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On an Algorithm for Finding Derivations of Associative Algebras

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Abstract

The paper addresses derivation of associative algebras in dimension two and three. Our approach to the derivation is based on an algorithm that uses the result on classification of finite dimensional associative algebras. At the end, derivation algebras with their dimensions for two and three dimensional complex associative algebras are tabulated.

Keywords: Associative algebra, derivation, right and left multiplication operators

1 Introduction

Associative algebras are one of the classical algebras that have extensively been studied and found to be related to other classical algebras like Lie and Jordan algebras. The classification of low dimensional associative algebras is believed to have been first investigated by Peirce [1]. In 1916, Hazlett classify nilpotent algebras of dimension ≤ 4 [2]. Mazzola published his result in 1979 on algebraic and geometric classification of associative algebras in dimension five

[3]. Most classification problems of finite dimensional associative algebras have been studied for certain property(s) of associative algebras while the complete classification of associative algebras in general is still an open problem. This study centers on derivation of low dimensional associative algebras. We use the results obtained in [4] on classification of two and three dimensional associative algebras to achieve our goal [5], [6].
This study is organized as follows. In the first section, we introduce the subject alongside with some previously obtained results. In Section 2, we provide some basic concepts needed for this study. subsequent Sections are dedicated to main result, acknowledgement and references respectively.

Definition 1.1. *Let A be a vector space(finite or infinite) over the complex $\mathbb{C}$. Suppose a binary operation xy $(x, y \in A)$ called multiplication is defined on A, i.e. for any $x, y \in A$, there exists a unique $w \in A$ such that $w = xy$. If for any $x, y, z \in A$ and $\lambda \in \mathbb{C}$, the conditions:*

$$x(y+z) = xy + xz, (y+z)x = yx + zx \quad (1)$$

$$x(yz) = (xy)z \quad (2)$$

$$(\lambda x)y = x(\lambda y) = \lambda(xy) \quad (3)$$

Definition 1.2. *Let A_1 and A_2 be two associative algebras over F. A homomorphism between A_1 and A_2 is an F-linear mapping $\varphi : A_1 \longrightarrow A_2$ such that*

$$\varphi(xy) = \varphi(x)\varphi(y)$$

for all $x, y \in A_1$.

The set of all homomorphism of A is denoted by $Hom(A)$.
In what follows, we define

$$[\cdot,\cdot] : A \times A \longrightarrow A$$

by $[x, y] = xy - yx$ for all $x, y \in A$.

Definition 1.3. *A derivation of associative algebra A is a linear transformation*
$d : A \to A$ where

$$d(x \cdot y) = d(x)y + xd(y) \qquad \forall \quad x, y \in A$$

The set of all derivations of a associative algebra A we denote by $Der(A)$. The $Der(A)$ is an associative algebra with respect the composition operation $\circ$ and it is a Lie algebra with respect to the bracket $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$.

Theorem 1.1. *If d is a derivation of an associative algebra A, then d is a derivation of A as a lie algebra.*

2 Main Results

Proposition 2.1. *Let A be an associative algebra. Then $Der(A)$ is a Lie algebra with respect to the bracket $[f,g] = f \circ g - g \circ f$.*

In an associative algebra we consider the right r_a, and the left l_a, multiplication operators defined as follows:

$$r_a(b) := b \cdot a$$
$$l_x(b) := a \cdot b$$

Lemma 2.1. *The sets $r(A) = \{r_a | a \in A\}$, $l(A) = \{l_a | a \in A\}$ are subalgebras of the associative algebra $Der(A)$.*

Proof. The proof can be easily derived from the following identities

$$r_{a \cdot b} = r_b r_a, \quad l_{a \cdot b} = l_b l_a,$$

□

Lemma 2.2. *Let $(A, \cdot)$ be an associative algebra and $d \in Hom(A)$, then the following conditions are equivalent:*

$$i.\ d \in Der(A) \quad ii.\ [d, l_a] = l_{d(a)} \quad and \quad iii.\ [d, r_a] = R_{d(a)}$$

Definition 2.1. *The sets $Ann_R(A)$ and $Ann_L(A)$ defined by*

$$Ann_R(A) = \{x \in A | A \cdot x = 0\}$$

and

$$Ann_L(A) = \{x \in A | x \cdot A = 0\}$$

of an associative algebra A are called the right and the left annihilators of A respectively.

Theorem 2.1. *Let $f : A_1 \to A_2$ be an isomorphism of complex algebras A_1 and A_2. Then the mapping $\rho : EndA_1) \to End(A_2)$ defines by $\rho(d) = f \circ d \circ f^1$, is an isomorphism of vector spaces $Der(A_1)$ and $Der(A_2)$, i.e*

$$\rho(Der(A_1)) = Der(A_2).$$

Proof. Suppose we have $A_1 = (V_1, \cdot)$ and $A_2 = (V_2, *)$. The isomorphism relation $f(x \cdot y) = f(x) \cdot f(y)$ holds for all $x, y \in A_1$, implies that for all $x, y \in A_2$

$$x * y = f\big(f^{-1}(x) \cdot f^{-1}(y)\big)$$

By rewriting the definition (1.3) we have $d \in Der(A_1)$

$$d\big(f^{-1}(x) \cdot f^{-1}(y)\big) = d(f^{-1}(x)) \cdot f^{-1}(y) + f^{-1}(x) \cdot d(f^{-1}(y)).$$

Applying the mapping f on this equation we have

$$f \circ d \circ f^{-1}(x * y) = \big(f \circ d \circ f^{-1}(x)\big) * y + x * \big(f \circ d \circ f^{-1}(y)\big),$$

i. e. $f \circ d \circ f^{-1} \in Der(A_2)$. □

3 An algorithm for finding derivations

Provided that $\{e_1, e_2, \cdots, e_n\}$ are the basis of A as n-dimensional complex associative algebra, then the $e_i e_j$ components with $i, j = 1, 2, \cdots, n$ can be referred to as the A structure constants on the basis $\{e_1, e_2, \cdots, e_n\}$. if

$$e_i e_j = \sum_{k=1}^{n} \gamma_{ij}^k e_k$$

then

$$\{\gamma_{ij}^k, i, j, k, \leq n\}$$

is denoted the set of structure constants of A. Further all the algebras considered are supposed to be over the field of complex numbers $\mathbb{C}$. It is important to discuss the associative algebras derivations. So, in matrix, $d = (d_{ij})_{i,j=1,2,\cdots,n}$ with the basis as $\{e_1, e_2, \cdots, e_n\}$. If the structure constants $\{\gamma_{ij}^k\}$ are given then we form a system of equations with respect to d_{ij} and solving this system we get the descriptions of the derivations. We get the system in the form presented below:

$$\sum_{k=1}^{n} \gamma_{ij}^k d_{tk} = \sum_{k=1}^{n} (d_{ki}\gamma_{kj}^t + d_{kj}\gamma_{ik}^t) \tag{4}$$

for $1 \leq i, j, t \leq n$.

The next sections are devoted to the applications of the algorithm to low-dimensional associative algebras cases.

3.1 Two-dimensional associative algebras

The classification of all two-dimensional associative algebras has been given by [4]. The list of isomorphism classes is given by the following theorem.

Theorem 3.1. *Let A be 2-dimensional complex associative algebra. Then it is isomorphic to one of the following pairwise non-isomorphic associative algebras:*
$As_2^1 : e_1e_1 = e_2;$
$As_2^2 : e_1e_1 = e_1,\ e_1e_2 = e_2;$
$As_2^3 : e_1e_1 = e_1,\ e_2e_1 = e_2;$
$As_2^4 : e_1e_1 = e_1,\ e_2e_2 = e_2;$
$As_2^5 : e_1e_1 = e_1,\ e_1e_2 = e_2,\ e_2e_1 = e_2.$

Let us describe the derivations of 2-dimensional complex associative algebras. In the description, we use the classification result stated in Theorem 3.1 above.

Table 1: Derivations of two-dimensional associative algebras

IC	Derivation	Dim	IC	Derivation	Dim
As_2^1	$\begin{pmatrix} d_{11} & 0 \\ d_{21} & 2d_{11} \end{pmatrix}$	2	As_2^2	$\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$	2
As_2^3	$\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$	2	As_2^4	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0
As_2^5	$\begin{pmatrix} 0 & 0 \\ 0 & d_{22} \end{pmatrix}$	1			

3.2 Three-dimensional associative algebras

The classification of all three-dimensional complex associative algebras has been given in [4]. The following theorem gives the list of isomorphism classes.

Theorem 3.2. *Any* 3*-dimensional non decomposable complex associative algebra* A *is isomorphic to one of the following pairwise non-isomorphic algebras.*

$As_3^1:\quad e_1e_3 = e_2,\quad e_3e_1 = e_2;$
$As_3^2(\alpha):\quad e_1e_3 = e_2,\quad e_3e_1 = \alpha e_2,\quad \alpha \in \mathbb{C}\backslash\{1\};$
$As_3^3:\quad e_1e_1 = e_2,\quad e_1e_2 = e_3,\quad e_2e_1 = e_3;$
$As_3^4:\quad e_1e_3 = e_2,\quad e_2e_3 = e_2,\quad e_3e_3 = e_3;$
$As_3^5:\quad e_2e_3 = e_2,\quad e_3e_1 = e_1,\quad e_3e_3 = e_3;$
$As_3^5:\quad e_2e_3 = e_2,\quad e_3e_1 = e_1,\quad e_3e_3 = e_3;$
$As_3^6:\quad e_3e_1 = e_2,\quad e_3e_2 = e_2,\quad e_3e_3 = e_3;$
$As_3^7:\quad e_1e_2 = e_1,\quad e_2e_2 = e_2,\quad e_3e_1 = e_1,\quad e_3e_3 = e_3;$
$As_3^8:\quad e_1e_3 = e_1,\quad e_2e_3 = e_2,\quad e_3e_1 = e_1,\quad e_3e_3 = e_3;$
$As_3^9:\quad e_2e_3 = e_2,\quad e_3e_1 = e_1,\quad e_3e_2 = e_2,\quad e_3e_3 = e_3;$
$As_3^{10}:\quad e_1e_3 = e_1,\quad e_2e_3 = e_2,\quad e_3e_1 = e_1,\quad e_3e_2 = e_2,\quad e_3e_3 = e_3;$
$As_3^{11}:\quad e_1e_3 = e_2,\quad e_2e_3 = e_2,\quad e_3e_1 = e_2,\quad e_3e_2 = e_2,\quad e_3e_3 = e_3;$
$As_3^{12}:\quad e_1e_1 = e_2,\quad e_1e_3 = e_1,\quad e_2e_3 = e_2,\quad e_3e_1 = e_1,\quad e_3e_2 = e_2,$
$\quad e_3e_3 = e_3;$
$As_3^{13}:\quad e_1e_1 = e_1,\quad e_2e_2 = e_2,\quad e_3e_3 = e_3;$
$As_3^{14}:\quad e_1e_2 = e_1,\quad e_2e_1 = e_1,\quad e_2e_2 = e_2,\quad e_3e_3 = e_3;$
$As_3^{14}:\quad e_1e_2 = e_1,\quad e_2e_1 = e_1,\quad e_2e_2 = e_2,\quad e_3e_3 = e_3;$
$As_3^{15}:\quad e_1e_2 = e_1,\quad e_2e_2 = e_2,\quad e_3e_3 = e_3;$
$As_3^{16}:\quad e_2e_1 = e_1,\quad e_2e_2 = e_2,\quad e_3e_3 = e_3;$
$As_3^{17}:\quad e_1e_1 = e_2,\quad e_3e_3 = e_3.$

In the Table 2, we give the description of derivation of three-dimensional complex associative algebras.

Theorem 3.3. *The derivations of Three dimensional complex associative algebras are given as follows*

Table 2: Derivations of three-dimensional associative algebras

IC	Derivation	Dim	IC	Derivation	Dim
As_3^1	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & m & d_{23} \\ 0 & 0 & d_{33} \end{pmatrix}$	4	$As_3^2(\alpha)$	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & d_{22} & d_{23} \\ 0 & 0 & m_1 \end{pmatrix}$	4
As_3^3	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & 2d_{11} & 0 \\ d_{31} & 2d_{21} & 3d_{11} \end{pmatrix}$	3	As_3^4	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & m_2 & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$	3
As_3^5	$\begin{pmatrix} d_{11} & 0 & d_{13} \\ 0 & d_{22} & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$	4	As_3^6	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & m_2 & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$	3
As_3^7	$\begin{pmatrix} d_{11} & d_{12} & -d_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2	As_3^8	$\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$	3
As_3^9	$\begin{pmatrix} d_{11} & 0 & d_{13} \\ 0 & d_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	3	As_3^{10}	$\begin{pmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	4

IC	Derivation	Dim	IC	Derivation	Dim
As_3^{11}	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & m_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2	As_3^{12}	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & 2d_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2
As_3^{13}	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0	As_3^{14}	$\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	1
As_3^{15}	$\begin{pmatrix} d_{11} & d_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2	As_3^{16}	$\begin{pmatrix} d_{11} & d_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2
As_3^{17}	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & 2d_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2			

where $m = d_{11} + d_{33}, \quad m_1 = d_{22} - d_{11}, \quad m_2 = d_{11} + d_{21}.$

Proof. Let us consider As_3^1 and by using the system of equation as mentioned (4) we get $d_{12} = d_{13} = d_{31} = d_{32} = 0$ and $d_{22} = d_{11} + d_{33} = m$.

Therefore the derivations of As_3^1 are given as follows $d := \begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & m & d_{23} \\ 0 & 0 & d_{33} \end{pmatrix}$.

Clearly,

$$d_1(e_1) = e_1 \quad \text{and} \quad d_1(e_2) = e_1;$$

$$d_2(e_2) = e_2 \quad \text{and} \quad d_2(e_3) = e_3;$$

$$d_3(e_1) = e_2 \quad \text{and} \quad d_4(e_3) = e_2.$$

So, $d_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $d_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $d_3 := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $d_4 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

is a basis of $Der(A)$ and $dimDer(A) = 4$.
The derivations of the rest in Theorem 3.2 can be given similarly as shown above. □

Corollary 3.1. *i There are one characteristically nilpotent classes in the list of isomorphism classes of three- dimensional associative algebras.*

ii The dimensions of the derivation algebras in this case very between zero and four.

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