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Application of Annihilator Extension's Method to Classify Zinbiel Algebras

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1 Introduction

Lie algebras are one of the significant entities of the modern theory of non-associative algebras. After several researches and explorations in the field of Lie algebras, some generalisations of these algebras have been observed, namely Lie superalgebras, Binary Lie algebras, Leibniz algebras, Malcev algebras, and so on.

According to the study [2] carried out by J. L. Loday on categorical characteristics of Leibniz algebras, it is concluded that Zinbiel algebra (read Leibniz in the reverse order) is a new object in this connection. Sometimes Zinbiel algebras are also referred to as dual Leibniz algebras because the category of Zinbiel algebras is Koszul dual to the Leibniz algebras [1].

Some interesting characteristics of Zinbiel algebras are highlighted in the works [3, 4]. Specifically, the nilpotency of an arbitrary complex Zinbiel algebra with finite dimensions was established in the work [4]. The examples of Zinbiel algebras are provided in the works [4, 2, 1].

In quantum mechanics, the central extensions play a significant role. The

Wigner's theorem used in one of the earlier encounters indicates that a symmetry of a quantum mechanical system ascertains an anti-unitary conversion of a Hilbert space. The quantum theory of conserved currents of a Lagrangian is another field of physics where central extensions are applied. These extensions are closely associated to the so-called affine Kac Moody algebras, which are referred to as the universal central extensions of loop algebras.

In general, since the symmetry group of a quantised system is a pivotal extension of the classical symmetry group, it makes it necessary to have central extensions in physics. Similarly, the corresponding symmetry Lie algebra of the quantum system is a key extension of the classical symmetry algebra. The representation of symmetry groups in a unified superstring theory is achieved by Kac Moody algebras. In the quantum field theory, the centrally extended Lie algebras have a dominant role to play, specifically with respect to the M-theory, string theory, and conformal field theory.

According to the theory of Lie groups, and representation of Lie algebras, a Lie algebra extension refers to an expansion caused by a Lie algebra h to a particular Lie algebra g. There are numerous ways in which extensions can occur. By considering the direct sum of two Lie algebras, a trivial extension can arise. Central extension and split extension are some of the other types of extension. Moreover, there is a likelihood that extensions can occur naturally when developing a Lie algebra using representations of the projective group. An extension occurring due to a derivation of a polynomial loop algebra over finite-dimensional simple Lie algebra and a central extension renders a non-twisted affine Kac Moody algebra to a Lie algebra which is isomorphic [5].

A current algebra over two space-time dimensions can be constructed using the centrally extended loop algebra. The Heisenberg algebra refers to the central extension of a commutative Lie algebra, and the Virasoro algebra refers to the universal central extension of the Witt algebra [5, 6, 8].

To classify Zinbiel algebras over finite fields, an approach is explained in this paper, which is comparable to the Skjelbred-Sund method is applied for the classification of Lie algebras. 2-cocycles and the corresponding annihilator extensions are recommended in all the cases. The base change action is defined as an action that is applicable for automorphism group of small algebras on cocycles. The recommended approach is new and provides a detailed list of Zinbiel algebras in dimension 5.

2 Extension of Zinbiel algebra via annihilator

In this section we introduced the concept of an annihilator extension of Zinbiel algebras.

Definition 2.1. *Zinbiel algebra is a vector space R together with a binary operation $\circ : R \times R \longrightarrow R$ satisfying the condition :*

$$(x \circ y) \circ z = x \circ (y \circ z) + x \circ (z \circ y), \quad \textit{for all} \quad x, y, z \in R \tag{1}$$

Homomorphism of Zinbiel algebras is a linear transformation preserving operations. The category of Zinbeil algebras as denoted by **Zinb**.
An ideal I of a Zinbiel algebra A satisfy $IR \subseteq I$ and $RI \subseteq I$.

For a given Zinbiel algebra R, we define the following sequence:

$$R^1 = R, \quad R^{k+1} = R \circ R^k; k \geq 1$$

Definition 2.2. *A Zinbiel algebra R is called nilpotent if there exists $s \in N$ such that $R^s = 0$. The minimal number s satisfying this property is called index of nilpotency or nilindex of the algebra R.*

Definition 2.3. *For a given R, the ideal*

$$Ann(R) = \left\{a \in R : R \circ a = a \circ R = 0\right\} \tag{2}$$

describe the annihilator of a R.

Definition 2.4. *Let R_1, R_2 and R_3 be Zinbiel algebras. The algebra R_2 is called an extension of R_3 by R_1 if there exist homomorphism $\alpha : R_1 \to R_2$ and $\beta : R_2 \to R_3$ such that the following sequence*

$$0 \longrightarrow R_1 \xrightarrow{\alpha} R_2 \xrightarrow{\beta} R_3 \longrightarrow 0 \cdots$$

Definition 2.5. *An extension is called **trivial** if there exists an ideal I of R_2 complementary to $ker\beta$.*

It may happen that there exist several extensions of R_3 by R_2. To classify extensions the notion of equivalent extensions is defined.

Definition 2.6. *The sequence*

$$0 \longrightarrow R_1 \xrightarrow{\alpha} R_2 \xrightarrow{\partial_2} R_3 \longrightarrow 0 \cdots$$

*is called **an annihilator extension** if the kernal of β is contained in the annihilator of R_2. That is $ker\beta \subset Ann(R_2)$.*

Definition 2.7. *Two sequences*

$$0 \longrightarrow R_1 \overset{\alpha}{\longrightarrow} R_2 \overset{\beta}{\longrightarrow} R_3 \longrightarrow 0$$

$$0 \longrightarrow R_1 \overset{\alpha^{'}}{\longrightarrow} R_2 \overset{\beta^{'}}{\longrightarrow} R_3 \longrightarrow 0$$

are called ***equivalent extensions*** *if there exists an isomorphism* $f : R_2 \to R_2^{'}$ *such that* $f \circ \alpha = \alpha^{'}$ *and* $\beta^{'} \circ f = \beta$

Proposition 2.1. *Let* ϕ *be an isomorphism between two Zinbiel algebras* R_1 *and* R_2*. Then for any* $k \in \mathbb{N}$*;* $\varphi(Ann(R_1^k)) = Ann(R_2^k)$*.*

Proof. Considering an isomorphism, it implies

$$\begin{aligned}
\varphi(Ann(R_1^k)) &= \varphi\Big\{x \in R_1 | x \circ R_1^k = R_1^k \circ x = 0\Big\} \\
&= \Big\{\varphi(x) \in \varphi(R_1) | \varphi(x) \circ \varphi(R_1^k) = \varphi(R_1^k) \circ \varphi(x) = 0\Big\} \\
&= \Big\{y \in R_2 | y \circ R_2^k = R_2^k \circ y = 0\Big\} \\
&= Ann(R_2^k).
\end{aligned}$$

□

3 Cocycles on Zinbiel algebras

In this section we propose the concept of 2-cocycle for Zinbiel algebras.

Definition 3.1. *Let* R *be an algebra and* V *be a vector space over a field* $\mathbb{K}$*. A bilinear function* $\varphi : R \times R \to V$ *is said to be a Zinbiel* 2*-cocycle of* R *if*

$$\varphi(x \circ y, z) = \varphi(x, y \circ z) + \varphi(x, z \circ y) \tag{3}$$

for all x*,* y *and* $z \in R$

The set of all 2-cocycle on Zinbiel algebra R with values in V is denoted by $Z^2(R, V)$.
It is easy to see that $Z^2(R, V)$ is a vector space if one defines the vector space operations as follows:

$$\begin{aligned}
(\varphi_1 \oplus \varphi_2)(x, y) &= \varphi_1(x, y) + \varphi_2(x, y) \\
(k, \varphi)(x) &= k\varphi(x),
\end{aligned}$$

φ_1, φ_2 and $\varphi \in Z^2(R, V)$, $k \in \mathbb{K}$.
A linear combination of 2-cocycles again is a 2-cocycle. A special type of 2-cocycles given by the following proposition is called 2-coboundries.

Proposition 3.1. *Let $\eta : R \to V$ be a linear map, define $\varphi(x,y) = \upsilon(\lambda(x,y))$. Then φ is a cocycle.*

Proof. Let Us check the axioms:

$$\begin{aligned}\varphi(x \circ y, z) &= \upsilon((x \circ y) \circ z) \\ &= \upsilon(x \circ (y \circ z) + x \circ (z \circ y)) \\ &= \upsilon(x \circ (y \circ z)) + \upsilon(x \circ (z \circ y)) \\ &= \varphi(x, y \circ z) + \varphi(x, z \circ y)\end{aligned}$$

□

Let $\upsilon : R \to R$ be a linear map. Define $\eta(x,y) = \upsilon(\lambda(x,y))$. Then η is a 2-cocycle called coboundary on R with values in V is denoted by $B^2(R,V)$ and $H^2(R,V) = Z^2(R,V)/B^2(R,V)$ is called the second group of cohomologies with values in V.

Theorem 3.1. *Let $e_1, e_2, \cdots, e_k$ be a basis of V, and $0 \in Z^2(A,V)$. Then, φ can be uniquely written as $\varphi(x,y) = \sum\limits_{k=1}^{n} \varphi_i(x,y)e_k$, where $\varphi_k \in Z^2(A,\mathbb{K})$.*

Proof. Consider any x, $y \in R$ then $\varphi(x,y)$ can be uniquely written as $\varphi(x,y) = \sum\limits_{k=1}^{n} \alpha_k e_k$, where $\alpha_i \in \mathbb{K}$. For each $k = 1, 2, \cdots, n$, define a bilinear form $\varphi_k : A \times A \to \mathbb{K}$ by $\varphi_k(x,y) = \alpha_k$. Then, $\varphi(x,y) = \sum\limits_{k=1}^{n} \varphi_k(x,y)e_k$. Moreover, we have,

$$\begin{aligned}\sum_{k=1}^{n} \varphi_k(x,y)e_k = \theta(x,y) = \varphi(y,x) &= \sum_{k=1}^{n} \varphi_k(y,x)e_k, \\ \sum_{k=1}^{n} \varphi_k((x \circ y) \circ z)e_k &= \sum_{k=1}^{n} \varphi(x \circ (y \circ z) + x \circ (z \circ y))e_k\end{aligned}$$

Therefore, for every $k = 1, \cdots, s$ $\varphi_k(x,y) = \varphi_k(y,x)$ and $\varphi_k((x \circ y) \circ z) = \varphi(x \circ (y \circ z) + x \circ (z \circ y))$. Consequently, $\varphi_k \in Z^2(R,\mathbb{K})$ for $k = 1, \cdots, n$. For uniqueness, let $\varphi(x,y) = \sum\limits_{k=1}^{n} \vartheta_k(x,y)e_k$. Then $\sum\limits_{k=1}^{n} (\varphi_k - \vartheta_k)(x,y)e_k = 0$. Since $e_1, \cdots, e_k$ are linearly independent, it follows that $(\varphi_k - \vartheta_k)(x,y) = 0$ for all $x, y \in R$, $k = 1, \cdots, n$. Hence $\varphi_k - \vartheta_k = 0$ for $k = 1, \cdots, n$.

□

Let $\varphi \in Z^2(A,V)$. The set $\varphi^{\perp} = \left\{x \in A : \varphi(x,y) = 0, \quad \text{for all} \quad y \in A\right\}$ is called the radical of φ.

Theorem 3.2. *Let* (R, λ) *be a Zinbiel algebra,* V *a vector space over* $\mathbb{K}$*,*

$$\varphi : R \times R \to V$$

be a bilinear map. Let $R_\varphi = R \oplus V$*. For* $x,\ y \in R,\ v,\ w \in V$ *we define*

$$\lambda_\varphi(x + v, y + w) = \lambda(x, y) + \varphi.$$

Then R_φ *is a Zinbiel algebra if and only if* φ *is a* 2*-cocycle of* R*.*

Proof. The proof is straightforward by using definitions of 2-cocycle of R. □

4 Methods

There is an action of $\mathrm{Aut}(L)$ on $Z^2(R, \mathbb{K})$ as follows: let $\phi \in \mathrm{Aut}(R)$ and $\theta \in Z^2(R, \mathbb{K})$ then

$$(\phi, \theta)(x, y) = \theta(\phi(x), \phi(y))$$

Let assume that $G_m(H^2(R, \mathbb{K}))$ be the Grassmanian of subspaces of dimension m in $HL^2(R, \mathbb{K})$. The action above can be extended to $G_m(H^2(R, \mathbb{K}))$ as follows: suppose that $\phi \in \mathrm{Aut}(L)$ and

$$T = (\theta_1, \theta_2, \theta_3, \cdots, \theta_s) \in G_m(H^2(R, \mathbb{K}))$$

.
We define

$$\phi \cdot T(\phi\theta_1 \cdot \phi\theta_2 \cdot \phi\theta_3, \cdots, \phi\theta_s)$$

It follows that, $\phi{\cdot}T \in G_m(HL^2(L, \mathbb{K}))$. Let express the orbit of $T \in G_m(HL^2(L, \mathbb{K}))$ under the action of group automorphism $\mathrm{Aut}L$ on $G_m(HL^2(L, \mathbb{K}))$ as $\mathrm{Orb}(T)$.
We have the following lemmas.

Lemma 4.1. *Let* T_1 *and* T_2 *be two elements of* $G_m(H^2(R, \mathbb{K}))$ *defined by* $T_1 =< \theta_1, \theta_2, \theta_3, \cdots, \theta_m >$ *and* $T_2 =< \vartheta_1, \vartheta_2, \vartheta_3, \cdots, \vartheta_m >$*. If* $T_1 = T_2$*. then*

$$\cap_i^m = 1^{\theta_i^\perp} \cap Ann(R) = \cap_i^m = 1^{\vartheta_i^\perp} \cap Ann(R)$$

As a consequence of Lemma 4.1 above, we define the subspace
$W_m(L) = \{T =< \theta_1, \theta_2, \theta_3, \cdots, \theta_m >\in G_s(H^2(R, \mathbb{K})) : \cap_i^m = 1^{\theta_i^\perp} \cap \mathrm{Ann}(R) = 0\}$

Example 4.1. *The Zinbiel algebra R with non-zero $e_1e_1 = e_2$ between the basis elements e_1, e_2 is central extension of the abelian algebra $Ke_1 \times Ke_1$ by Ke_2, given bilinear form*

$$\Delta_{11} : (x_1e_1 + x_2e_2, y_1e_1 + y_2e_2) \to x_1y_1$$

Example 4.2. *The Zinbiel algebra with non-zero $e_1e_1 = e_3$ between the basis elements e_1, e_2 and e_3 is a central extension of R from the previous example: $Aut(R)$ consists of all operators*

$$\Phi = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{11}^2 \end{bmatrix}$$

$H^2(R, \mathbb{K})$ consist of $\theta = a\Delta_{21}$. Moreover, $\theta^{\perp} \cap \mathrm{Ann}(R) = 0$ if and only if $a \neq 0$. The automorphism group acts as follows $a \to aa_{11}^2$. Then we choose $a_{11} = \frac{1}{\sqrt[3]{a}}$ such that a is mapped to 1 this yield $e_1e_1 = e_3$.

5 Applications

Let R be a Zinbiel algebra and $Ann(R^{'})$ be its annihilator which we suppose to be nonzero. Set $V = Ann(R^{'})$ and $R = R^{'}/Ann(R^{'})$ Then there is a such that $R^{'} = R_\theta$. We conclude that any Zinbiel algebra with a nontrivial annihilator can be obtained as an annihilator extension of a Zinbiel algebra of smaller dimension. So in particular, all nilpotent Zinbiel algebras can be constructed by this way.

Procedure: Let R be a Zinbiel algebra of dimension $n - s$. The procedure outputs all nilpotent Zinbiel algebras $R^{'}$ of dimension n such that $R^{'}/Ann(R^{'})$. It runs as follows:

- Determine $Z^2(R, \mathbb{K})$, $B^2(R, \mathbb{K})$ and $H^2(R, \mathbb{K})$
- Determine the orbits of $Aut(R)$ on s-dimensional subspaces of $H^2(R, \mathbb{K})$.
- For each of the orbits let θ be the cocycle corresponding to a representative of it, and construct R_θ.

Finally, let us introduce some notation. Let R be a Zinbiel algebra with a basis $e_1, e_2, e_3, \cdots, e_n$. Then by $\Delta_{ij} : R \times R \to \mathbb{C}$ with $\Delta_{ij}(e_s, e_t) = 1$ if $\{i, j\} = \{s, t\}$ and $\Delta_{ij}(e_s, e_t) = 0$ if $\{i, j\} \neq \{s, t\}$. Then the set $\{\Delta_{ij} : 1 \leq i \leq j \leq n\}$ is a basis for the vector of the bilinear forms on R. Then every $\theta \in Z^2(R, \mathbb{K})$ can be uniquely written as $\theta = \sum\limits_{1 \leq i \leq j \leq n} c_{ij}\Delta_{ij}$, where $c_{ij} \in \mathbb{C}$. In this part we make use the description of three and four dimensional Zinbiel algebras over $\mathbb{C}$ in [8]

6 Central extension of three dimensional Zinbiel algebras

The algebraic classification of three-dimensional Zinbiel algebras. There exists only one three-dimensional nilpotent Zinbiel algebra. We have the list of all three-dimensional nilpotent Zinbiel algebras as follows: The annihila-

R	Isomorphism class	$H^2(R,\mathbb{C})$	$\mathrm{Aut}(R)$
$R_{3,3}$	$e_1e_1 = e_3,\ e_1e_1 = e_3$	$\Delta_{12}, \Delta_{13}, \Delta_{33}$	$\begin{pmatrix} a_{22}a_{33} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$
$R_{3,6}$	$e_1e_1 = e_3, e_1e_2 = e_3, e_2e_2 = e_3$	Δ_{11}, Δ_{12}	$\begin{pmatrix} a_{33}^2 & 0 & a_{13} \\ a_{13}a_{33} & 1 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$

tor Extension of $R_{3,3}$: $e_1e_1 = e_3,\ e_1e_1 = e_3$. The basis of $HL^2(L,\mathbb{K}) = \mathrm{span}\{\Delta_{1,3}, \Delta_{2,2}, \Delta_{3,2}, \Delta_{3,3}\}$. The annihilator $\mathrm{Ann}(R_{3,3}) = \mathrm{span}\{e_1\}$. By Theorem, we need to find the representatives of $\mathrm{Aut}(R_{3,3})$-orbit on $W_4(R_{3,3})$. Choose an arbitrary subspace $T \in W_9(R_{3,3})$.
$\theta = [a, b, c, d] = a\Delta_{1,2} + b\Delta_{2.2} + c\Delta_{3,2} + d\Delta_{3,3}$, such that $\theta^{\perp} \cap \{e_1\} = 0$.
The automorphism group, $\mathrm{Aut}(A_{3,9})$ consist of matric of the form

$$\phi = \begin{pmatrix} a_{22}a_{33} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \text{with} \quad a_{22}a_{33} \neq 0$$

The automorphism group ϕ act on T as follows:
$a \to aa_{22}a_{33}^2$, $b \to ba_{22}^2$, $c \to ca_{22}a_{33}$ and $d \to da_{33}^2$
We now consider the following cases:
Case 1**:** If $b \neq 0$.

Case 1.1**:** Let $a = 0$. By taking $a_{33} = 1$ and $a_{22}\dfrac{1}{c}$, we have $c \to 1$. To fix $c = 1$ requires $a_{22} = 1$. Then
$a \to 0$, $b \to \frac{b}{c^2} = a$, $c \to 1$ and $d \to d$.
In this case we obtain the representative $T_1 = [a, b, c, d] = [0, a, 1, 0]$.
Hence we get the algebra

$$e_2e_2 = ae_4, e_2e_3 = e_1, e_3e_2 = e_4$$

.
Case 1.2**:** Let $a \neq 0$. By Taking $a_{22} = \frac{1}{aa_{33}^2}$, we have $a \to 1$ and to fix it require that $a_{22} = 1$. So we have
$a \to 1$, $b \to \frac{b}{a_{33}}$, $c \to \frac{c}{a_{33}}$ and $d \to \frac{d}{a_{33}}$

Case 1.2.1 : Assume that $c = 0$. Setting $a_{33} = b$, we have
$a \to 1$, $b \to 1$, $c \to 0$ and $d \to \frac{d}{b}$
If $\frac{d}{b} = 0$ we get
$a \to 1$, $b \to 1$, $c \to 0$ and $d \to 0$
This, give rise to the following representative $T_2 = [a, b, c, d] = [1, 1, 0, 0]$. Hence we get the algebra

$$e_1e_3 = e_4, e_2e_2 = e_4, e_2e_3 = e_1$$

.
Case 1.2.2**:** Assume that $c \neq 0$. Setting $a_{33} = c$, we have
$a \to 1$, $b \to \frac{b}{c}$, $c \to 1$ and $d \to \frac{d}{c}$
Taking $\dfrac{d}{c} = 0$ we have
$a \to 1$, $b \to \frac{b}{c} = a$, $c \to 1$ and $d \to 0$
Thus, we get the following representative $T_4 = [a, b, c, d] = [0, a, 1, 0]$. Hence we get the algebra

$$e_1e_3 = e_4, e_2e_2 = ae_4, e_2e_3 = e_1, e_3e_2 = e_4$$

Case 2**:** By taking $b = 0$. Setting $a_{33} = 1, \dfrac{1}{2}$ we obtain $a \to 1$ and to fix it requires $a_{22} = 1$. Hence
$a \to 0$, $b\dfrac{b}{c}$, $c \to c$ and $d \to d$
Now get we have the following representative $T_4 = [a, b, c, d] = [0, \alpha, 1, 0]$. Hence we get the algebra

$$e_2e_2 = \alpha e_4, e_2e_3 = e_1, e_3e_2 = e_4$$

.
Similarly algebra for $\beta = 0, 1$. Then we the following algebras:

$$e_1e_3 = e_4, e_2e_1 = \alpha e_4, e_2e_3 = e_1, e_3e_2 = e_4$$

For $\alpha \in \mathbb{C}$, we have the following algebras:
$R_{4,1}:\ \ e_1e_3 = e_4,\ \ e_2e_3 = e_1,\ \ e_3e_2 = e_4,$
$R_{4,2}:\ \ e_1e_2 = e_4,\ \ e_2e_1 = e_4,\ \ e_2e_2 = e_1,\ \ e_3e_2 = e_4,$ and for $\beta = 2$ in equa-
$R_{4,3}:\ \ e_1e_3 = e_4,\ \ e_2e_1 = e_4,\ \ e_2e_3 = e_1,\ \ e_3e_2 = e_4.$
tion 4, we have
$R_{4,4} : e_2, e_2 = e_4,\ \ e_2, e_3 = e_1\ \ e_3, e_2 = 2e_4$ The annihilator extension of $R_{3,6}$: $e_1e_1 = e_3, e_1e_2 = e_3, e_2e_2 = e_3$. Here we get the basis of $HL^2(R, \mathbb{K}) = \text{span}\{\Delta_{2,3}\}$. Furthermore, the annihilator $\text{Ann}(R_{3,6}) = \text{span}\{e_2\}$.

According to Theorem we need to find the representatives of $\mathrm{Ann}(R_{3,6})$-orbit on $W_6(R_{3,6})$. Consider an arbitrary subspace $T \in W_6(R_{3,6})$ i.e, a subspace spanned by

$$\theta(\alpha) = a\Delta_{2,3} \quad \text{such that} \quad \theta^{\perp} \cap \{e_1\} = \{0\}$$

The automorphism group, $\mathrm{Aut}(R_{3,6})$ consists of matrices of the form

$$\phi = \begin{pmatrix} a_{33}^2 & 0 & a_{13} \\ a_{13}a_{33} & 1 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \text{with} \quad a_{33} \neq 0$$

Then, θ acts on T as follows:
$a \to aa_{33}$

Assume that, $a \neq 0$. If $a_{33}\frac{1}{a}$, then we have $a \to 1$. To fix $a = 1$, we require $a_{33} = 1$. Then we get a representative $T_1 = [\alpha] = [1]$. So we get the following algebra:
$R_{4,5} : e_1e_3 = e_2, \quad e_2e_3 = e_4 \quad e_3e_3 = e_1$

7 Central extension of four dimensional Zinbiel algebras

The algebraic classification of four dimensional Zinbiel algebras as follows: The

R	Isomorphism class	$H^2(R,\mathbb{C})$	$\mathrm{Aut}(R)$
$R_{4,4}$	$e_1e_2 = e_3, e_1e_3 = e_4, e_2e_1 = -e_3$	$\Delta_{12}, \Delta_{21}, \Delta_{22}, \Delta_{41}$	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ \alpha_{21} & a_{22} & 0 & 0 \\ a_{31} & 0 & a_{11}^2 & 0 \\ a_{41} & a_{42} & a_{31} & a_{11}^3 \end{pmatrix}$
$R_{4,7}$	$e_1e_2 = e_3, e_1e_2 = e_4$	$\Delta_{12}, \Delta_{22}, \Delta_{44}$	-
$R_{4,12}$	$e_1e_2 = e_3, e_2e_1 = e_4$	$\Delta_{11}, \Delta_{21}, \Delta_{22}, \Delta_{23}$	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{11}a_{22} & 0 \\ a_{41} & a_{42} & a_{11}a_{32} & a_{11}^2a_{22} \end{pmatrix}$
$R_{4,16}$	$e_1e_2 = e_4, e_2e_1 = -e_4, e_3e_3 = e_4$	$\Delta_{12}, \Delta_{21}, \Delta_{32}$	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11}^2 & 0 & 0 \\ a_{31} & 0 & a_{11}^2 & 0 \\ a_{41} & a_{42} & a_{11}a_{21} & a_{11}^3 \end{pmatrix}$

central extension of $R_{4,4} : e_1e_2 = e_3, e_1e_3 = e_4, e_2e_1 = -e_3$. Here we get the basis of $HR^2(R,\mathbb{K}) = span\{\Delta_{12}, \Delta_{21}, \Delta_{22}, \Delta_{41}\}$. The automorphism group,

$Aut(R_{4,4})$ consists of matrices of the form

$$\theta = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ \alpha_{21} & a_{22} & 0 & 0 \\ a_{31} & 0 & a_{11}^2 & 0 \\ a_{41} & a_{42} & a_{31} & a_{11}^3 \end{pmatrix}.$$

Then, ϕ acts on T as follows $a \to aa_{11}^3$.

Suppose that, $a \neq 0$. Choosing $a_{11}^3 = \frac{1}{a}$, then we have $a \to 1$. To fix $a = 1$, we require $a_{11} = 1$. Then we get a representative $T_1 = [a] = [1]$. So we get the following algebra:

$R_{5,1}: e_1e_1 = e_3 \quad e_2e_1 = e_3 \quad e_3e_1 = e_4 \quad e_4e_1 = e_5 \quad e_2e_2 = e_4 - 2e_5.$

The algebraic classification of 5-dimensional nilpotent Zinbiel algebras as follows:

$$\begin{array}{llllll}
R_{5,1}: e_1e_1 = e_3 & e_2e_1 = e_3 & e_3e_1 = e_4 & e_4e_1 = e_5 & e_2e_2 = e_4 - 2e_5 \\
R_{5,2}: e_1e_2 = e_3 & e_2e_1 = -e_3 & e_1e_3 = e_4 & e_3e_1 = -e_4 & e_1e_1 = e_5 \\
 & e_2e_2 = e_5 & e_1e_4 = e_5 & e_4e_1 = -e_5 & \\
R_{5,3}: e_1e_2 = e_3 & e_2e_1 = -e_3 & e_1e_3 = e_4 & e_3e_1 = -e_4 & e_2e_2 = e_5 \\
 & e_1e_4 = e_5 & e_4e_1 = -e_5 & & \\
R_{5,4}: e_1e_2 = e_3 & e_2e_1 = -e_3 & e_1e_3 = -e_4 & e_3e_1 = -e_4 & e_2e_1 = e_5 \\
 & e_1e_4 = e_5 & e_4e_1 = -e_5 & & \\
R_{5,5}: e_1e_2 = e_3 & e_2e_1 = -e_3 & e_1e_3 = e_4 & e_3e_1 = -e_4 & e_1e_1 = e_5 \\
 & e_1e_4 = e_5 & e_4e_1 = -e_5 & &
\end{array}$$

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