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Features of Propositional Logic

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Abstract

Logic is an old branch of knowledge; we shall be occupied in this paper with the symbolic logic (or also known as mathematical logic). Propositional logic is a branch of symbolic logic which based on bivalence of classical logic. which it was focused in the beginning on two central problems of logic as formal. Namely, how to decide the given conclusion derived from certainly premises is valid or invalid argument. Recently, it plays role in applications in computer sciences and Engineering. To be able to deciding some known facts about mathematical argument, structure programming. We need logic information's. In This present paper, we highlight features of propositional logic, validity, deduction, consistence, sounds, completing and reducible in propositional logic system (PLS), by strict manner mathematical proofs.

Keywords: Propositional logic, validity, deduction, inference rules, sound, complete and reducible.

1 Introduction

Classical logic is usually divided into three logics, proposition (or statement) logic, monadic, and polyadic. The division is based on the introduction of variables into the language concerned. Proposition logic involves no variables, monadic logic consists of predicates applied for single variables, and polyadic logic has predicates of several variables including the equality for more information see [7,15,18,20,21,22]. In the present paper we introduced known concepts about propositional logic, but introduced them in a strict mathematical and demonstrative manner, with proofs of some theories and characteristics and their clarification. The aim of this work to understand the connection between Boolean algebras and logic. This discipline developed by work's Bool in [4], who is worth nickname father of Algebraic logic. At a later stage, The American mathematicians (of Hungarian origin) Halmos (1916-2006) published the sequence papers to make algebra out of logic by the existing theory of Boolean algebras see [9,10,11,12,13,14]. In [1] we have investigated and extension existential and universal quantifiers operators on Boolean Algebras and their properties and in [2] we studied monadic properties such as ideals, filters associated with homomorphism and simple monadic algebra.

2 Propositional Logic System (PLS)

Definition 2.1.A *proposition* is a statement (statement is *declarative sentence*) that is true or false but not both.

Definition 2.2. The *language* L of propositional logic system consists of:
1. Symbols (vocabulary) $A_1, A_2, A_3, \ldots$(for *simple* (or *atomic*) proposition);
2. Symbols $\neg, \wedge, \vee, \longrightarrow, \leftrightarrow$ and $\oplus$ (for connective proposition) and
3. Punctuation (,).

Remark. Special symbols 0 and 1 may be added to the language L to indicted special proposition. Also in some books, the logical connectives are calling the truth- functional propositional connectives.

Definition 2.3. A *well- formed formula* (**wff**) is defined as follows:
1. $A_1, A_2, A_3, \ldots$ are well- formed formulas.
2. Rules of information, if A and B are **wffs**, then $\neg A$, $A \wedge B$, $A \vee B$, $A \longrightarrow B$, $A \leftrightarrow B$ and $A \oplus B$ are **wffs**.

Remark.

i. The symbols $\neg, \wedge, \vee, \longrightarrow, \leftrightarrow$ indicted to *negation*, conjunction (wedge in latex), disjunction (vee in latex), conditional, biconditional and exclusive nor, respectively.
ii. Formulas are nothing else but string of symbols. They don't have content or meaning so far. In the definition 2.1.1. they will associate with the meaning from the point of view bivalence of classical logic.
iii. PLS does not consider any proposition that has not been constructed by it.

As a matter of fact, PLS distinguishes between the propositions which considers in propositional logic as usable in argument and sets and those that grammarians reject in the same way for sentences that break the normative grammar.

Definition 2.4. An *argument form* is a finite sequence of **wffs** $A_1, A_2, A_3, \ldots, A_n$is called *premises* followed by a **wff** B called *conclusion.* This is written as follows: $A_1, A_2, A_3, \ldots, A_n, \therefore B$.The main problem how to check whether or not the conclusion B is derived from the given premises argument form $A_1, A_2, A_3, \ldots, A_n$. Usually, there are two different ways which called validity and deduction to do this.

Remark.

i. One of the main tasks assigned to logical systems is the task of classifying arguments into valid and invalid arguments.
ii. Argument consists of the following objects:
    - Set of **wffs** which called premises that lead to conclusion.
    - Set of **wff** which called conclusion that derived from premises.
    - Set of logical system which applicable to get conclusion from premises.

2.1 Validity of Language Propositional System Logic (VLSPL)

The truth value of any **wff** in the language $LSPL$ is considered true (T) or values (F) but not both. This is the main principle of bivalence of classical logic.

Definition 2.1.1. The semantic (or *meaning*) of propositional logic consist of truth valuations. A *valuation* (or *truth assignment* or *interpretation*) v in the language L is a function from the set of simple statement letters into the set $\{T, F\}$. I. e,

$$v(A) = \begin{cases} T, if\ A\ is\ true \\ F, if\ A\ is\ false \end{cases}$$

which satisfies the following conditions:

1. $v(\neg A) \neq v(A)$
2. $v(A \wedge B) = T \leftrightarrow v(A) = v(B) = T$
3. $v(A \vee B) = F \leftrightarrow v(A) = v(B) = F$
4. $v(A \rightarrow B) = F \leftrightarrow v(A) = T \wedge v(B) = F$
5. $v(A \leftrightarrow B) = T \leftrightarrow v(A) = v(B)$
6. $v(A \oplus B) = T \leftrightarrow v(A) \neq v(B)$

All interpretation of a **wff** can be viewed by a truth table**.**

Remark. Often the language machine of computer and logical digit treatment with binary system depend on 0 and 1, usually use 1 instead of "T" and 0 instead of "F". Moreover, the number of probabilities (or assignments) in truth table is = 2^n, where n is the number of simple proposition in **wff**.

Definition 2.1.2. Any *formula* A which accurse in anther formula B is called a *sub formula* of B.

Definition 2.1.3. An *argument form* $A_1, A_2, \dots, A_n, \therefore B$ is called *valid* if there is no *interpretation* v such that $v(A_1) = v(A_2) = \cdots = v(A_n) = T\ and\ v(B) = F$. The valid argument form is denoted by $A_1, A_2, \dots, A_n \vDash B$, otherwise it is called *invalid* and denoted by $A_1, A_2, \dots, A_n \nvDash B$.

Theorem 2.1.4. $A_1, A_2, \dots, A_n \vDash B$ if and only if $(A_1 \wedge A_2 \wedge \dots \wedge A_n) \longrightarrow B$ is true" T" for all interpretation v.

Proof. Let $A_1, A_2, \dots, A_n \vDash B$ be *valid argument*. Suppose that $(A_1 \wedge A_2 \wedge \dots \wedge A_n) \longrightarrow B$ is not *valid* for some $v(A_i), 1 \leq i \leq n$ in L. This *contradiction* with hypothesis. So that $(A_1 \wedge A_2 \wedge \dots \wedge A_n) \longrightarrow B$ is true" T" for all *interpretation* v. Conversely, suppose that $(A_1 \wedge A_2 \wedge \dots \wedge A_n) \longrightarrow B$ is true" T" for all interpretation v. To show that $A_1, A_2, \dots, A_n \vDash B$ is valid argument. Suppose that $A_1, A_2, \dots, A_n \vDash B$ is not valid argument, therefore there exists interpretation v such that: $v(A_1) = v(A_2) = \cdots v(A_n) = T$ and $and\ v(B) = F$. Hence $A_1, A_2, \dots, A_n \vDash B$,so $v((A_1 \wedge A_2 \dots \wedge A_n) \longrightarrow B) = F$. This contradiction with hypothesis, consequently is valid argument ■.

Definition 2.1.5. A **wff** B is called:

1. Valid if $\vDash B$,i.e., $v(B) = T$ for any interpretation v.
2. Satisfiability (contingent) if $v(B) = T$ for some interpretation v.
3. Un satisfiability (contradiction) if $v(B) = F$ for any interpretation v.

Theorem 2.1.6. If $\vDash A$ and $\vDash A \longrightarrow B$, then $\vDash B$.

Proof. Consider $\vDash A$ and $\vDash A \longrightarrow B$. Suppose that $\nvDash B$. Then there exists some interpretation v in L Such that $v(B) = F$,but $\vDash A$, implies that $v(A) = T$ for all interpretation v in L.Hence $v(A \longrightarrow B) = F$, therefore $\nvDash A \longrightarrow B$ is invalid. This contradiction with premises $\vDash A \longrightarrow B$, consequently, $\vDash B$■.

Theorem 2.1.7. If $\vDash A$ if and only if , $\neg A$ is a contradiction or Un satisfiability.

Proof. Let $\vDash A$ be a valid wff. Suppose that $\vDash \neg A$, therefore $v(\neg A) = T$ for all interpretation v in L. But $\vDash A$, therefore $v(A) = T$ for all interpretation v in L. Hence $v(A) = v(\neg A) = T$. This contradiction with definition (2.1.1) part 1. So that , $\neg A$ is a contradiction and $v(\neg A) = F$ for all interpretation v in L. Conversely, let , $\neg A$ be a contradiction and Assume that $\nvDash A$, implies that $v(A) = F$ fore some interpretation v in L. Hence $v(\neg A) = T$, since $v(A) \neq v(\neg A)$. But $v(\neg A) = F$, this contradict. Hence $\vDash A$■.

Theorem 2.1.8. $A_1, A_2, \dots, A_n \vDash A \longrightarrow B$ if and only if $A_1, A_2, \dots, A_n, A \vDash B$.

Proof. Let $A_1, A_2, \dots, A_n \vDash A \longrightarrow B$. Suppose that $A_1, A_2, \dots, A_n, A \nvDash B$,implies that $v(\ (A_1 \wedge A_2 \wedge \dots \wedge A_n \wedge A\ ) \rightarrow B) = F$, for some interpretation v in L (by theorem 2.1.4). Therefore, $(\neg\ (A_1 \wedge A_2 \wedge \dots \wedge A_n \wedge A\ ) \vee B) = F$

$\Longrightarrow v(\neg(A_1 \wedge A_2 \wedge \dots \wedge A_n)\ \vee (\ \neg A \vee B)) = F$

$\Longrightarrow v(\neg(A_1 \wedge A_2 \wedge \dots \wedge A_n)\ \vee (\ A \rightarrow B))$

$\Longrightarrow v(\ (A_1 \wedge A_2 \wedge \dots \wedge A_n)\ \rightarrow (\ A \rightarrow B)) = F$

$\Longrightarrow v\ (A_1 \wedge A_2 \wedge \dots \wedge A_n)) = T$ and $v(\ A \rightarrow B) = F$

$\Longrightarrow v\ (A_1) = v(A_2) = \cdots v(A_n) = T$ and $v(\ A \rightarrow B) = F$

$\Longrightarrow A_1, A_2, \dots, A_n \nvDash A \longrightarrow B$ is invalid argument, this contradict with hypothesis. We deduce $A_1, A_2, \dots, A_n, A \vDash B$. Conversely, consider

$A_1, A_2, \dots, A_n, A \vDash B$. Suppose that $A_1, A_2, \dots, A_n \nvDash A \longrightarrow B$, hence $v((A_1 \wedge A_2 \wedge \dots \wedge A_n) \to (A \longrightarrow B)) = F$ for some interpretation v in L. Therefore $v(\neg(A_1 \wedge A_2 \wedge \dots \wedge A_n) \vee (A \longrightarrow B)) = F$, hence $v(\neg(A_1 \wedge A_2 \wedge \dots \wedge A_n \wedge A) \vee B) = F$,we get $v((A_1 \wedge A_2 \wedge \dots \wedge A_n \wedge A) \to B) = F$, so $v\ (A_1 \wedge A_2 \wedge \dots \wedge A_n \wedge A) = T$ and $v(B) = F$, we have $v\ (A_1) = v(A_2) = \cdots = v(A_n) = v(A) = T$ and $v(B) = F$, consequently, $A_1, A_2, \dots, A_n, A \nvDash B$. This contradiction the negation is true $A_1, A_2, \dots, A_n \vDash A \longrightarrow B$■.

Theorem 2.1.8. $A_1, A_2, \dots, A_n \vDash B$ if and only if $A_1, A_2, \dots, A_n, \neg B \vDash 0$.

Proof. Let $A_1, A_2, \dots, A_n \vDash B$. Suppose that $A_1, A_2, \dots, A_n, \neg B \nvDash 0$, implies that $v((A_1 \wedge A_2 \wedge \dots \wedge A_n \wedge \neg B) \to 0) = F$ for some interpretation v in L. Hence $v(\neg(A_1 \wedge A_2 \wedge \dots \wedge A_n \wedge \neg B) \vee 0) = F$

$\Longrightarrow v(\neg(A_1 \wedge A_2 \wedge \dots \wedge A_n) \vee (B \vee 0)) = F$

$\Longrightarrow v(\neg(A_1 \wedge A_2 \wedge \dots \wedge A_n) \vee B) = F$

$\Longrightarrow v(\ (A_1 \wedge A_2 \wedge \dots \wedge A_n) \to B) = F$

$\Longrightarrow v(A_1 \wedge A_2 \wedge \dots \wedge A_n) = T$ and $v(B) = F$

$\Longrightarrow v(A_1) = v(A_2) = \cdots = v(A_n) = T$ and $v(B) = F$

$\Longrightarrow A_1, A_2, \dots, A_n \nvDash B$. This contradiction so that, $A_1, A_2, \dots, A_n, \neg B \vDash 0$. Conversely, consider $A_1, A_2, \dots, A_n, \neg B \vDash 0$ and assume that $A_1, A_2,, \dots, A_n \nvDash B \Longrightarrow v((A_1 \wedge A_2 \wedge \dots \wedge A_n) \to B\ ) = F$, for some interpretation v in L. Therefore $v(\neg(A_1 \wedge A_2 \wedge \dots \wedge A_n) \vee B\ ) = F$

$\Longrightarrow v(\neg(A_1 \wedge A_2 \wedge \dots \wedge A_n) \vee (B \vee 0)) = F$

$\Longrightarrow v(\neg(A_1 \wedge A_2 \wedge \dots \wedge A_n \wedge \neg B) \vee (0)) = F$

$\Longrightarrow v(\ (A_1 \wedge A_2 \wedge \dots \wedge A_n \wedge \neg B) \to 0) = F$

$\Longrightarrow v(A_1 \wedge A_2 \wedge \dots \wedge A_n \wedge \neg B) = T$ and $v(0) = F$

$\Longrightarrow v(A_1) = v(A_2) = \cdots = v(A_n) = v(\neg B) = T$ and $v(0) = F$

$\Longrightarrow A_1, A_2, \dots, A_n, \neg B \nvDash 0$. We get contradiction with hypothesis. Hence,$A_1, A_2, \dots, A_n \vDash B$■.

Definition 2.1.9. Let A and B be two **wffs**, then:

i. A is said to be *logically implies* to B in the language L of propositional logic, if $A \vDash B$ and denoted by $A \Longrightarrow B$.

ii. A is said to be *logically equivalent* to B in the language L of propositional logic, if $A \Longrightarrow B$ and $B \Longrightarrow A$,this is denoted by $A \Longleftrightarrow B$.

Definition 2.1.10. Consider a set $S = \{A : A \text{ is a wff}\}$ of all wffs. Define the logically equivalent relation $\Leftrightarrow$ on S as follows: A $\Leftrightarrow B$ **Iff** $\vDash A \leftrightarrow B$.

Theorem 2.1.11. The *logically equivalent* relation $\Leftrightarrow$ is an equivalence relation. It

Proof. 1. The logically equivalent relation $\Leftrightarrow$ is reflexive relation. I.e. $A \Leftrightarrow A, \forall A \in S$. Since, $A \Leftrightarrow A$ **iff** $\vDash A \leftrightarrow A$ **iff** $\vDash (A \to A) \wedge \vDash (A \to A)$ is true for

all interpretation v in L with respect to A.
2. The logically equivalent relation $\Leftrightarrow$ is symmetric relation. I.e. If $A \Leftrightarrow B$, then $B \Leftrightarrow A, \forall A, B \in S$. Since $A \Leftrightarrow B$ **Iff** $\vDash A \leftrightarrow B$ **Iff** $\vDash (A \leftrightarrow B) \rightarrow (B \leftrightarrow A)$ is true for all interpretation v in L.Hence $\vDash (B \leftrightarrow A)$.
3.The logically equivalent relation $\Leftrightarrow$ is transitive relation. I.e. If $A \Leftrightarrow B$ and $B \Leftrightarrow C$, then $A \Leftrightarrow C, \forall A, B, C \in S$. It easy verify $\vDash (A \leftrightarrow B) \wedge (B \leftrightarrow C) \rightarrow (A \leftrightarrow C)$ and deduce that $\vDash (A \leftrightarrow C)$. So $A \Leftrightarrow C$. From the above argument the logically equivalent relation is an equivalence relation on S ■.

Definition 2.1.12. Consider a set $S = \{p: p \text{ is a wff}\}$ of all wffs. Define the logically implies relation $\Leftrightarrow$ on S as follows: A $\Longrightarrow B$ **Iff** $\vDash A \longrightarrow B$.

Theorem 2.1.13. The logically implies relation $\Longrightarrow$ is an ordering relation on S.

2.1.1 Ordering of Propositional Logic System (OPLS)

Definition 2.1.1.1 Consider a set $S = \{p: p \text{ is a wff}\}$ of all **wffs**. Define the logically implies relation $\Leftrightarrow$ on S as follows: A $\Longrightarrow B$ **Iff** $\vDash A \longrightarrow B$.

Theorem 2.1.1.2. The logically implies relation $\Longrightarrow$ is an ordering relation on S.

Proof. Similar argument in theorem 2.1.11.■.

Definition 2.1.1.3. Let $S = \{A: A \text{ is a } \mathbf{wff}\}$ be set of all **wffs** and $\Leftrightarrow$ is an equivalence relation on S. Consider the quotient set $\Gamma = S/_{\Leftrightarrow} = \{[A]: A \in S\}$, where $[A] = \{B \in S: B \Leftrightarrow A\}$ is the set of equivalence class. Define a binary relation B on Γ as following: $A \Longrightarrow B$ **iff** $[A] \leq [B]$.

Theorem 2.1.1.4. The order pair $(\Gamma, \leq)$ is total order set.

Proof. Firstly, show that the relation is will-defined. Suppose that $[A] = [A']$ and $[B] = [B']$where $[A] \leq [B]$. Since $[A] = [A']$,implies that $A \Leftrightarrow A'$ and $[B] = [B']$, implies that $B \Leftrightarrow B'$. Also we have $[A] \leq [B]$, therefore $A \Longrightarrow B$, hence $A' \Longrightarrow B'$,consequently, $[A'] \leq [B']$. Hence the relation $\leq$ is well-defined. Secondly, to show that $\leq$ is total order relation on Γ.
1.The relation $\leq$ is reflexive, because $[A] \leq [A]$ **iff** $A \Longrightarrow A$ **iff** $\vDash A \longrightarrow A$.
2.The relation $\leq$ is antisymmetric. Assume that $[A] \leq [B]$ and $[B] \leq [A]$for all $[A], [B] \in \Gamma$. Now, $[A] \leq [B] \wedge [B] \leq [A] \rightarrow [A] \Leftrightarrow [B]$ **iff** $(A \Longrightarrow B \wedge B \Longrightarrow A) \rightarrow [A] \Leftrightarrow [B]$ **iff** $\vDash (A \rightarrow B \wedge B \rightarrow A) \rightarrow A \leftrightarrow B$.
3.The relation $\leq$ is transitive. Assume that $[A] \leq [B]$ and $[B] \leq [C]$for all $[A], [B], [C] \in \Gamma$. $[A] \leq [B] \wedge [B] \leq [C] \rightarrow [A] \leq [C]$ **iff** $A \Longrightarrow B \wedge B \Longrightarrow C \rightarrow A \Longrightarrow C$ **iff** $\vDash (A \rightarrow B \wedge B \rightarrow C) \rightarrow (A \rightarrow C)$.
4.we see that $[A] \leq [B] \vee [B] \leq [A]$ **iff** $A \Longrightarrow B \vee \boldsymbol{B} \Longrightarrow A$ **iff** $\vDash A \rightarrow B \vee \boldsymbol{B} \rightarrow A$.
From pervious condition the relation $\leq$ is total order relation on Γ. Hence $(\Gamma, \leq)$

is a total order set ■.

2.2 Deduction of Propositional Logic System (DPLS)

Definition 2.2.1. Let $B = \{\alpha_i : \alpha(\text{wff}), i = 1,2,\dots,n\}$ be set of all finite sequences of wffs (premises) and Consider $B_1 = \{\beta : \beta(\text{wff})\}$ the set of conclusion derived from premises. A rule of inference is a mapping that maps asset (possibly empty) of wff $\alpha_1, \alpha_2, \dots, \alpha_n$ into a wff β. It is written as follows:

$$\begin{array}{c} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \\ \beta \end{array}$$

Remark. We will use the following abbreviation for some words, namely**.** Premises [pre]**,** Assumption[Ass], Elimination [E] and Introduction [I**}**. For more information about The rules of inference of propositional logic see [7,18,20,21,22,24]. Any assumption used in a proof must be discharged. The following methods are used to discharged assumption in natural deduction of propositional logic.

i. The assumption A is made an antecedent of a conditional $A \to B$.
ii. The assumption A leads to a contradiction 0, then $\neg A$ is considered.
iii. Assumption used as in the rule $\vee -$ Elimination ($\vee - E$).

Definition 2.2.2. Let $A_1, A_2, A_3, \dots, A_n, \therefore B$ be an argument form, we say that the conclusion $B$is deducible from the premises $A_1, A_2, A_3, \dots, A_n$ if there is a finite sequence of wffs such that:

i. Each wff either belong to $\{A_1, A_2, A_3, \dots, A_n\}$ or is derived from pervious wffs in by an inference rules.
ii. The last wff of the sequence is B. The finite sequence of wffs is called natural deduction(proof) in L. This is denoted by: $A_1, A_2, A_3, \dots, A_n \vdash B$(this is called sequent). B is called a theorem in L, if $\vdash B$.

Definition 2.2.3. A and B are called provably equivalent if $A \vdash B$ and $B \vdash A$, this denoted by $A \dashv\vdash B$.

To illustrate the above definitions and how to applied the deduction in L and inference rule we will prove the following theorem.

Theorem 2.2.4. Prove that Demorgan's theorem are provably equivalent. I.e. $\neg(A \wedge B) \dashv\vdash \neg A \vee \neg B$.

Proof. $\neg A \vee \neg B \vdash \neg(A \wedge B)$**.**

Line #	wff	Reason	
1.	$\neg A \vee \neg B$	Pre	
2.	$A \wedge B$	Ass	
3.	$\neg A$	Ass	
4.	A	2,$\wedge -E$	
5.	0	3,4, $\neg - E$	
6.	$\neg B$	Ass	
7.	B	2,$\wedge -E$	
8.	0	6,7, $\neg - E$	
9.	0	1,3,5,6,8,$\vee -E$	Discharge 3,6.
10.	$\neg(A \wedge B)$	2,9, $\neg - I$	Discharge 2.

Conversely, $\neg(A \wedge B) \vdash \neg A \vee \neg B$.

Line #	wff	Reason	
1.	$\neg(A \wedge B)$	Pre	
2.	$\neg(\neg A \vee \neg B)$	Ass	
3.	$\neg A$	Ass	
4.	$\neg A \vee \neg B$	3,$\vee -I$	
5.	0	2,4, $\neg - E$	
6.	$\neg\neg A$	3,5, $\neg - I$	Discharge 3.
7.	A	6, DN	
8.	$\neg B$	Ass	
9.	$\neg A \vee \neg B$	8,$\vee -I$	
10.	0	2,9, $\neg - E$	
11.	$\neg\neg B$	8,10, $\neg - I$	Discharge 8.
12.	B	11, DN	
13.	$A \wedge B$	7,12,$\wedge -I$	
14.	0	1,13, $\neg - E$	
15.	$\neg\neg(\neg A \vee \neg B)$	2,14, $\neg - I$	Discharge 2.
16.	$\neg A \vee \neg B$	6, DN.	

We deduced that Demorgan's theorem are provably equivalent ■.

Theorem 2.2.5.

i. $A \vdash B$ **iff** $\neg B \vdash \neg A$.

ii. $A_1, A_2, A_3, \dots, A_n \vdash A \rightarrow B$ **iff** $A_1, A_2, A_3, \dots, A_n, A \vdash B$.

iii. $A_1, A_2, A_3, \dots, A_n \vdash B$ **iff** $A_1, A_2, A_3, \dots, A_n, \neg B \vdash 0$.

iv. $A_1, A_2, A_3, \dots, A_n \vdash B$ **iff** $(A_1 \wedge A_2 \wedge A_3 \wedge \dots \wedge A_n) \rightarrow B$.

Definition 2.2.6. A system $\boldsymbol{S}$ of inference rules is called sound, if $A_1, A_2, A_3, \dots, A_n \vdash B$, implies that $A_1, A_2, A_3, \dots, A_n \vDash B$.

Definition 2.2.7. A system $\boldsymbol{S}$ of inference rules is called complete, if $A_1, A_2, A_3, \dots, A_n \vDash B$. implies that $A_1, A_2, A_3, \dots, A_n \vdash B$.

Theorem 2.2.8. [7] The system of natural inference rules of propositional logic is both sound and complete. i.e.; $A_1, A_2, A_3, \dots, A_n \vdash B$ **iff** $A_1, A_2, A_3, \dots, A_n \vDash B$.

Corollary 2.2.9. $\vdash B$ **iff** $\vDash B$. Notice that we get in particular; i. $A \vdash B$ **iff** $A \Longrightarrow B$ and ii. $A \dashv\vdash B$ **iff** $A \Longleftrightarrow B$.

Corollary 2.2.10. $\vdash B$ **iff** $\vDash B$. Notice that we get in particular; i. $A \vdash B$ **iff** $A \Longrightarrow B$ and ii. $A \dashv\vdash B$ **iff** $A \Longleftrightarrow B$.

Definition 2.2.11. A System of logic $\boldsymbol{S}$ is called consistent, if there is now wff A such that both $\vdash A$ and $\vdash \neg A$**.**

Definition 2.2.12. A System of logic $\boldsymbol{S}$ is called complete, if for any wff $A,$ we have either $\vdash A$ or $\vdash \neg A$**.**

Definition 2.2.13. A System of logic $\boldsymbol{S}$ is called decidable, if there is an effective method (algorithm) for deciding given any wff A of $\boldsymbol{S}$, whether or not $\vdash A$ **.**

Theorem 2.2.15. [15] Propositional calculus is consistent, incomplete and decidable.

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