta)} \notin \eta$ and so $S(\eta) = x_{\alpha(\mu,\eta)} \notin \lambda$. Hence S is $N\Omega$-converges to x_α.

Theorem 4.5. Let $S = \{S(n) : n \in D\}$ and $T = \{T(n) : n \in D\}$ be a molecular nets in an L-ts (L^X, τ) such that $T(n) \geq S(n)$ for each $n \in D$ and $x_\alpha \in M(L^X)$. Then the following results are true :

(i) S is $N\Omega$-converges to x_α, then T is $N\Omega$-converges to x_α.

(ii) x_α is $N\Omega$-cluster point of S, then x_α is $N\Omega$-cluster point of T .

Proof. (i) Let $x_\alpha \in M(L^X)$ such that S be $N\Omega$-converges to x_α, then for each $\lambda \in N\Omega BR_{x_\alpha}$ there exists $n \in D$ such that for each $m \in D$ and $m \geq n$ then $S(m) \notin \lambda$. Since $T(n) \geq S(n) > \lambda$, and so for each $\lambda \in N\Omega BR_{x_\alpha}$ there exists $n \in D$ such that for each $m \in D$ and $m \geq n$ then $T(m) \notin \lambda$. This shows that T is $N\Omega$-converges to x_α.

(ii) Let $x_\alpha \in M(L^X)$ such that x_α is $N\Omega$-cluster point of S, then for each $\lambda \in N\Omega BR_{x_\alpha}$ and each $n \in D$ there exists $m \in D$ such that $m \geq n$ then $S(m) \notin \lambda$. Since $T(n) \geq S(n)$ for each $n \in D$, then $T(n) \geq S(n) > \lambda$. Thus for each $\lambda \in N\Omega BR_{x_\alpha}$ and for each $n \in D$ there exists $m \in D$ such that $m \geq n$ then $T(m) \notin \lambda$. This shows that x_α is $N\Omega$-cluster point of T.

Theorem 4.6. Assume that $S = \{S(n) : n \in D\}$ is a molecular net in an L-ts (L^X, τ) and $x_\alpha \in M(L^X)$. Then the following results are true :

(i) x_α is $N\Omega$-cluster point of S iff there exists a subnet T of S such that T is $N\Omega$-converges to x_α.

(ii) If x_α is $N\Omega$-cluster point of S, then T is $N\Omega$-converges to x_α for each subnet T of S.

Proof. (i) Provided that $S = \{S(n) : n \in D\}$ and x_α is $N\Omega$-cluster point of S, then for each $\lambda \in N\Omega BR_{x_\alpha}$ and each $n \in D$ there is $k \in D$ such that $S(k) \notin \lambda$ and $k \geq n$. Taking $k = g(n, \lambda)$, we get a mapping $g : D \times N\Omega BR_{x_\alpha} \to D$ with $S(g(n, \lambda)) \notin \lambda$. Put $E = D \times N\Omega BR_{x_\alpha}$ and we define the relation $\leq$ on E as follows : $(n_1, \lambda_1) \leq (n_2, \lambda_2)$ iff $n_1 \leq n_2$ and $\lambda_1 \leq \lambda_2$, then $(E, \leq)$ is a directed set. For each $(n, \lambda) \in E$, we choose $T(n, \lambda) = S(g(n, \lambda))$, then $T = \{T(n, \lambda) : (n, \lambda) \in E\}$ is a subnet of S. Because:

(*) There exists mapping $f : E \to D$ define as follows $f(n, \lambda) = n$ and $T = S \circ f$.

(**) Let $n_1 \in D$, then there exists $(n_1, \lambda_1) \in E$ and $(n_1, \lambda_1) \leq (n_2, \lambda_2) \in E$ iff $n_1 \leq n_2$ and $\lambda_1 \leq \lambda_2$, $f(n_2, \lambda_2) = n_2 \geq n_1$. Now we prove that T is $N\Omega$-converges to x_α, let $\lambda \in N\Omega BR_{x_\alpha}$ and $n \in D$, so $(n, \lambda) \in E$. Therefore for each $(n, \lambda) \in E$ and $(n, \lambda) \leq (m, \eta)$ then $T(m, \eta) = S(g(m, \eta)) \notin \eta$ and $\lambda \leq \eta$, so $T(m, \eta) \notin \lambda$. Thus T is $N\Omega$-converges to x_α.

Conversely, it follows directly from Definition 2.9.

(ii) It follows directly from Definition 2.9.

Theorem 4.7. Let (L^X,τ) be an L-ts , $\alpha\in M(L)$ and $\mu\in L^X$. Then μ is $N..\Omega-$bounded iff every α-filter $\mathcal{F}$ containing μ as an element has a $\delta-$cluster point in X with height α.

Proof. Suppose that μ is $N..\Omega-$bounded and $\mathcal{F}$ is an α-filter containing μ as an element $(\alpha\in M(L))$, then $\lambda\wedge\mu\in\mathcal{F}$ for each $\lambda\in\mathcal{F}$, hence $\vee_{x\in X}(\lambda\wedge\mu)(x)\geq\alpha$ for each $\lambda\in\mathcal{F}$ and for each $x_\alpha\in M(L^X)$ there exists a molecule $x_{(\lambda,\alpha)}\in\lambda\wedge\mu$ with height α. Put $S(\mathcal{F})=\{x_{(\lambda,\alpha)}:(\lambda,\alpha)\in\mathcal{F}\times M(L)\}$. In $\mathcal{F}\times M(L)$ we define the relation that $(\lambda_1,\alpha_1)\geq(\lambda_2,\alpha_2)$ iff $\lambda_1\leq\lambda_2$ and $\alpha_1\geq\alpha_2$. Then $\mathcal{F}\times M(L)$ is a directed set with this relation and $S(\mathcal{F})$ is a constant molecular α-net in μ. Since μ is $N..\Omega-$bounded, then by Theorem 4.1., $S(\mathcal{F})$ has a $\delta-$cluster point in X with height α, say x_α. So by Theorem 2.20, $\mathcal{F}$ $\delta-$cluster to x_α as well.

Conversely, suppose that the condition is satisfied and $S=\{S(n):n\in D\}$ is a constant molecular α-net in μ. Let $\lambda_m=\vee(S(n))$ for each $m\in D$, $n\geq m$. Since D is a directed set, then the family $\{\lambda_m:m\in D\}$ can generate a filter $\mathcal{F}(S)$. Since S is a constant molecular α-net, then for each $\alpha\in M(L)$ $(\exists n\in D)$ $(\forall\, m\in D, m\geq n)$ $(\vee(S(m))=\alpha)$, hence $\vee(\lambda_m(x))=\vee(\vee(S(n)))=\alpha$, $n\geq m$ and so $\vee(\lambda_m(x))=\alpha$. Since $\mathcal{F}(S)$ is produced by $\{\lambda_m:m\in D\}$, then for each $\lambda\in\mathcal{F}(S)$ contains some λ_m and therefore $\vee(\lambda(x))=\alpha$. Hence $\mathcal{F}(S)$ is an α-filter. By assumption, $\mathcal{F}(S)$ has a $\delta-$cluster point in X with height α, say x_α. Thus for each $\mu\in R_{x_\alpha}$ and for each $\lambda\in\mathcal{F}$ (S). In particular, λ_m we have $\lambda_m\not\leq cl(\text{int}(\mu))$, and by Theorem 2.21 We have S has a $\delta-$cluster point x_α and by Theorem 4.1. we have μ is $N..\Omega-$bounded.

Theorem 4.8. If a set μ in an L-ts (L^X,τ) is $N..\Omega-$bounded, then every α-ideal I in L^X and $\mu\notin I$ has a $\delta-$cluster point in X with height α.

Proof. Let I be an α-ideal in L^X and $\mu\in L^X$ be an $N..\Omega-$bounded with $\mu\notin I$. Then for each $\eta\in I$ we have $\vee_{x\in X}(\eta)(x)<\alpha$, and then for each $\alpha\in M(L)$ there exists a molecule $S(\eta,\alpha)=x_{(\eta,\alpha)}\notin\eta$. Put $D(I)$ $=\{(\eta,\alpha):x_{(\eta,\alpha)}\in\mu,\ \eta\in I\ and\ x_{(\eta,\alpha)}\notin\eta\}$. In $D(I)$ we define the relation that $(\eta_1,\alpha_1)\geq(\eta_2,\alpha_2)$ iff $\eta_1\geq\eta_2$. Then $(D(I),\geq)$ is a directed set with this relation and $S(I)=\{S(\eta,\alpha)=x_{(\eta,\alpha)}:(\eta,\alpha)\in D(I)\}$ is a constant molecular α-net in μ. Since μ is $N..\Omega-$bounded, then by Theorem 4.1., $S(I)$ has a $\delta-$cluster point in X with height α, say x_α, by Theorem 2.22., we have x_α is also a $\delta-$cluster point of I.

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