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Use of a Childhood Toy, the Yo-Yo, to Present a Mechanical Proof of the Namesake Theorem of Pythagoras

Amitava Biswas

SHS School, University of Southern Mississippi
Hattiesburg, MS 39406, USA

Abhishek Bisaria

Spectral D&T Inc, Bridgewater, NJ 08807, USA

Abstract

Conventional expositions of the Pythagorean theorem generally lack any need for common toys, although it is common in vast areas of mathematics, engineering, and sciences. Since this theorem is usually mastered by many children, this article connects it to a childhood toy, the yo-yo. As the yo-yo revolves, the points on its string may appear to sweep rings, involutes, or circles, depending on viewpoints. The equivalency among these viewpoints, exposes equivalency between sums of circles and rings, proving the fundamental equation of the Pythagorean theorem. Like other alternative methods, this unusual interpretation of the old theorem could help instructors maintain critical thinking among the young, utilizing their usual toys, and prepare them for more unusual applications.

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1. Introduction

The namesake theorem attributed to Pythagoras of Samos is documented extensively since distant past [9, 10, 16-20, 35, 42, 45, 46]. This theorem is essential for advanced users as well as young learners in not only geometry but also many other fields [11, 13, 14, 21-23, 28, 29, 36-39, 44, 53]. In spite of innumerable commonly known proofs and applications of this theorem [3, 4, 25, 32-34, 49-52], direct references to common toys used by children are not commonly reported. Frequent association with common objects is important for learners [7, 12, 24]. Therefore, here is another attempt based upon a simple, inexpensive, mechanical toy, the yo-yo [48], depicted in Figure 1. This exposition could be of interest for critical thinking, as this ordinary toy has certain extraordinary kinematic and dynamic features [15, 43].

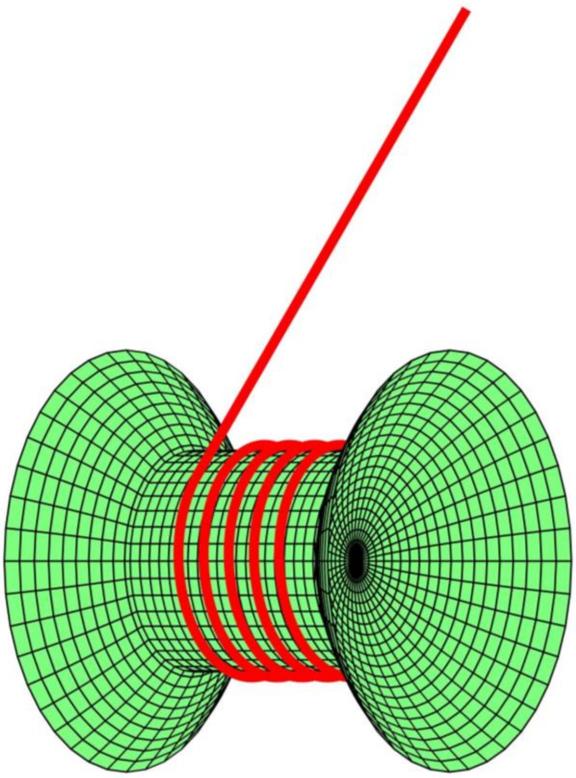

Fig 1. Wireframe model of a childhood toy, the yo-yo.

2. Setup

Many yo-yos have two side caps permanently bonded to an axle [48]. Figure 2 shows simplified features of a yo-yo. The smaller circle is the axle for partially winding a thin and flexible string ***GDB***. The larger circle is the outer border of a side cap. These two coplanar and concentric circles share center ***C***, and enclose an annular area. The string is coiled properly to have frictional grip around the axle. It is under some tension. Therefore, the portion ***GD*** is straight, free from any unconstrained bend or slack, and thereby tangential to the inner circle at the point of contact ***D***. It intersects the outer circle at ***F***. The angle $\angle CDF$ is a right angle and the side ***CF*** is the hypotenuse.

By the Pythagorean theorem,

$$CF^2 = CD^2 + DF^2 \qquad (1)$$

Since the area of any circle is the square of its radius times the constant π [30, 41], equation (1) gives:

$$\pi DF^2 = \pi CF^2 - \pi CD^2 \qquad (2)$$

Therefore, the Pythagorean theorem is proved when:

Area of the circle of radius DF = Area of the annulus (3)

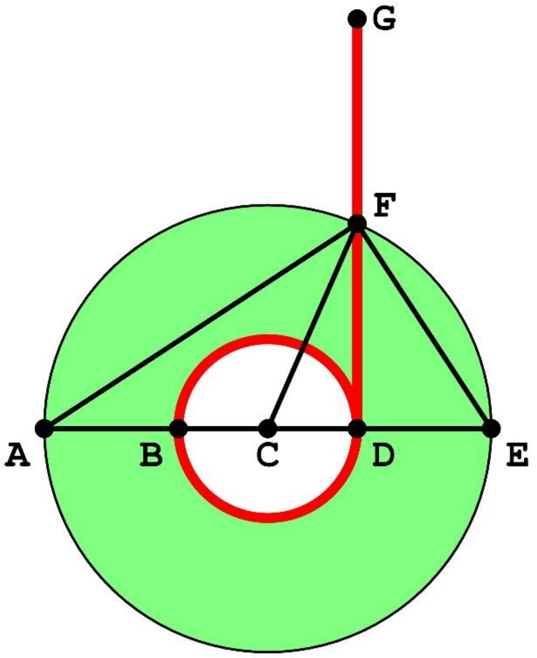

Fig 2. String GDB crosses the outer circle at F and touches the inner circle at D.

3. Traditional

In Figure 2, following traditional methods [1, 2, 8, 26, 27, 40], the edges ***AC, EC,*** and ***FC*** are equal to each other, being the radius of the outer circle. Therefore, the angle $\angle CAF$ is equal to the angle $\angle CFA$. Similarly, the angle $\angle CEF$ is equal to the angle $\angle CFE$. The angles $\angle ADF$ and $\angle EDF$ are right angles. Therefore, the angle $\angle AFE$ is a right angle, and the triangles ΔEFD, ΔFAD are similar. The respective sides are proportional as follows:

$$\frac{DF}{DA} = \frac{DE}{DF} \qquad (4)$$

Now the circular areas get involved, when the equation (2) follows from (4):

$$\pi DF^2 = \pi DA * DE = \pi (AC+CD) * (AC-CD) = \pi CF^2 - \pi CD^2 \qquad (5)$$

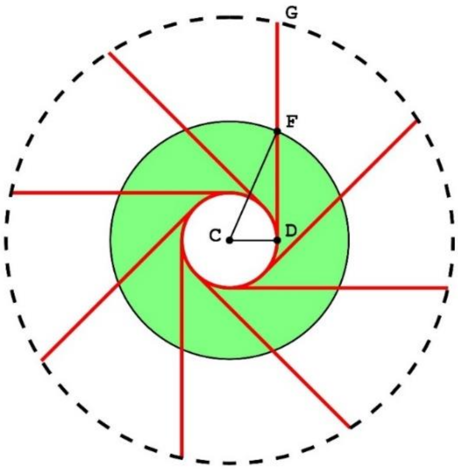

Fig 3. Observing from a revolving yo-yo as the string slips around the axle.

4. When the string may slip around the axle

Two yo-yos of identical dimensions will be analyzed. The first yo-yo is shown in Figure 3. Here, the frictional grip between the string and the axle of the yo-yo is limited and the far end point ***G*** of the string is held at a fixed distance from the center ***C***. When the string is taut, further turning of the yo-yo causes the string to slide over the axle. The point ***D*** of the string, remains and slides along the inner circle of the yo-yo. If this yo-yo is uniformly and monotonously turned by exactly one full rotation, then the straight part ***DF*** of the string must sweep all and only points of the annular area, between the inner and outer circles of the yo-yo, exactly once. From this viewpoint, at any instant, the rate of sweeping this annular area per revolution of the yo-yo is:

$$\pi CF^2 - \pi CD^2$$

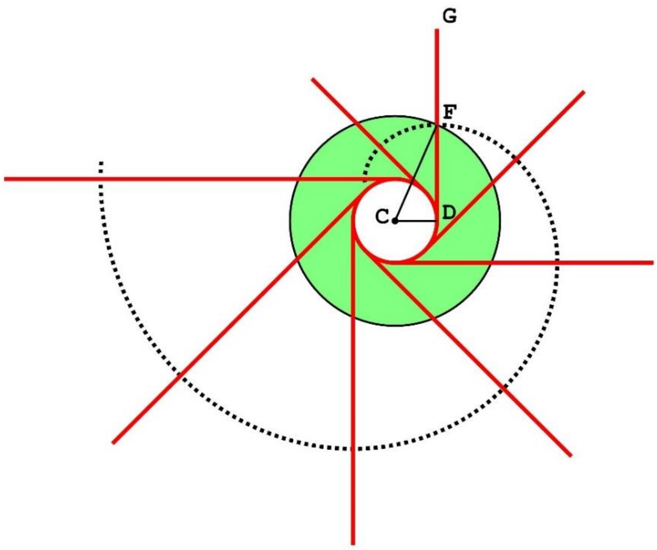

Fig 4. If the string does not slip, any point on the string traces an involute.

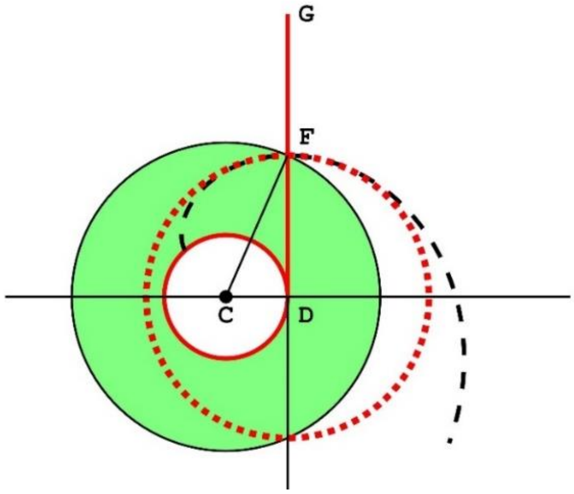

Fig 5. At the point ***F***, the radius of curvature of the involute is ***DF***.

5. When the string may not slip around the axle

The second yo-yo is shown in Figures 4 and 5. Here, the frictional grip between the string and the axle of the yo-yo is high enough to avoid any slips. The end point ***G*** of the string is supported by a small and limited tension. The string remains taut and gets progressively wrapped (or unwrapped) around the inner circle when the yo-yo is turned. Depending upon wrapping or unwrapping, the part ***DF*** of the string is not stationary, and the point ***D*** of the string moves away from the point of contact. Nevertheless, there is always a part of the string, say ***DF'***, delimited by the inner and outer circles, that is straight, equal to the length of ***DF,*** and occupying the original position of ***DF***. If this yo-yo is uniformly and monotonously turned by exactly one full rotation, then the part ***DF'*** of the string must sweep all and only points of the annular area, between the inner and outer circles of the yo-yo, exactly once. From this viewpoint, at any instant, like the first yo-yo, the rate of sweeping the annular area per revolution of the second yo-yo is also:

$$\pi CF^2 - \pi CD^2$$

As shown in Figure 4, in the absence of any slips between the string and the axle, any point fixed on the unwrapped portion of the string traces an involute, relative to the inner circle. To be consistent with the definition of involutes [47], the radius of curvature at any point of this involute is equal to its tangential distance from the inner circle. As shown in Figure 5, the instantaneous motion of the point ***F*** on the string, relative to the annulus, must be along a circle with center ***D*** and radius ***DF*** [15, 43]. Consequently, whenever during wrapping or unwrapping, a point of the string crosses the outer circle, its instantaneous motion must be along a circle of radius ***DF***. Relative to the axle, the instantaneous motion of the point ***D*** on the string is zero. Instantaneous motion of any point of the string within ***DF*** must preserve the length and straightness of ***DF*** and follow a circular path with center ***D***. From these considerations, the rate of sweeping the annular area by the string per revolution of this yo-yo is:

$$\pi DF^2$$

Comparing synchronous motions of the two yo-yos, the rate of sweeping areas of the annulus and the circle must be the same, proving the equation (2).

6. Conclusions

This article showed that a mechanistic demonstration of the Pythagorean theorem can follow from a revolving yo-yo, naturally sweeping circular areas instead of squares. There are other similar examples where circles appear naturally linked to the Pythagorean theorem [3, 4, 31]. One obvious application is seen when analyzing circular symmetry of the Gaussian probability distribution in two dimensions [5, 6].

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