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Multiplicative Invertibility Characterization on Star-like Cyclicpoid $C_yP\omega_n^*$ Finite Partial Transformation Semigroups

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Abstract

The star-like finite semigroup $\alpha\omega_n^*$ is a new classical transformation semigroup. This current paper standardize the co-existence relations of some operator algebras and transformation semigroup. The multiplicative invertibility characterization of cyclicpoid transformation $C_yP\omega_n^*$ on this new class of transformation semigroup establishes a major link between the group theory and semigroup theory together with functional analysis theory.

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1 Introduction

Given any bijective partial star-like transformation α_i^* of n having the set $\{1, 2, \ldots n\}$ both as its domain and codomain, its cardinality is $n!$. So for any

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integer $i \in \{1, 2, \dots n\}$ there exists exactly one integer $j \in \{1, 2, \dots n\}$ for which $\alpha_i^*(j) = i$. The determinant function of a star-like matrix space is a unique partial rule, $|\cdot| : M_{n\times n}^*(F^n) \longrightarrow F^n$ that is linear in the rows of the matrix, we observe that its zero when the matrix space is not invertible, and such that $|I_n^*| = 1$ then:

$$det(\alpha_i^*) = \sum_{\alpha_i^* \in C_y P\omega_n^*}^{n} (\alpha_i^*)a_1, \alpha_i^*(1)a_2, \alpha_i^*(2)a_3 \dots, \alpha_i^*(n)a_n \tag{1}$$

Any finite cyclicpoid partial transformation $C_yP\omega_n^*$ is said to be star-like if

$$|\alpha^* k_i - w_{i+1}| \leq |\alpha^* k_{i+1} - w_i| \tag{2}$$

for all $w_i \in D(\alpha^*)$ and $k_i \in I(\alpha^*)$ such that

$$C_yP\omega_n^* = \begin{pmatrix} w_1 & w_2 & \dots & w_i \\ \alpha^* k_1 & \alpha^* k_2 & \dots & \alpha^* k_i \end{pmatrix} \tag{3}$$

where $i \in \mathbb{N} \cup \{0\} \in \mathbb{R}$ where $\emptyset$ is assume to be zero for all α^* in $C_yP\omega_n^*$
Factorization of assertion about the relationships that exist between metric spaces, normed linear spaces, and inner product spaces was made by [5]. In the study of [7] they shown that $\omega - OCP_n \in M_m(N)$ is differentiable such that $T(t)$ is an exponentially bounded one parameter semigroup generated by matrix $M_n(N)$ for some $A \in \omega - OCP_n$. Matrix factorization was used by [3] to characterize invertible matrices. The Hannming distance function was used by [1] to show that a mapping is a linear transformation in semigoup such that any transformation semigroup is metricizable. Thus, this paper establishes that any given star-like transformation is invertible and it is a cyclicipoid. For the fact that in linear algebra the expression of a matrix as the product is a general method for solving system of equations, we characterize multiplicative invertibility on $C_yP\omega_n^*$. We refer to [4] and [2] for a general introduction to semigroup theory. Also, for introductory functional analysis with application to algebra we refer to [6], [8] and [9]

2 Preliminary Notes

We give some basic preliminaries that we shall need in the later section:
Definition 1: Star-like Semigroup. Let $X_n = \{1, 2, 3, \dots\}$ be finite n order non-negative integers, then a finite semigroup is said to be star-like if $|\alpha^* k_i - w_{i+1}| \leq |\alpha^* k_{i+1} - w_i|$ such that $\mathbb{N} \cup \{0\} \in \mathbb{R}$ for all $k_i, w_i \in \alpha\omega_n^*$ then $\alpha\omega_n^*$ must satisfies the following axiomatic properties: (i) $0\alpha^* = 0$ (positivity), (ii)

$\alpha^*e^* = e^*\alpha^*$ (identity), (iii) $\alpha(k+w)^* = \alpha k^* + \alpha w^*$ for all $w \in D(\alpha^*)$(linearity), (iv) $g^{-1*}(\beta^*) = \alpha^*$, for all $\alpha^*, \beta^* \in \alpha\omega_n^*$, (v) $f(\alpha^*) \leq I(\alpha^*)$, for all $\alpha^* \in \alpha\omega_n^*$

Definition 2: Star-like Partial Monogenic Invertible Semigroup. Given any star-like finite semigroup $\alpha\omega_n^*$ is a partial monogenic invertible $C_yP\omega_n^*$ if there exist a star-like generator $\beta^* \in \alpha\omega_n^*$:

$$\beta^{q*} : \left\{\alpha^{-1*} = \beta^{-1*} \Longleftrightarrow \alpha^{-1*} = \beta^{-1*}\alpha^{-1*}\beta^{-1*}; \alpha^*, \beta^* \in C_yP\omega_n^*\right\}$$

where $\mathbb{N} \cup \{0\} \in \mathbb{R}$ and $\emptyset$ is assume to be zero such that $C_yP\omega_n^*$ satisfies the following axiomatic properties: (i) $0\beta^{-1*} = 0$ (positivity), (ii) if $\beta^{q*} = \emptyset$, $\alpha^{n*} = \alpha^*$ then $\alpha^{-1*} = \beta^{-1*}$ (Nildempotency), (iii) $g^{-1*}(\beta^*) = \alpha^*$, then $g^{-1*}(\alpha^*) = \beta^*$, for all $\alpha^*, \beta^* \in C_yP\omega_n^*$, (iv) $\beta^{(k+w)*} = \beta^{k*} + \beta^{w*}$ for all $w \in D(\beta^*)$(linearity), (v) $f(\alpha^*) \leq I(\beta^*)$, for all $\alpha^*, \beta^* \in C_yP\omega_n^*$

Definition 3: Cyclicpoid. A cyclicpoid is a set G_{cy}^* in $C_yP\omega_n^*$ with a unary operation $(^{-}1) : G_{cy}^* \longrightarrow G_{cy}^*$ and a star-like partial function $(\cdot) : G_{cy}^* \times G_{cy}^* \longrightarrow G_{cy}^*$ that satisfy the following axiomatic properties for all $\alpha^*, \beta^*, \gamma^* \in C_yP\omega_n^*$: (i) If $\alpha^* \cdot \beta^*$ and $\beta^* \cdot \gamma^*$ are defined, then $(\alpha^* \cdot \beta^*) \cdot \gamma^*$ and $\alpha^* \cdot (\beta^* \cdot \gamma^*)$ are defined and are equal, (ii) $\alpha^{-1*} \cdot \alpha^*$ and $\alpha^* \cdot \alpha^{-1*}$ are defined for any β^* in $\alpha\omega_n^*$, (iii) if $\alpha^* \cdot \beta^*$ is defined then $\alpha^* \cdot \beta^*\beta^{-1*} = \alpha^*$ and $\alpha^{-1*} \cdot \alpha^* \cdot \beta^* = \beta^*$. From these axioms (i - iii), if β^* is a generator such that $\beta^* \in C_yP\omega_n^*$, two easy and convenient properties follows: (iv) If $\beta^{m*} = \beta^{p*} \Longrightarrow m = p$; $(\beta^{-1*})^{-1} = \beta^*$, (v) $\exists\, m \neq p : \beta^{m*} = \beta^{p*}$ where $\alpha^* \cdot \beta^*$ is defined then $(\alpha^* \cdot \beta^*)^{-1} = \beta^{-1*} \cdot \alpha^{-1*}$

3 Main Results

Theorem 1: For any given star-like transformation $\alpha^* \in C_yP\omega_n^*$ the following axiomatic must hold: (i) $0\alpha^* = 0$ for all $\alpha^*, 0 \in C_yP\omega_n^*$, (ii) if $e\alpha^* = \alpha^*e = \alpha^*$ then $g^{-1}(\alpha^*) = \beta^*$, (iii) $\alpha^*\beta^* = \alpha\gamma^* \Longrightarrow \beta^*\gamma^*$ for all $\alpha^*, \beta^*, \gamma^* \in C_yP\omega_n^*$, (iv) if $\alpha^*\beta^* = \beta^*\alpha^*$, then there exist $c \in \mathbb{R} : \alpha^*\beta^* = c(\alpha^* + \gamma^*)$ for all $\alpha^*, \beta^*, \gamma^* \in V^*$

Proof: Let $\alpha^* = \begin{pmatrix} w_1 & w_2 & \dots & w_n \\ k_1 & k_2 & \dots & k_n \end{pmatrix}$, $\beta^* = \begin{pmatrix} w_1 & w_2 & \dots & w_n \\ q_1 & q_2 & \dots & q_n \end{pmatrix}$, and $\gamma^* = \begin{pmatrix} w_1 & w_2 & \dots & w_n \\ l_1 & l_2 & \dots & l_n \end{pmatrix}$ for all $\alpha^*, \beta^*, \gamma^* \in C_yP\omega_n^*$

(i) it is obvious for any star-like transformation semigroup: $\alpha^*\beta^* \in C_yP\omega_n^*$ then $0\alpha^* = 0$ for all $\alpha^*C_yP\omega_n^*$ in $\mathbb{N} \cup \{0\} \in \mathbb{R}$

(ii) suppose $e\alpha^* = \alpha^*e = \alpha^*$ then $g^{-1}(\alpha^*) = \beta^*$, since $C_yP\omega_n^*$ is a monoid star-like abelian semigroup with $\emptyset$ in $\mathbb{N} \cup \{0\} \in \mathbb{R}$ then $e\alpha^* = \alpha^*$ or $\alpha^*e = \alpha^*$ for all $\alpha^*, e \in C_yP\omega_n^*$ there exist at most one identity $e \in C_yP\omega_n^*$ such that

$$\begin{aligned}\alpha^*\beta^* = e &\Longleftrightarrow \beta^*\alpha^* = e \\ &\Longleftrightarrow \beta^* = \alpha^{-1*}. \text{ Thus, } g^{-1}(\alpha^*) = \beta^*\end{aligned}$$

(iii) if $\alpha^*\beta^* = \alpha^*\gamma^* \Longrightarrow \beta^* = \gamma^*$ and
$\beta^*\alpha^* = \gamma^*\alpha^* \Longrightarrow \beta^* = \gamma^*$ for all $\alpha^*, \beta^*, \gamma^* \in C_yP\omega_n^*$.
Indeed $\alpha^*\beta^* = \alpha^*\gamma^* \Longrightarrow \alpha^{-1*}(\alpha^*\beta^*) = \alpha^{-1*}(\alpha^*\gamma^*)$
implies $(\alpha^{-1*}\alpha^*)\beta^* = (\alpha^{-1*}\alpha^*)\gamma^* \Longrightarrow e\beta^* = e\gamma^* \quad \Longrightarrow \beta^* = \gamma^*$
similarly, $\beta^*\alpha^* = \gamma^*\alpha^* \Longrightarrow \beta^* = \gamma^*$ for all $\alpha^*, \beta^*, \gamma^* \in C_yP\omega_n^*$ then, $\beta^*\alpha^* = \gamma^*\alpha^* \Longrightarrow \beta^*\alpha^*(\alpha^{-1*}) = \gamma^*\alpha^*(\alpha^{-1*})$ implies $\beta^*(\alpha^*\alpha^{-1*}) = \gamma^*(\alpha^*\alpha^{-1*}) \Longrightarrow \beta^*e = \gamma^*e$. Therefore, $\alpha^*\beta^* = \alpha^*\gamma^*$
(iv) suppose $\alpha^*\beta^* = \beta^*\alpha^*$, there exist real number $c \in \mathbb{R}$:$c(\alpha^*+\gamma^*) = c\alpha^*+c\gamma^*$.
We consider some elements $\alpha^* = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $\beta^* = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$ then by definition (1) we have $\alpha^*\beta^* = \beta^*\alpha^* = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$ hence, for any $c \in \mathbb{R} : \alpha^*\beta^* \in C_yP\omega_n^*$ in $\mathbb{N} \cup \{0\} \in \mathbb{R}$ there exist $\gamma^* = \begin{pmatrix} 1 & 2 \\ 1 & \emptyset \end{pmatrix}$ such that

$$c\left(\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & \emptyset \end{pmatrix}\right) = c\left(\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\right) + c\left(\begin{pmatrix} 1 & 2 \\ 1 & \emptyset \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

now, we see that $c(\alpha^* + \gamma^*) = \beta^*\alpha^*$ (if $c = 1 = D(\alpha^* or \gamma^*)$) which gives linear combination in $C_yP\omega_n^*$

Theorem 2: Let m be the index and p the period of an element $\beta^* \in C_yP\omega_n^*$, then $K_{\beta^*} = \{\beta^{m*}, \beta^{(m+1)^*}, \beta^{(m+2)^*} \ldots \beta^{(m+p-1)^*}\}$ if and only if $\beta^* \in C_yP\omega_n^*$ is cyclicpoid ($C_yP\omega_n^*$ contain a subsemigroup that is a group) and $C_yP\omega_n^*$ is a star-like semigroup.

Proof

$\Longrightarrow$ we need to show that:
(i) K_{β^*} is closed under operation in $C_yP\omega_n^*$
(ii) K_{β^*} has unique identity
(iii) For any element of K_{β^*}, there exist its inverse for which their product is also in K_{β^*}. By definition (3), we see that there exists a star-like monogenic semigroup of every $|K_{\beta^*}| = p$ (period) and $\beta^{m*} = \beta^{x*}$ (index) of K_{β^*}. Indeed lets consider the element:

$$\beta^*_{m,p} = \begin{pmatrix} 1 & 2 & 3 & \ldots & m & m+1 & \ldots & m+p-1 & \ldots & m+p \\ 2 & 3 & 4 & \ldots & m+1 & m+2 & \ldots & m+p & \ldots & m+1 \end{pmatrix} \quad (4)$$

with a rule $T_{m+p} : m+p \longrightarrow m+p$ which take from a $m+p$ element into itself, for some positive integers m, p. The second row in equation (4) tells us what is ascribed to the element above it. The element β^* generates the star-like $C_yP\omega_n^*$ monogenic semigroup of index m and period p such that β^* is cyclicpoid. So let see what it looks like for $m = 7$ and $p = 14$:

$$|C_yP\omega_3^*| = \beta^*_{7,14} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & \emptyset \end{pmatrix}$$

The diagraph of β^* looks like this:

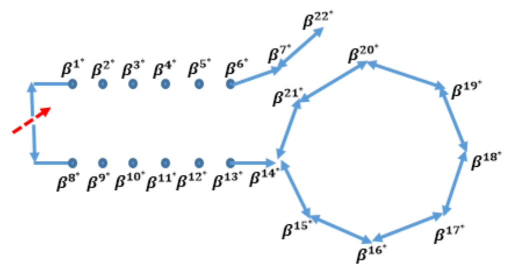

Figure 1: **star-like monogenic order of small** $n(C_y P\omega_3^*)$

The cycle in the diagraph of Figure (1) is the representation of the cyclicpoid generated by a certain elements one of $\{\beta^{14*}, \beta^{15*}, \beta^{16*}, \beta^{17*}, \beta^{18*}, \beta^{19*}, \beta^{20*}, \beta^{21*}\}$. Now, suppose there exist an identity element $e \in P\omega_n^* : C_y P\omega_n^* \subseteq P\omega_n^*$ which idempotent (that is $e^2 = e$) for all $e \in P\omega_n^*$. We consider:

$$(\beta^{7+k})^2 = \beta^{7+k} \tag{5}$$

for $k \in \{0, 1, 2, \dots 7\}$. We see from Figure (1) that equation (5) is equivalent to: $2(7+k) \equiv 7+k \; mod \; 8 \Longrightarrow k \equiv -7 \; mod \; 8$ so $k = 7$ and we get that $\beta^{7+7} = \beta^{14}$ represent the idempotent sets in $C_y P\omega_n^*$ which is the only idempotent contained in the cycle. It is now a matter of a simple check to see that it is indeed an identity element in $\{\beta^{14*}, \beta^{15*}, \beta^{16*}, \beta^{17*}, \beta^{18*}, \beta^{19*}, \beta^{20*}, \beta^{21*}\}$. We need to show that for every $k \in \{0, 1, 2, \dots 7\}$, there is $l \in \{0, 1, 2, \dots 7\}$ such that

$$\beta^{7+k} \cdot \beta^{7+l} = \beta^{14} \tag{6}$$

Again, we see that equation (6) is equivalent to: $7 + k + 7 + l \equiv \; 14 mod \; 8 \Longrightarrow l \equiv -k \; mod \; 8$. We have shown that K_{β^*} is a subsemigroup of $C_y P\omega_n^*$ which is a group. But we need to show also, that K_{β^*} is monogenic (cyclic) by which we find a generator $g \in \{0, 1, 2, \dots 7\}$ such that for any $k \in \{0, 1, 2, \dots 7\}$ there exists $l \in \{0, 1, 2, \dots 7\} : (\beta^{7+g})^l = \beta^{7+k}$ then $l(7 + g) \equiv \; 7 + k \; mod \; 8 \Longrightarrow$ $7 + g \equiv 1 \; mod \; 8$

Which is obviously exists in the set $\{\beta^{14*}, \beta^{15*}, \beta^{16*}, \beta^{17*}, \beta^{18*}, \beta^{19*}, \beta^{20*}, \beta^{21*}\}$

Thus, K_{β^*} is cyclicpoid

Conversely, $\Longleftarrow$

since $C_y P\omega_n^*$ is a subsemigroup of $P\omega_n^*$ then by theorem (1) we have that $C_y P\omega_n^*$ is a star-like semigroup. Therefore, $C_y P\omega_n^*$ is a cyclicpoid star-like partial transformation semigroup.

Proposition 3: If α^* is an invertible star-like matrix space, then the linear system $\alpha^* \vec{z} = \vec{b}$ has a unique solution given by $\vec{z} = \alpha^{-1*} \vec{b}$

Proof

To establish the existence of the solution, we consider;

$$\alpha^*(\alpha^{-1*}\vec{b}) = (\alpha^*\alpha^{-1*})\vec{b} = I_n^*\vec{b} = \vec{b}$$

So $\alpha^{-1*}\vec{b}$ is a solution. We see that for any solution $\vec{w}$:

$$\vec{w} = I_n^*\vec{w} = (\alpha^*\alpha^{-1*})\vec{w} = \alpha^{-1*}(\alpha^*\vec{w})$$

$$= (\alpha^*\alpha^{-1*})\vec{b}$$

Lemma 4: Let $C_yP\omega_n^*$ be star-like cyclicpoid invertible semigroup, such that m of $V^* \in C_yP\omega_n^*(n, F)$, then $det(m) \neq 0$ if and only if m is star-like invertible matrix space.

Proof

Suppose m is a star-like invertible matrix space, Let $\alpha_1^*, \alpha_2^*, \alpha_3^*, \ldots$ in m are sub star-like vectors then,

$$\begin{bmatrix} m\alpha_1^* \\ m\alpha_2^* \\ m\alpha_3^* \\ \vdots \\ m\alpha_n^* \end{bmatrix}^T = M^*_{n\times n^*}(F^n)$$

where $m = \begin{bmatrix} \alpha_{11}^* & \alpha_{12}^* & \alpha_{13}^* & \ldots & \alpha_{1n}^* \\ \alpha_{21}^* & \alpha_{22}^* & \alpha_{23}^* & \ldots & \alpha_{2n}^* \\ \alpha_{31}^* & \alpha_{32}^* & \alpha_{33}^* & \ldots & \alpha_{3n}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1}^* & \alpha_{n2}^* & \alpha_{n3}^* & \ldots & \alpha_{nn}^* \end{bmatrix}$, $I_n^* = \begin{bmatrix} \mathrm{e} & 0 & 0 & \ldots & 0 \\ 0 & \mathrm{e} & 0 & \ldots & 0 \\ 0 & 0 & \mathrm{e} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \mathrm{e} \end{bmatrix}$

such that $[m : I_n^*] \Longrightarrow [I_n^* : m^{-1}]$., we transform m to a star-like triangular matrix space to find $det(m)$:

$m = \begin{bmatrix} \alpha_{11}^* & \alpha_{12}^* & \alpha_{13}^* & \ldots & \alpha_{1n}^* \\ 0 & \alpha_{22}^* & \alpha_{23}^* & \ldots & \alpha_{2n}^* \\ 0 & 0 & \alpha_{33}^* & \ldots & \alpha_{3n}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \alpha_{nn}^* \end{bmatrix}$, thus $det(m) = \alpha_i^* \neq 0$

conversely, suppose $[m : I_n^*] \Longrightarrow [I_n^* : m^{-1}]$ then $\begin{bmatrix} m\alpha_1^* \\ m\alpha_2^* \\ m\alpha_3^* \\ \vdots \\ m\alpha_n^* \end{bmatrix}^T \neq 0$, we have

$m^{-1} = \frac{Adj(m)^T}{|m|}$,

such that $[m : I_n^*] = \begin{bmatrix} \alpha_{11}^* & \alpha_{12}^* & \alpha_{13}^* & \dots & \alpha_{1n}^* & \vdots & e & 0 & 0 & \dots & 0 \\ \alpha_{21}^* & \alpha_{22}^* & \alpha_{23}^* & \dots & \alpha_{2n}^* & \vdots & 0 & e & 0 & \dots & 0 \\ \alpha_{31}^* & \alpha_{32}^* & \alpha_{33}^* & \dots & \alpha_{3n}^* & \vdots & 0 & 0 & e & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1}^* & \alpha_{n2}^* & \alpha_{n3}^* & \dots & \alpha_{nn}^* & \vdots & 0 & 0 & 0 & \dots & e \end{bmatrix}$

implies $[I_n^* : m^{-1}] = \begin{bmatrix} e & 0 & 0 & \dots & 0 & \vdots & \alpha_{11}^* & -\alpha_{21}^* & -\alpha_{31}^* & \dots & -\alpha_{n1}^* \\ 0 & e & 0 & \dots & 0 & \vdots & -\alpha_{12}^* & \alpha_{22}^* & -\alpha_{32}^* & \dots & -\alpha_{n2}^* \\ 0 & 0 & e & \dots & 0 & \vdots & -\alpha_{13}^* & -\alpha_{23}^* & \alpha_{33}^* & \dots & -\alpha_{n3}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e & \vdots & -\alpha_{1n}^* & -\alpha_{2n}^* & -\alpha_{3n}^* & \dots & \alpha_{nn}^* \end{bmatrix}$.

Thus, m is star-like invertible.

Proposition 5:

Let $\alpha^*, \beta^* \in M_{n\times n^*}^*(F^n)$, then $\alpha^* \cdot \beta^*$ is invertible, if β^* is invertible

Proof

Suppose that β^* is not invertible, then the linear transformation $L_{\beta^*} : F^n \longrightarrow F^n$, $L_{\beta^*}(z) = \beta^* z$ is then not invertible as the matrix space representation of L_{β^*} is not invertible. However, since it is a linear transformation and the domain and co-domain of α^* and β^* are each of the same finite dimensions, its follows that L_{β^*} is not injective and not surjective. As L_{β^*} is not injective there exists some $z \in F^n$ where $L_{\beta^*}(z) = \beta^* z = 0$; $z \neq 0$, then if $\beta^* z = 0$ we have

$$[(\alpha^* \cdot \beta^*)^{-1}]\alpha^* \cdot \beta^* = [(\alpha^* \cdot \beta^*)^{-1}\alpha^*] \cdot 0$$

implies $z = 0$ which is a contradiction. But by theorem (2) we deduced that $det(\alpha^* \cdot \beta^*) \neq 0$ such that $z \in det(\alpha^* \cdot \beta^*)$ where $det(\alpha^*)$ and $det(\beta^*)$ are non-zero. If $\alpha^*[\beta^*(\alpha^* \cdot \beta^*)^{-1}] = I_n^*$ (I_n^* is a star-like unit vector)
then $(\alpha^* \cdot \beta^*)^{-1}\alpha^* \cdot \beta^* = I_n^* \Longrightarrow (\alpha^* \cdot \beta^*)^{-1*}\alpha^*\beta^* \cdot \beta^{-1*} = \beta^{-1*} = (\alpha^* \cdot \beta^*)^{-1*}\alpha^*$
if we multiply both sides by β^*, we get $[\beta^*(\alpha^* \cdot \beta^*)^{-1}]\alpha^* = I_n^*$.
Thus β^* is invertible.

Lemma 6:

Let $P\omega_n^*$ be a star-like partial semigroup, and let $\lambda^* \in P\omega_n^*$ be a partial star-like bijective function, then λ^* has at least one inverse.

Proof

suppose λ^* is a partial function, $\lambda^* : \mathbb{R} \longrightarrow \mathbb{R}$ which we define as $\lambda^*(z) = 2^z$. If $\lambda^*(z) = 2^z$ is onto, then there exist some real value $c \in \mathbb{N} \cup \{0\} \in \mathbb{R}$ so that there is no other real value of z for equation (7):

$$\lambda^*(z) = 2^z = c \tag{7}$$

taking $\log_2$ on both sides, we get

$$z = \log_2 c \tag{8}$$

since $c > 0$, there is a real, defined value of z which contradict the condition in equation (7). Thus λ^* is onto.
Now, we need to prove that λ^* is one-one.
Then, by contradiction principle we assume λ^* many-one so that we consider

$$\lambda^* \in P\omega_n^* = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}$$

such that we can take two real value $z_1, z_2 \in \mathbb{N} \cup \{0\} \in \mathbb{R}$ where $z_1 \neq z_2$ with $\lambda^*(z_1) = \lambda^*(z_2) : 2^{z_1} = 2^{z_2} \quad \Longrightarrow 2^{z_1 - z_1} = 1$

$$\Longrightarrow z_1 - z_2 = 0$$
$$\Longrightarrow z_1 = z_2$$

since $z_1 = z_2$ we see that our assumption is incorrect, thus λ^* is one-one. Consequently, λ^* is a bijective partial star-like function since it is both one-one and onto, then we can find the inverse, such that we take $c = \lambda^*(z)$ and then interchange z and c to get $z = 2^c$ implies $\log_2(z) = c$. Therefore the inverse function of $\lambda^*(z)$ is $\lambda^*(z) = \log_2 z$.

Theorem 7:
Given any α_i^*, β_j^* and γ_k^* in $M_{n\times n^*}^*(F^n)$ matrix space, the following are equivalent:

(sa_1) the matrix space α_i^* is star-like
(sa_2) the matrix space β_j^* is cyclicpoid
(sa_3) the matrix space γ_k^* is star-like invertible
(sa_4) there is an $(n \times n)$-matrix space $\beta_j^* : \beta_j^* \cdot \alpha_i^* = I_n^*$
(sa_5) there is an $(n \times n)$-matrix space $\gamma_k^* : \alpha_i^* \cdot \gamma_k^* = I_n^*$
(sa_6) the matrix space α_i^{T*} is star-like invertible
(sa_7) for all $\overrightarrow{b} \in F^n$, the linear system $\alpha_i^* \overrightarrow{z} = \overrightarrow{b}$ has a unique solution
(sa_8) for all $\overrightarrow{b} \in F^n$, the linear system $\alpha_i^* \overrightarrow{z} = \overrightarrow{b}$ is consistent
(sa_9) the homogenous linear system $\alpha_i^* \overrightarrow{z} = \overrightarrow{0}$ has only one solution
(sa_{10}) the reduced row echelon form of any α_i^*, β_j^* and γ_k^* in $M_{n\times n^*}^*(F^n)$ is the star-like identity matrix space I_n^*
(sa_{11}) the matrix space β_j^* is a product of cyclicpoid class
(sa_{12}) the rank of the any matrix space α_i^*, β_j^* and γ_k^* in $M_{n\times n^*}^*(F^n)$ is n
(sa_{13}) the rows of any matrix space α_i^*, β_j^* and γ_k^* in $M_{n\times n^*}^*(F^n)$ are linearly

independent
(sa_{14}) the columns of any matrix space α_i^*, β_j^* and γ_k^* in $M^*_{n\times n^*}(F^n)$ are linearly independent
(sa_{15}) the rows of any matrix space α_i^*, β_j^* and γ_k^* in $M^*_{n\times n^*}(F^n)$ span $\mathbb{R}^n$
(sa_{16}) the columns of any matrix space α_i^*, β_j^* and γ_k^* in $M^*_{n\times n^*}(F^n)$ span $\mathbb{R}^n$

Proof

$sa_1 \Longrightarrow sa_2$: if $\alpha_i^* \in M^*_{n\times n^*}(F^n)$ is star-like, then there exist $\beta_j^* \in M^*_{n\times n^*}(F^n)$: $(\beta_j^{-1*})^{-1} = \beta_j^*$; $\beta_j^{m*} = \beta_j^{p*}$, if $m = p$ and $(\alpha_j^* \cdot \beta_j^*)^{-1} = \beta_j^{-1*} \cdot \alpha_j^{-1*}$; $\beta_j^{m*} = \beta_j^{p*}$ if $m \neq p$
$sa_1 \Longleftrightarrow sa_2 \Longleftrightarrow sa_3$: if $\alpha_i^* \in M^*_{n\times n^*}(F^n)$ is invertible then the transformation $\beta_j^* := \alpha_i^{-1*}$
$sa_3 \Longleftrightarrow sa_4 \Longleftrightarrow sa_5$: if $\alpha_i^* \in M^*_{n\times n^*}(F^n)$ is invertible then the transformation $\gamma_k^* := \alpha_i^{-1*}$
$sa_1 \Longleftrightarrow sa_6$: the of star-like invertible matrix space include
$(\alpha_i^{-1*})^T = (\alpha_i^{*T})^{-1}$; so the matrix space α_i^* is invertible if and only if matrix space α_i^{*T} is also invertible
$sa_4 \Longrightarrow sa_7$: suppose that $\alpha_i^* \cdot \beta_j^* = I_n^*$;
if $\vec{w} \in M^*_{n\times n^*}(F^n)$ is a solution to $\alpha_i^* \vec{z} = \vec{b}$, then it follows that

$$\vec{w} = I_n^* \vec{w} = (\beta_j^* \cdot \alpha_i^*) \vec{w} = \beta_j^*(\alpha_i^* \vec{w}) = \beta_j^*(\alpha_i^* \vec{b})$$

$sa_5 \Longrightarrow sa_8$: if $\alpha_i^*\gamma_k^* = I_n^*$; then we see that $\alpha_i^*(\gamma_k^* \vec{b}) = (\alpha_i^*\gamma_k^*) \vec{b} = \vec{b}$
so, the vector space $\gamma_k^* \vec{b} \in M^*_{n\times n}(F^n)$ is a solution to the linear system $\alpha_i^* \vec{z} = \vec{b}$
$sa_8 \Longleftrightarrow sa_{10} \Longleftrightarrow sa_{12} \Longleftrightarrow sa_{15}$: since $\alpha_i^* \in M^*_{n\times n^*}(F^n)$ is a square matrix space, the characterizations of universal consistency of the augmented matrices establishes these equivalences.
$sa_7 \Longrightarrow sa_9$: since $\alpha_i^* \in M^*_{n\times n^*}(F^n)$ is a cyclicpoid, then there exist a unique solution in the spacial case $\vec{b} = \vec{0}$
$sa_9 \Longleftrightarrow sa_{10} \Longleftrightarrow sa_{12} \Longleftrightarrow sa_{13}$: since $\alpha_i^* \in M^*_{n\times n^*}(F^n)$, then the characterizations of a unique solution prove the equivalences.
$sa_{10} \Longrightarrow sa_{11}$: this equivalence follows from the generalization of star-like cyclicpoid classes as its both right and left multiplicative
$sa_{11} \Longleftrightarrow sa_3 \Longleftrightarrow sa_1$: since every cyclicpoid $G^*_{cy}(\alpha_i^*)$ classes are star-like, and invertible, the properties of invertible matrices space show that α_i^* is invertible.
$sa_6 \Longleftrightarrow sa_{14}$: since we have already established that $'sa_6'$ is equivalent to $'sa_{13}'$, the transpose version also holds.
$sa_6 \Longleftrightarrow sa_{16} \Longleftrightarrow sa_1$: since we have already also established that $'sa_3'$ equivalent to $'sa_{15}'$, the transpose version also holds. Alternatively, Figure (2) gives the structural illustration of the proof (theorem 7) of the multiplicative invertibility characterization of star-like cyclicpoid $C_yP\omega_n^*$ transformation semigroups:

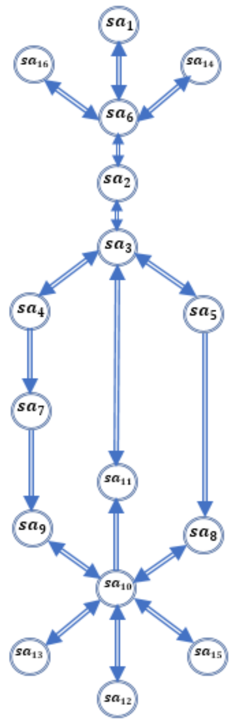

Figure 2: **Multiplicative Invertibility Characterization of** $C_yP\omega_n^*$

4 Concluding Remarks.

Remark 1:
If the hypothesis of periodicity is neglected then we can no longer guarantee the existence of idempotent in any given star-like cyclicpoid semigroup.

Remark 2:
The characterization of invertible matrix space shows that any star-like transformation α_i^*, β_j^* and γ_k^* in $M_{n\times n^*}^*(F^n)$ is invertible if and only if its reduced row echelon form equals the star-like identity matrix space I_n^*. Therefore if $C_yP\omega_n^*$ is a product of cyclicpoid class G_{cy}^* such that $G_{cy}^*(\alpha_i^*) = I_n^*$ then $G_{cy}^*(\beta_j^*) = \beta_j^{-1*}$

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