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(t,r) Broadcast Domination Number of the Join and Corona of Graphs[1]

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Abstract

Let G be a graph and $u, v \in V(G)$. If $t \in \mathbb{N}$, then the reception strength of u with respect to t and v is given by $r_v^t(u) = t - d(u,v)$ if $t \geq d(u,v)$ and $r_v^t(u) = 0$ if $t < d(u,v)$. If $S \subseteq V(G)$, then the reception strength of u with respect to t and S is given by $\sum_{v \in S} r_v^t(u)$. We say that S is a (t,r) broadcast dominating set in G if the reception strength of u with respect to t and S is greater than or equal r for all vertices u. The (t,r) broadcast domination number of G is the minimum cardinality of a (t,r) broadcast dominating set of G.

In this paper, we gave the (t,r) broadcast domination number of paths, cycles, complete graphs, the join of two arbitrary graphs, and the corona of two arbitrary graphs.

Mathematics Subject Classification: 05C12

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1 Introduction

Let G be a graph and $u, v \in V(G)$. If $t \in \mathbb{N}$, then the reception strength of u with respect to t and v is given by $r_v^t(u) = t - d(u, v)$ if $t \geq d(u, v)$ and $r_v^t(u) = 0$ if $t < d(u, v)$. If $S \subseteq V(G)$, then the reception strength of u with respect to t and S is given by $\sum_{v \in S} r_v^t(u)$. S is called a (t, r) broadcast dominating set if the reception strength of u with respect to t and S is greater than or equal r for any vertex u. The (t, r) broadcast domination number of G is the minimum cardinality of a (t, r) broadcast dominating set of G.

Let X and Y be sets. The *disjoint union* of X and Y, denoted by $X \dot{\cup} Y$, is found by combining the elements of X and Y, treating all elements to be distinct. Thus, $|X \dot{\cup} Y| = |X| + |Y|$. The *join* of two graphs G and H, denoted by $G + H$, is the graph with vertex-set $V(G + H) = V(G) \dot{\cup} V(H)$ and edge-set $E(G + H) = E(G) \dot{\cup} E(H) \dot{\cup} \{uv : u \in V(G), v \in V(H)\}$. For example, the join of paths $P_3 = (a, b, c)$ and $P_2 = (x, y)$ is shown in Figure 1.

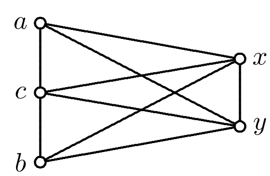

Figure 1: The graph $P_3 + P_2$

Let G be a graph of order n, and H be any graph. The *corona* of G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G and n copies of H, and then joining the *ith* vertex of G to every vertex of the ith copy of H. For example, the corona $P_2 \circ C_3$ of cycle $C_3 = [a, b, c]$ and path $P_2 = (x, y)$ is shown in Figure 2.

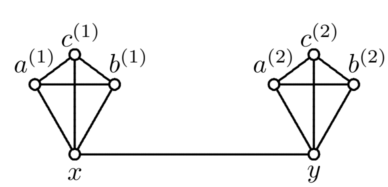

Figure 2: The graph $P_2 \circ C_3$

Hereafter, please refer to [5] for the other concepts.

Blessing et al. [1] introduced the concept (t, r) broadcast domination number in graphs, and gave the (t, r) broadcast domination number of small grids. Moreover, they gave an upper bound for the (t, r) broadcast domination numbers for large grids, and conjectured that these bounds are tight for sufficiently large grids. Furthermore, they conjectured that these bounds are tight for sufficiently large grids.

As mentioned in [1], the concept (t, r) broadcast domination can be applied in multi-agent security and pursuit, city planning (e.g. placement of hospitals, radio stations, nuclear reactors), routing in communication networks, and sensor placement in power networks. Since then many are investigating the concept. Some of these investigations are the following.

Crepeau et al. [3] presented the (t, r) broadcast domination number of paths, grid graphs, the slant lattice, and the king's lattice.

Herrman and Hintum [7] proved a conjecture by Drews, Harris, and Randolph about the minimal density of towers in $\mathbb{Z}_2$ that provide a $(t, 3)$ domination broadcast for $t \geq 17$ and explore generalizations. They also determined the (t, r) broadcast domination number of powers of paths, $P_n(k)$ and powers of cycles, $C_n(k)$.

Buathong and Krityakierne [2] proposed an optimization approach of finding the (t, r) broadcast domination number of a network to help overcome the limitations of existing theoretical approaches in graph theory.

Harris et al. [6] gave some upper bounds for the (t, r) broadcast domination number of triangular matchstick graphs when $(t, r) \in \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (t, t)\}$.

Drews et al. [4] introduced the concept density of a (t, r) broadcast, which allowed them to provide optimal (t, r) broadcasts on the infinite grid graph for all $t \geq 2$ and $r = 1, 2$ and gave bounds for the density of the optimal $(t, 3)$ broadcast for all $t \geq 2$. In addition, they gave a couple of counterexamples to the conjecture of Blessing et al. [1] that the optimal (t, r) and $(t + 1, r + 2)$ broadcasts are identical for all $t \geq 1$ and $r \geq 1$ on the infinite grid.

2 (t,r)-Broadcast Domination Number of Paths

In this section we characterize the (t, r)-broadcast domination number of paths.

Theorem 2.1 *Let $t, r \in \mathbb{N}$. If P_n is a path of order $n \geq 3$, then $\gamma_{t,r}(P_n) = 1$ if and only if $t - r \geq \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof: Assume that $\gamma_{t,r}(P_n) = 1$ and $t - r < \left\lfloor \frac{n}{2} \right\rfloor$. Let S be a minimum (t, r) broadcast dominating set in P_n. Then S is a singleton, say $S = \{u\}$. Consider the following cases:
Case 1. n is odd.

Consider $u = \left\lceil \frac{n}{2} \right\rceil$. Then, $r_u^t(n) = t - d(\left\lceil \frac{n}{2} \right\rceil, n) = t - \left\lfloor \frac{n}{2} \right\rfloor$. Since S is a (t, r) broadcast dominating set, $r_u^t(n) \geq r$. Hence,

$$\begin{aligned} r &\leq t - \left\lfloor \tfrac{n}{2} \right\rfloor \\ t - r &\geq \left\lfloor \tfrac{n}{2} \right\rfloor. \end{aligned}$$

This is a contradiction.
Case 2. n is even

Consider $u = \frac{n}{2}$. Then $r_u^t(n) = t - d(\frac{n}{2}, n) = t - \frac{n}{2}$. Since S is a (t, r) broadcast dominating set, $r_u^t(n) \geq r$. Hence,

$$\begin{aligned} r &\leq t - \tfrac{n}{2} \\ t - r &\geq \tfrac{n}{2} = \left\lfloor \tfrac{n}{2} \right\rfloor. \end{aligned}$$

This is a contradiction. Accordingly, $t - r \geq \left\lfloor \frac{n}{2} \right\rfloor$.

Conversely, assume that $t - r \geq \left\lfloor \frac{n}{2} \right\rfloor$. Consider the following cases:
Case 1. n is odd.

If n is odd, then let $S = \left\{ \left\lceil \frac{n}{2} \right\rceil \right\}$. Then, $r^t_{\left\lceil \frac{n}{2} \right\rceil}(1) = t - d(\left\lceil \frac{n}{2} \right\rceil, 1) = t - \left\lfloor \frac{n}{2} \right\rfloor$. Since $t - r \geq \left\lfloor \frac{n}{2} \right\rfloor$, $t - \left\lfloor \frac{n}{2} \right\rfloor \geq r$. Hence, $r^t_{\left\lceil \frac{n}{2} \right\rceil}(1) \geq r$. Similarly, $r^t_{\left\lceil \frac{n}{2} \right\rceil}(n) \geq r$. Since, $r_u^t(1) < r_u^t(2) < \cdots < r_u^t(\left\lfloor \frac{n}{2} \right\rfloor)$ and $r_u^t(n) < \cdots < r_u^t(\left\lceil \frac{n}{2} \right\rceil + 1)$, and $r_u^t(v) \geq r$ for all $u \in V(P_n)$. Therefore, S is (t, r) broadcast dominating set.
Case 2. n is even.

If n is even, then let $S = \left\{ \frac{n}{2} \right\}$. By the same argument as in the proof of *Case 1*, S is a (t, r) broadcast dominating set.

Accordingly, $\gamma_{t,r}(P_n) = 1$. □

3 (t,r)-broadcast domination Number of Cycles

In this section we independently give the (t, r)-broadcast domination number of cycles.

Theorem 3.1 *Let* $t, r \in \mathbb{N}$. *If* C_n *is a cycle of order* $n \geq 3$, *then* $\gamma_{t,r}(C_n) = 1$ *if and only if* $t - r \geq \left\lfloor \frac{n}{2} \right\rfloor$.

Proof: Assume that $\gamma_{t,r}(C_n) = 1$ and $t - r < \left\lfloor \frac{n}{2} \right\rfloor$. Let S be a minimum (t, r) broadcast dominating set in C_n. Then S is a singleton, say $S = \{u\}$. Consider the following cases:

Case 1. n is odd

Consider $u = \left\lceil \frac{n}{2} \right\rceil$. Then, $r_u^t(n) = t - d(\left\lceil \frac{n}{2} \right\rceil, n) = t - \left\lfloor \frac{n}{2} \right\rfloor$. Since S is a (t, r) broadcast dominating set, $r_u^t(n) \geq r$. Hence,

$$\begin{aligned} r &\leq t - \left\lfloor \frac{n}{2} \right\rfloor \\ t - r &\geq \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

This is a contradiction.

Case 2. n is even

Consider $u = \frac{n}{2}$. Then $r_u^t(n) = t - d(\frac{n}{2}, n) = t - \frac{n}{2}$. Since S is a (t, r) broadcast dominating set, $r_u^t(n) \geq r$. Hence,

$$\begin{aligned} r &\leq t - \frac{n}{2} \\ t - r &\geq \frac{n}{2} = \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

This is a contradiction.

Accordingly, $t - r \geq \left\lfloor \frac{n}{2} \right\rfloor$.

Conversely, assume that $t - r \geq \left\lfloor \frac{n}{2} \right\rfloor$. Without loss of generality, assume that n is odd. Then let $S = \{\left\lceil \frac{n}{2} \right\rceil\}$. Then, $r_{\left\lceil \frac{n}{2} \right\rceil}^t(1) = t - d(\left\lceil \frac{n}{2} \right\rceil, 1) = t - \left\lfloor \frac{n}{2} \right\rfloor$. Since $t - r \geq \left\lfloor \frac{n}{2} \right\rfloor$, $t - \left\lfloor \frac{n}{2} \right\rfloor \geq r$. Hence, $r_{\left\lceil \frac{n}{2} \right\rceil}^t(1) \geq r$. Similarly, $r_{\left\lceil \frac{n}{2} \right\rceil}^t \geq r$. Since, $r_u^t(1) < r_u^t(2) < \cdots < r_u^t(\left\lfloor \frac{n}{2} \right\rfloor)$ and $r_u^t(n) < \cdots < r_u^t(\left\lceil \frac{n}{2} \right\rceil + 1)$, and $r_u^t(v) \geq r$ for all $u \in V(P_n)$. Therefore, S is (t, r) broadcast dominating set. Accordingly, $\gamma_{t,r}(C_n) = 1$. □

4 (t,r)-broadcast domination Number of Complete Graphs

In this section we independently give the (t, r)-broadcast domination number of complete graphs.

Theorem 4.1 *Let $t, r \in \mathbb{N}$. If K_n is a complete graph of order $n \geq 3$, then $\gamma_{t,r}(K_n) = 1$*

Proof: Let $V(K_n) = \{1, 2, \ldots, n\}$. Let $S = \{1\}$ and $v \in \{2, 3, \ldots, n\}$. Then $r_1^t(v) = t - 1 \geq r$. Since v is arbitrary, $r_1^t(w) \geq r$ for all $w \in V(K_n)$. Hence, S is (t, r)- broadcast dominating set. Therefore, $\gamma_{t,r}(K_n) = 1$. □

5 (t,r)-broadcast domination Number of the Join of Graphs

In this section, we give the (t, r)-broadcast domination number of the join of graphs.

Theorem 5.1 *Let $t, r \in \mathbb{N}$. If K_n is a complete graph of order n, and G is any graph, then $\gamma_{t,r}(K_n + G) = 1$.*

Proof: Let $t, r \in \mathbb{N}$ with $r < t$. Let $v \in V(K_n)$ and $S = \{v\}$. Let $u \in V(K_n + G)\backslash\{v\}$. If $u \in V(K_n)$, then $r_v^t(u) = t - 1 \geq r$. Hence, S is a (t, r) broadcast dominating set. Therefore, $\gamma_{t,r}(K_n + G) = 1$. □

Theorem 5.2 *Let $t, r \in \mathbb{N}$. If G and H are non-complete graphs, then $\gamma_{t,r}(G + H) \leq 2$. If $t - r = 1$, then $\gamma_{t,r}(G + H) = 2$.*

Proof: Let $u \in V(G)$ and $v \in V(H)$. Let $S = \{u, v\}$ and $w \in V(G + H)\backslash S$. If $w \in V(G)$, then $r_v^t(w) = t - 1 \geq r$. Hence, $r_s^t(w) \geq r$. If $w \in V(H)$, then $r_u^t(w) = t - 1 \geq r$. Hence, $r_s^t(w) \geq r$. Therefore, S is a (t, r) broadcast dominating set. This implies that $\gamma_{t,r}(G + H) \leq 2$.

Suppose that $t - r = 1$ and $\gamma_{t,r}(G + H) = 1$. Let S be a minimum (t, r) broadcast dominating set say, $S = \{v\}$ (without loss of generality, let $v \in V(G)$). Since G is not a complete graph, there exists $u \in V(G)$ such that $uv \notin E(G)$, that is, $d(u, v) > 1$. Note that $r_v^t(u) = t - d(u, v) < t - 1 = r$. This is contradiction since S is a (t, r) broadcast dominating set. Hence, $\gamma_{t,r}(G+H) = 2$. □

6 (t,r)-broadcast domination Number of the Corona of Graphs

In this section, we give the (t, r)-broadcast domination number of the corona of graphs.

Theorem 6.1 *Let $t, r \in \mathbb{N}$, G be a connected graph and H be any graph. If $t - 1 \geq diam(G) + r$, then $\gamma_{t,r}(G \circ H) = 1$.*

Proof: Let $v \in V(G)$ and $u \in V(H)$. Then $d\,(u, v) \leq diam(G)+1$. Let $S = \{v\}$. Then

$$\begin{array}{rcl} r_s^t(u) & = & t - d\,(u, v) \\ & \geq & t - (diam(G) + 1) \\ & = & (t - 1) - diam(G) \\ & \geq & (diam(G) + r)\,t - diam(G) \\ & = & r \end{array}.$$

Hence, S is a (t, r) broadcast dominating set. Accordingly, $\gamma_{t,r}(C_n \circ H) = 1$. □

In particular, if G is a path P_n or a cycle C_n, then $\gamma_{t,r}(C_n \circ H) = 1$ and $\gamma_{t,r}(C_n \circ H) = 1$, respectively.

Theorem 6.2 *Let $\overline{K}_n$ be the empty graph of order n and H be a graph. Then, $\gamma_{t,r}(\overline{K}_n \circ H) = n$.*

Proof: Let $V(\overline{K}_n) = \{1, 2, \ldots, n\}$. Let $S = \{1, 2, \ldots, n\}$. We would like to show that S is a (t, r) broadcast dominating set in $\overline{K}_n + H$. Let $v \in V\left(\overline{K}_n \circ H^{(i)}\right)$ and consider the following cases:
Case 1. $v \in V\left(\overline{K}_n\right)$

If $v \in V(\overline{K}_n)$, then $r^t_s(v) = t > r$.
Case 2. $v \in V(H^{(i)})$ for some $i \in \{1, 2, \ldots, n\}$

If $v \in V(H^{(i)})$ for some $i \in \{1, 2, \ldots, n\}$, then $r^t_s(v) = t - 1 \geq r$.

This shows that S is a (t, r) broadcast dominating set in $\overline{K}_n \circ H$. Hence, $\gamma_{t,r}(\overline{K}_n \circ H) \leq n$

Suppose that $\gamma_{t,r}(\overline{K}_n \circ H) < n$. Let S' be a minimum (t, r) broadcast dominating set of $\overline{K}_n \circ H$, that is, $|S'| = \gamma_{t,r}\left(\overline{K}_n \circ H\right) < n$. If there are $n - 1$ objects to be placed in n boxes, then by the Pigeon-hole Principle at least 1 box is empty, say without loss of generality $V\left(2 + H^{(2)}\right) \cap S = \varnothing$. Consider $u \in V\left(H^{(2)}\right)$. Then $r^t_s(u) = 0 < r$. This is a contradiction since S' is a (t, r) broadcast dominating set. Accordingly, the assertion follows. □

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