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Soft Characterisations of Regular Ordered Semigroups

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Abstract

Let U be an initial universe set and let S be an ordered semigroup. The concepts of int-soft left, int-soft right ideals, int-soft quasi-ideal and int soft bi-ideal on S were introduced in [5]. In this paper, we study these ideas by providing an equivalent definition of soft right, soft left ideals and soft quasi-ideals. Based on this idea, we show that S is regular if and only if $(\alpha \circ \beta, S) = (\alpha \Cap \beta, S)$ for every soft right ideal (α, S) and every soft left ideal (β, S) over U, and the soft right and the soft left ideals over U are idempotent and for each soft right ideal (α, S) and each soft left ideal (β, S) over U, the soft set $(\alpha \circ \beta, S)$ is a soft quasi-ideal over U.

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1. Introduction

Many theories, fuzzy set theory, rough set theory and others, were introduced as tools to model the uncertainty in phenomena without sharp boundaries. In this direction, Molodtsov developed the theory of soft sets in [12]. In [11], some operations of soft sets and their properties studied by Maji et al. Because of having a lot of parameters to deal with uncertainty, the soft sets have been applied in a wide variety of fields. In the field of algebra theory, Aktas and Cagman[1] explained the notion of soft groups as a parametrized family of subgroups of a group. Following this definition, many researchers ([2],[4]-[7],[10],[16]) studied the algebraic properties of soft sets. Cagman et al. [3] introduced a different group structure on a soft set based on the inclusion relation and intersection of sets. The new groups are called soft int-groups [8, 13, 14]. Based on soft int-semigroup, Song et al [15] gave a soft description of Ideal theory in semigroups. Hamouda introduced the concepts of int-soft left, int-soft right ideals, int-soft quasi-ideal and int soft bi-ideal on an ordered semigroup in [5]. In this paper, We continued to study these concepts by setting equivalent definitions of soft left (resp., right) ideals, soft quasi-ideal in ordered semigroups are introduced. We prove two main results. First: An ordered semigroup $(S, ., \leq)$ is regular if and only if

$$(\alpha \circ \beta, S) = (\alpha \Cap \beta, S)$$

for every soft right ideal (α, S) and every soft left ideal (β, S) over U.

Second: S is regular if and only if the soft right and soft left ideals over U are idempotent and for each soft right ideal (α, S). For each soft left ideal (β, S) over U, the soft set $(\alpha \circ \beta, S)$ is a soft quasi-ideal over U.

2. Preliminaries

In this section, We denote by $(S, ., \leq)$ an ordered semigroup, that is, a semigroup $(S, .)$ with a simple order $\leq$ which satisfies the following condition:

$$\text{for } x, y, z \in S,\ x \leq y \text{ implies that } xz \leq yz \text{ and } zx \leq zy.$$

For $A, B \subseteq S$, we denote $(A] = \{t \in S : t \leq h\, for\, some\, h \in A\}$ and $AB = \{ab : a \in A; b \in B\}$. $A \subseteq S$ is called a subsemigroup of S if $AA \subseteq A$. A nonempty subset A of S is called a left (right) ideal of S if (i) $SA \subseteq A (AS \subseteq A)$ and (ii) $a \in A; S \ni b \leq a \Rightarrow b \in A$. $A \subseteq S$ is called a two-sided ideal (or simply ideal) of S if it is both left and right ideals of S. A non-empty subset Q of an ordered groupoid S is called a quasi-ideal of S if (1) $(QS] \cap (SQ] \subseteq Q$ and (2) $a \in Q; S \ni b \leq a \Rightarrow b \in Q$.

Let U be an initial universe set and $P(U)$ the power set of U.

Definition 2.1. [15] A soft set (α, S) over U is defined to be the set of ordered pairs $(\alpha, S) = \{(x, \alpha(x)) : x \in S, \alpha(x) \in P(U)\}$, where $\alpha : S \longrightarrow P(U)$ is a set-valued mapping.

The notation **Soft(S, U)** stands for the set of all soft sets over U.

Definition 2.2. [15] Let (α, S) and (β, S) be two soft sets. Then, (α, S) is a soft subset of (β, S), denoted by $(\alpha, S) \Subset (\beta, S)$, if $\alpha(x) \subseteq \beta(x)$ for all $x \in S$ and (α, S); (β, S) are called soft equal, denoted by $(\alpha, S) = (\beta, S)$, if and only if $\alpha(x) = \beta(x)$ for all $x \in S$.

Definition 2.3. [15] Let (α, S) and (β, S) be two soft sets. Then, union $(\alpha, A) \Cup (\beta, S)$ and intersection $(\alpha, S) \Cap (\beta, A)$ are defined by

$$(\alpha \Cup \beta)(x) = \alpha(x) \cup \beta(x),$$
$$(\alpha \Cap \beta)(x) = \alpha(x) \cap \beta(x),$$

respectively.

Definition 2.4. [15] Let S be a semigroup and $(\alpha, S) \in$ **Soft(S, U)**. Then, (α, S) is called an int- soft semigroup of S over U (or simply soft semigroup) if it satisfies

$$\alpha(xy) \supseteq \alpha(x) \cap \alpha(y) \qquad \text{for all } x, y \in S.$$

We denote by **Ssem(S,U)** to the set of all soft semigroups over U.

Example 2.1. *Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table:*

$*$	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	c	c	c	c
d	c	c	c	d

Let (α, S) be a soft set over $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ defined as follows:

$$\alpha : S \longrightarrow P(U), \quad x \longmapsto \begin{cases} U, & \text{if } x = a, \\ \{2, 4, 6, 8, 10\}, & \text{if } x = b, \\ \{3, 6, 9\}, & \text{if } x = c, \\ \{2, 6, 10\}, & \text{if } x = d. \end{cases}$$

Then (α, S) is not a soft semigroup over U since

$$\alpha(d \star b) = \alpha(c) = \{3, 6, 9\} \nsupseteq \{2, 6, 10\} = \alpha(d) \cap \alpha(b).$$

The soft set (β, S) over U which is given by

$$\beta : S \longrightarrow P(U), \quad x \longmapsto \begin{cases} U, & \text{if } x = a, \\ \{3, 6, 9\}, & \text{if } x = c, \\ \{6\}, & \text{if } x \in \{b, d\}. \end{cases}$$

is a soft semigroup over U.

For $a \in S$, set $A_a = \{(x, y) \in S \times S : a \leq xy\}$. The soft product of $(\alpha, S), (\beta, S) \in$ **Soft(S,U)** is a soft set $(\alpha \circ \beta, S)$ over U such that

$$(\alpha \circ \beta)(a) = \begin{cases} \bigcup_{(x,y) \in A_a} \quad \{\alpha(x) \cap \beta(y)\}, & \text{if } A_a \neq \phi, \\ \phi, & \text{otherwise.} \end{cases}$$

Proposition 2.1. [5] *Let $(S, ., \leq)$ be an ordered groupoid, (α_i, S) and $(\beta_i, S), i \in \{1, 2\}$ are soft sets over U such that*

$$(\alpha_1, S) \Subset (\beta_1, S)\ (\alpha_2, S) \Subset (\beta_2, S).$$

Then $(\alpha_1 \circ \alpha_2, S) \Subset (\beta_1 \circ \beta_2, S)$.

For an ordered semigroup S, let $(\theta, S) \in$ **Soft(S,U)** defined by $\theta(x) = U$ for all $x \in S$. It is clear from Proposition 2.1 that the set **Soft(S,U)** with the multiplication $\circ$ and the order $\Subset$ is an ordered groupoid and the soft set (θ, S) is the greatest element in $(\mathbf{Soft(S,U)}, \Subset)$.

Definition 2.5. [5] Let $(S, ., \leq)$ be an ordered semigroup. A soft set (α, S) is called a soft left ideal (resp. soft right ideal) over U if

(1) $\alpha(xy) \supseteq \alpha(x)(resp.\quad \alpha(xy) \supseteq \alpha(y)$ for every $x, y \in S$,
(2) $x \leq y \Longrightarrow \alpha(x) \supseteq \alpha(y)$.

Definition 2.6. [5] Let $(S, ., \leq)$ be an ordered semigroup. A soft set (α, S) is called a soft quasi- ideal over U if

(1) $(\alpha \circ \theta, S) \Cap (\theta \circ \alpha, S) \Subset (\alpha, S)$,
(2) $x \leq y \Longrightarrow \alpha(x) \supseteq \alpha(y)$.

3. A Characterization of S by Soft Left (right)Ideals

The main goal of this section is to prove that an ordered semigroup $(S, ., \leq)$ is regular if and only if

$$(\alpha \circ \beta, S) = (\alpha \Cap \beta, S)$$

for every soft right ideal (α, S) and every soft left ideal (β, S) over U.

For $\phi \neq A \subseteq S$, define a map $\chi_A : S \longrightarrow P(U)$ as follows:

$$\chi_A(x) = \begin{cases} U, & \text{if } x \in A, \\ \phi, & \text{otherwise.} \end{cases}$$

Then $(\chi_A, S) \in$ **Soft(S,U)** is called the characteristic soft set[15].

Proposition 3.1. *Let S be an ordered semigroup. Then $(\alpha, S) \in$ **Soft(S,U)** is a soft right ideal over U if and only if*

(i) $(\alpha \circ \theta, S) \Subset (\alpha, S)$,
(ii) $x \leq y \Longrightarrow \alpha(x) \supseteq \alpha(y)$.

Proof. Suppose (α, S) is a soft right ideal over U. Let $x \in S$. If $A_x = \phi$, then $(\alpha \circ \theta)(x) = \phi \subseteq \alpha(x)$. In case $A_x \neq \phi$, then

$$(\alpha \circ \theta)(x) = \bigcup_{(y,z) \in A_x} \{\alpha(y) \cap \theta(z)\} = \bigcup_{(y,z) \in A_x} \{\alpha(y)\}.$$

However, for all $(y, z) \in A_x, we have x \leq yz$ and $\alpha(x) \supseteq \alpha(yz) \supseteq \alpha(y)$. Hence $(\alpha \circ \theta)(x) \subseteq \alpha(x)$. Therefore, we have $(\alpha \circ \theta, S) \Subset (\alpha, S)$. Conversely, assume that (i) is hold. By lemma 3.11 in [15], we get $\alpha(xy) \supseteq (\alpha \circ \theta)(xy) \supseteq \alpha(x)$ for all $x, y \in S$.

□

Similarly, we prove the following result.

Proposition 3.2. *Let S be an ordered semigroup. Then $(\beta, S) \in$ **Soft(S,U)** is a soft left ideal over U if and only if*

(i) $(\theta \circ \beta, S) \Subset (\beta, S)$,

(ii) $x \leq y \Longrightarrow \beta(x) \supseteq \beta(y)$.

An ordered semigroup S is called regular if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$ [9].

Theorem 3.1. *Let S be a regular ordered semigroup and (α, S) a soft right ideal over U. Then $(\alpha \circ \alpha, S) = (\alpha, S)$*

Proof. Let (α, S) be a soft right ideal over U. By prop. we have $(\alpha \circ \alpha, S) \Subset (\alpha \circ \theta, S) \Subset (\alpha, S)$. Now let $x \in S$. : Since S is regular, there exists $a \in S$ such that $(xa, x) \in A_x \neq \phi$. Thus for all $(y, z) \in A_x$,

$$(\alpha \circ \alpha)(x) = \bigcup_{(y,z) \in A_x} \{\alpha(y) \cap \alpha(z)\} \supseteq \{\alpha(y) \cap \alpha(z)\}.$$

Since $(xa, x) \in A_x$ and (α, S) is a soft right ideal over U, then $(\alpha \circ \alpha)(x) \supseteq \alpha(xa) \cap \alpha(x) \supseteq \alpha(x)$, that is $(\alpha \circ \alpha, S) \Supset (\alpha, S)$.

□

Proposition 3.3. *Let $\phi \neq A \subseteq S$. Then A is an ideal of S if and only if (χ_A, S) is a soft ideal over U.*

Proof. Let A be an ideal of S, then item(1) of definition 2.5 is satisfied for (χ_A, S) by Theorem 7 in [15]. For $x, y \in S$ such that $x \leq y$, we have $\chi_A(y) \supseteq \chi_A(x)$. In fact, if $y \notin A$ then $\chi_A(y) = \phi$ and $\chi_A(x) \supseteq \chi_A(y) = \phi$. If $y \in A$ then $\chi_A(y) = U$. Since A is an ideal of S and $S \ni x \leq y$, then $x \in A$, so $\chi_A(x) = U$. Conversely, assume that (χ_A, S) is a soft ideal over U, then $SA \subseteq A$ and $AS \subseteq A$ by Theorem 7 in [15]. Let $y \in A, S \ni x \leq y$, then $\chi_A(x) \supseteq \chi_A(y)$. Thus $\chi_A(x) = U$, and so $x \in A$.

□

Lemma 3.1. *Assume $(S, ., \leq)$ is an ordered groupoid, (α, S) is a soft right ideal and (β, S) is a soft left ideal over U, then $(\alpha \circ \beta, S) \Subset (\alpha \Cap \beta, S)$.*

Proof. Let $x \in S$. If $A_x = \phi$, then $(\alpha \circ \beta)(x) = \phi \subseteq (\alpha \Cap \beta)(x)$. In case $A_x \neq \phi$, we have

$$(\alpha \circ \beta)(x) = \bigcup_{(y,z)\in A_x} \{\alpha(y) \cap \beta(z)\}.$$

Since (α, S) is a soft right ideal over U and $x \leqslant yz$ then $\alpha(x) \supseteq \alpha(yz) \supseteq \alpha(y) \supseteq \alpha(y) \cap \beta(z)$ for every $(y, z) \in A_x$. Hence

$$\alpha(x) \supseteq \bigcup_{(y,z)\in A_x} \{\alpha(y) \cap \beta(z)\} = (\alpha \circ \beta)(x).$$

Similarly, we get

$$\beta(x) \supseteq \bigcup_{(y,z)\in A_x} \{\alpha(y) \cap \beta(z)\} = (\alpha \circ \beta)(x).$$

So we have $(\alpha \circ \beta)(x) \subseteq (\alpha \Cap \beta)(x)$.

□

The following example shows that the inclusion $(\alpha \Cap \beta, S) \Subset (\alpha \circ \beta, S)$ in the lemma above does not hold in general.

Example 3.1. *Let $S = \{0, a, b, c\}$ be the ordered semigroup defined by the multiplication and the order below:*

*	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	a	0
c	0	0	a	a

$\leq := \{(0, 0), (a, a), (b, b), (c, c), (c, 0)\}$.

Let (α, S) be a soft set over $U = \{1, 2, 3, 4, 5, 6\}$ defined as follows:

$$\alpha : S \longrightarrow P(U), \quad x \longmapsto \begin{cases} U, & \text{if } x = 0, \\ \{2, 4, 6\}, & \text{if } x = a, \\ \{2, 6\}, & \text{if } x = b, \\ \{2\}, & \text{if } x = c. \end{cases}$$

Then (α, S) is a soft ideal over U. It is easy to check that $(\alpha \circ \alpha)(x) \subseteq (\alpha \Cap \alpha)(x)$ for every $x \in S$. But the converse is not true because $(\alpha \Cap \alpha)(b) = \{2, 6\} \nsubseteq \phi = (\alpha \circ \alpha)(b)$.

Theorem 3.2. *If $(S, ., \leq)$ is a regular ordered semigroup, then*

$$(\alpha \circ \beta, S) = (\alpha \Cap \beta, S)$$

for every soft right ideal (α, S) and every soft left ideal (β, S) over U.

Proof. By lemma 3.1, we have $(\alpha \circ \beta, S) \Subset (\alpha \Cap \beta, S)$. Let $x \in S$. Since S is regular then there exists $a \in S$ such that $x \leq xax$. Then $(xa, x) \in$

$A_x \neq \phi$. Since (α, S) is a soft right ideal over U then $\alpha(xa) \supseteq \alpha(x)$. Hence $\alpha(x) \cap \beta(x) \subseteq \alpha(xa) \cap \beta(x)$. This implies that

$$(\alpha \Cap \beta)(x) = \alpha(x) \cap \beta(x) \subseteq \alpha(xa) \cap \beta(x) \subseteq \bigcup_{(y,z) \in A_x} \{\alpha(y) \cap \beta(z)\} = (\alpha \circ \beta)(x).$$

Therefore, we get $(\alpha \Cap \beta, S) \Subset (\alpha \circ \beta, S)$.

□

Now, we give a soft characterization of regular ordered semigroups in terms of soft right(left) ideals over U.

Theorem 3.3. *An ordered semigroup $(S, ., \leq)$ is regular if and only if*

$$(\alpha \circ \beta, S) = (\alpha \Cap \beta, S) \tag{3.1}$$

for every soft right ideal (α, S) and every soft left ideal (β, S) over U.

Proof. Assume that $(S, ., \leq)$ is regular. Then $(\alpha \circ \beta, S) = (\alpha \Cap \beta, S)$ by Theorem 3.2. Suppose (1) is verified for every soft right ideal and every soft left ideal over U. For every $xinS$ there exists $a \in S$ such that $x \leq xax$ and so S is regular. Indeed, since R_x is a right ideal of S, χ_{R_x} is a soft right ideal over U by Proposition 3.3. Also χ_{L_x} is a soft left ideal over U. Since $x \in R_x \cap L_x$, then

$$(\chi_{R_x} \circ \chi_{L_x})(x) = (\chi_{R_x} \Cap \chi_{L_x})(x) = \chi_{R_x}(x) \cap \chi_{L_x}(x) = U.$$

This means that there exists at least an element $(y, z) \in A_x$ where $\chi_{R_x}(y) \cap \chi_{L_x}(z) = U$. As a result, we have $\chi_{R_x}(y) = U \Rightarrow y \in R_x$ and $\chi_{L_x}(z) = U \Rightarrow z \in R_x$. Thus $y = xa, z = bx$ for some $a, b \in S$. Then $x \leq yz = (xa)(bx) = x(ab)x$. Therefore, S is regular.

□

4. A Characterization of S by Soft quasi ideals

The main goal in this section is to show that $(S, ., \leq)$ is regular if and only if the soft right and the soft left ideals over U are idempotent and for each soft right and left ideals (α, S) and (β, S) respectively over U, the soft set $(\alpha \circ \beta, S)$ is a soft quasi-ideal over U.

Proposition 4.1. *Let S be an ordered semigroup, $A, B \subseteq S$. Then*

(1) *$A \subseteq B$ iff $(\chi_A, S) \Subset (\chi_B, S)$,*
(2) *$A = B$ iff $(\chi_A, S) = (\chi_B, S)$,*

Proof. Straightforward.

□

Proposition 4.2. *Let S be a groupoid and $\{A_i : i \in I\}$ a family of subsets of S. Then we have $(\chi_{\bigcap_{i \in I} A_i}, S) = \Cap_{i \in I}(\chi_{A_i}, S)$*

Proof. We need to prove that $\chi_{\bigcap_{i\in I} A_i}(x) = \bigcap_{i\in I} \chi_{A_i}(x)$ for all $x \in S$. If $x \in \bigcap_{i\in I} A_i$, then $\chi_{\bigcap_{i\in I} A_i}(x) = U$. Since $x \in A_i$ for each $i \in I$, then $\chi_{A_i}(x) = U$, and so $\bigcap_{i\in I} \chi_{A_i}(x) = U$. Now consider the case $x \notin \bigcap_{i\in I} A_i$, it follows that $\chi_{\bigcap_{i\in I} A_i}(x) = \phi$. Since $x \notin A_i$ for all $i \in I$, then $\chi_{A_i}(x) = \phi$, consequently $\bigcap_{i\in I} \chi_{A_i}(x) = \phi$.

□

Proposition 4.3. *Let $(S, ., \leq)$ be an ordered semigroup, $A, B \subseteq S$. Then $(\chi_A \circ \chi_B, S) = (\chi_{(AB]}, S)$.*

Proof. we only need to prove that $(\chi_A \circ \chi_B)(x) = \chi_{(AB]}(x)$. Let $x \in (AB]$, then $x \leq ab$ for some $a \in A$ and $b \in B$. Hence $(a, b) \in A_x \neq \phi$. Thus

$$(\chi_A \circ \chi_B)(x) = \bigcup_{(y,z)\in A_x} \{\chi_A(y) \cap \chi_B(z)\} \supseteq \chi_A(a) \cap \chi_B(b) = U$$

Therefore, we have $(\chi_A \circ \chi_B)(x) = \chi_{(AB]}(x) = U$. Suppose $x \notin (AB]$, then $\chi_{(AB]}(x) = \phi$. If $A_x = \phi$, then $(\chi_A \circ \chi_B)(x) = \phi = \chi_{(AB]}(x)$. However, if $A_x \neq \phi$ then

$$(\chi_A \circ \chi_B)(x) = \bigcup_{(y,z)\in A_x} \{\chi_A(y) \cap \chi_B(z)\}.$$

We claim that $\chi_A(y) \cap \chi_B(z) = \phi$ for all $(y, z) \in A_x$. Indeed, since $(y, z) \in A_x$, we have $x \leq yz$. Because $x \in (AB]$, then $y \notin A$ or $z \notin B$. If $y \notin A$, $\chi_A(y) = \phi$, and so $\chi_A(y) \cap \chi_B(z) = \phi$. In addition, we get the same result in the case of $z \notin B$. Therefore, we conclude that $\chi_A(y) \cap \chi_B(z) = \phi$ for all $(y, z) \in A_x$.

□

Theorem 4.1. *Let $(S, ., \leq)$ be an ordered semigroup, then $Q \subseteq S$ is a quasi-ideal of S if and only if (χ_Q, S) is a soft quasi-ideal over U.*

Proof. Let Q be a quasi-ideal of S, then (χ_Q, S) is a soft quasi-ideal over U because

$$\begin{aligned}(\chi_Q \circ \theta, S) \Cap (\theta \circ \chi_Q, S) &= (\chi_Q \circ \chi_S, S) \Cap (\chi_S \circ \chi_Q, S)\\ &= (\chi_{(QS]}, S) \Cap (\chi_{(SQ]}, S)\\ &= ((\chi_{(QS]\cap(SQ]}, S)\\ &\Subset ((\chi_Q, S)\end{aligned}$$

Moreover, for $x, y \in S$ such that $x \leq y$, we have $\chi_Q(y) \supseteq \chi_Q(x)$. In fact, if $y \notin Q$ then $\chi_Q(y) = \phi$ and $\chi_Q(x) \supseteq \chi_Q(y) = \phi$. If $y \in Q$, then $\chi_Q(y) = U$. Since Q is a quasi-ideal of S and $S \ni x \leq y$, then $x \in Q$, so $\chi_Q(x) = U$. Thus $\chi_Q(x) \supseteq \chi_Q(y)$. Conversely, assume that (χ_Q, S) is a soft Quasi-ideal over U, then $(\chi_Q \circ \theta, S) \Cap (\theta \circ \chi_Q, S) \Subset (\chi_Q, S) \Rightarrow (\chi_Q \circ \chi_S, S) \Cap (\chi_S \circ \chi_Q, S) \Subset (\chi_Q, S)$. By proposition 4.2, we have $(\chi_{(QS]}, S) \Cap (\chi_{(SQ]}, S) \Subset (\chi_Q, S) \Rightarrow (\chi_{(QS]\cap(SQ]}, S) \Subset (\chi_Q, S)$. By Proposition 4.1, we get $(SQ] \cap (QS] \subseteq Q$. Let $y \in Q, S \ni x \leq y$, then $\chi_Q(x) \supseteq \chi_Q(y) = U$. Thus $\chi_Q(x) = U$, and so $x \in Q$. This completes the proof.

□

Proposition 4.4. *Let S be an ordered groupoid. A soft right (left) ideal (α, S) over U is idempotent if and only if $(\alpha, S) \Subset (\alpha \circ \alpha, S)$.*

Proof. Suppose (α, S) is idempotent, that is, $(\alpha, S) = (\alpha \circ \alpha, S)$. Then $(\alpha, S) \Subset (\alpha \circ \alpha, S)$. Conversely, let $(\alpha, S) \Subset (\alpha \circ \alpha, S)$. Since $(\alpha, S) \Subset (\alpha, S)$ and $(\alpha, S) \Subset (\theta, S)$, then $(\alpha \circ \alpha, S) \Subset (\alpha \circ \theta, S) \Subset (\alpha, S)$, by proposition 2.1. Thus $(\alpha, S) = (\alpha \circ \alpha, S)$.

□

Theorem 4.2. [5] *Let $(S, ., \leq)$ be an ordered semigroup. A soft set (α, S) is a soft quasi-ideal over U if and only if there exist a soft right ideal (β, S) and a soft left ideal (γ, S) over U such that $(\alpha, S) = (\beta \Cap \gamma, S)$.*

By theorem 4.2 and theorem 3.3, we have the following result.

Theorem 4.3. *Let $(S, ., \leq)$ be a regular ordered semigroup, (α, S) a soft right ideal over U and (β, S) a soft left ideal over U. Then $(\alpha \circ \beta, S)$ is a soft quasi-ideal over U.*

Finally, we give another soft characterization of regular ordered semigroups.

Theorem 4.4. *Let $(S, ., \leq)$ be an ordered semigroup, then S is regular if and only if the soft right and the soft left ideals over U are idempotent and for each soft right ideal (α, S) and each soft left ideal (β, S) over U, the soft set $(\alpha \circ \beta, S)$ is a soft quasi-ideal over U.*

Proof. ($\Rightarrow$) This direction comes directly from theorem 3.2 and theorem 4.3. ($\Leftarrow$) Directly from theorem 4.2 and theorem 3.3.

□

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