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# On Hypercyclicity $\infty$-Tuples of Commutative Bounded Linear Operators

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**Abstract**

In this paper we study the epsilon hypercyclicity on $\infty$-tuple makes with commutative bounded linear operators on Banach spaces.

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**Keywords:** Hypercyclic vector, $\infty$-tuple, Hypercyclicity Criterion, Banach space, Epsilon hypercyclicity

## 1 Introduction

If $T_1, T_2, ..., T_n, ...$ be commutative bounded linear mappings on a Banach space $\mathcal{X}$ and $\mathcal{T} = (T_1, T_2, ..., T_n, ...)$, by $\mathcal{T}(x)$ or $T_1^{m_1}T_2^{m_2}...T_n^{m_n}...(x)$ we mean

$$Sup_{n\to\infty}\{T_1^{m_1}T_2^{m_2}...T_n^{m_n}(x) : m_1, m_2, ..., m_n \geq 0\}$$

So

$$\begin{aligned}\mathcal{T}(x) &= T_1^{m_1}T_2^{m_2}...T_n^{m_n}...(x)\\ &= Sup_{n\to\infty}\{T_1^{m_1}T_2^{m_2}...T_n^{m_n}(x) : m_1, m_2, ..., m_n \geq 0\}\end{aligned}$$

**Definition 1.1** *Let $T_1, T_2, ..., T_n, ...$ be commutative bounded linear operators on a Banach space $\mathcal{X}$. For $\infty$-tuple $\mathcal{T} = (T_1, T_2, ..., T_n, ...)$, put*

$$\Gamma = \{T_1^{m_1} T_2^{m_2} ... T_n^{m_n} ... : m_1, m_2, ..., m_n, ... \geq 0\}$$

*For $x \in \mathcal{X}$, the orbit of $x$ under $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{S(x) : S \in \Gamma\}$, that is*

$$Orb(\mathcal{T}, x) = \{T_1^{m_1} T_2^{m_2} ... T_n^{m_n} ...(x) : m_1, m_2, ..., m_n ... \geq 0\}$$

*The vector $x$ is called hypercyclic vector for $\mathcal{T}$ and $\infty$-tuple $\mathcal{T}$ is called hypercyclic $\infty$-tuple, if the set $Orb(\mathcal{T}, x)$ is dense in $\mathcal{X}$, that is*

$$\overline{Orb(\mathcal{T}, x)} = \overline{\{T_1^{m_1} T_2^{m_2} ... T_n^{m_n} ...(x) : m_1, m_2, ..., m_n ... \geq 0\}} = \mathcal{X}$$

*Also if $\epsilon$ be a number in $(0, 1)$ and $x$ be a vector of $\mathcal{X}$. The vector $x$ is called $\epsilon$-hypercyclic vector for $\infty$-tuple $\mathcal{T} = (T_1, T_2, ..., T_n, ...)$ and the $\infty$-tuple $\mathcal{T}$ is called $\epsilon$-hypercyclic $\infty$-tuple, if for every non zero vector $y \in \mathcal{X}$, there exist some integers $m_1, m_2, ..., m_n, ...$ such that*

$$\| T_1^{m_1} T_2^{m_2} ... T_2^{m_n} ... x - y \| < \epsilon \| y \|$$

All operators in this paper are commutative operators on a Banach space $\mathcal{X}$. Readers can see $[1 - -10]$ for more information.

# 2 Main Results

The bellow theorem namely the Hypercyclicity Criterion is useful theorem in operator theory, that it used in papular theorem's proof, if $\infty$-tuple $\mathcal{T}$ satisfy this theorem then we say that it satisfying the The Hypercyclicity Criterion.

**Theorem 2.1 (The Hypercyclicity Criterion)** *Let $\mathcal{X}$ be a separable Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n, ...)$ is an $\infty$-tuple of continuous linear mappings on $\mathcal{X}$. If there exist two dense subsets $\mathcal{Y}$ and $\mathcal{Z}$ in $\mathcal{X}$, and $n$ strictly increasing sequences $\{m_{j,1}\}, \{m_{j,2}\}, ..., \{m_{j,n}\}, ...$ such that :*
*1. $T_1^{m_{j,1}} T_2^{m_{j,2}} ... T_n^{m_{j,n}} ... \to 0$ on $\mathcal{Y}$ as $j \to \infty$,*
*2. There exist function $\{S_j : Z \to \mathcal{X}\}$ such that for every $z \in Z$, $S_j z \to 0$, and $T_1^{m_{j,1}} T_2^{m_{j,2}} ... T_n^{m_{j,n}} ... S_j z \to z$,*
*then $\mathcal{T}$ is a hypercyclic $\infty$-tuple.*

**Theorem 2.2** *Let $\mathcal{X}$ be a separable Hilbert space and $\mathcal{T} = (T_1, T_2, ..., T_n, ...)$ be an $\infty$-tuple of commutative bounded linear operators on a Hilbert space $\mathcal{X}$. If for every $\epsilon > 0$, the $\infty$-tuple $\mathcal{T}$ is $\epsilon$-hypercyclic, then $\mathcal{T}$ is a hypercyclic .*

**Proof** . Note that, if $\mathcal{T}$ is a Hypercyclic $\infty$-tuple, then $\sigma_p(T^*) = \phi$, ($\mathcal{T}^* = (T_1^*, T_2^*, ..., T_n^*, ...)$), also all spaces that admitted some hypercyclic operator, are infinite dimensional spaces, so we can assume that $\mathcal{X}$ be infinite dimensional space. Suppose that $\mathcal{U}$ and $\mathcal{V}$ are subset of $\mathcal{X}$. Give $u \in \mathcal{U}, v \in \mathcal{V}$ two nonzero element and $\delta > 0$ so large that $B(u,\delta) \subset \mathcal{U}$ and $B(v,\delta) \subset \mathcal{V}$ so that $\delta < Max\{\| u \|, \| v \|\}$. Take $x$ such that $x$ be an $\epsilon$-hypercyclic for $\mathcal{T}$ with property $\epsilon < \frac{\delta}{6Max\{\|u\|,\|v\|\}}$, then there exist $m_{1,0}, m_{2,0}, ..., m_{n,0}, ...$ such that

$$\| T_1^{m_{1,0}} T_2^{m_{2,0}} ... T_n^{m_{n,0}} ... x - u \| < \epsilon \| u \| < \delta$$

thus we have

$$T_1^{m_1} T_2^{m_2} ... T_n^{m_n} ... x \in \mathcal{U}$$

Suppose on the contrary that there are only finitely many such integers

$$m_{1,1}, m_{2,1}, \ldots, m_{n,1}$$

$$m_{1,2}, m_{2,2}, \ldots, m_{n,2}$$

$$\ldots$$

$$m_{1,t}, m_{2,t}, \ldots, m_{n,t}$$

$$\ldots$$

As above, for each $u' \in \mathcal{X}$ with

$$\| u' - u \| < \frac{2\delta}{3}$$

there exist integers

$$m_1(u'), m_2(u'), ..., m_n(u'), ...$$

which satisfies

$$\| T_1^{m_1(u')} T_2^{m_2(u')} ... T_n^{m_n(u')} ... x - u' \| \leq \epsilon \| u' \| 2\epsilon \| u \| < \frac{\delta}{3}$$

Since

$$\| T_1^{m_1(u')} T_2^{m_2(u')} ... T_n^{m_n(u')} ... x - u' \| \leq \| T_1^{m_1(u')} T_2^{m_2(u')} ... T_n^{m_n(u')} ... x - u \| + \| u - u' \| < \delta$$

we have

$$m_k(u') \in \{m_{1,k}, m_{2,k}, ..., m_{n,k}, ...\}$$

for $k = 1, 2, ..., t, ...$ and the ball $B(u, \frac{2\delta}{3})$ is covered by a finite number balls

$$B(T_1^{m_{1,1}} T_2^{m_{2,1}} ... T_n^{m_{n,1}} x, \frac{\delta}{3}, ...)$$

$$B(T_1^{m_{1,2}}T_2^{m_{2,2}}...T_n^{m_{n,2}}x, \frac{\delta}{3}, ...)$$

$$\cdots$$

$$B(T_1^{m_{1,t}}T_2^{m_{2,t}}...T_n^{m_{n,t}}x, \frac{\delta}{3}, ...)$$

$$\cdots$$

Thus in an infinite dimensional space this is impossible. So there are infinitely many integers as $m_1, m_2, \ldots, m_n, ...$ with

$$\| T_1^{m_1}T_2^{m_2}...T_n^{m_n}...x - u \| < \delta$$

Then there exist $m_{i,k} > m_{i,0}$ for $k = 1, 2, ..., t, ...$ and $i = 1, 2, ..., n, ...$ such that

$$T_1^{m_1}T_2^{m_2}...T_n^{m_n}...x \in \mathcal{V}$$

Thus

$$T_1^{m_{1,1}-m_{1,0}}T_2^{m_{2,2}-m_{2,0}}...T_n^{m_{n,n}-m_{n,0}}...T_1^{m_{1,0}}T_2^{m_{2,0}}...T_n^{m_{n,0}}...x$$

is belong to

$$V \bigcap T_1^{m_{1,1}-m_{1,0}}T_2^{m_{2,2}-m_{2,0}}...T_n^{m_{n,n}-m_{n,0}}...(U)$$

that is

$$T_1^{m_{1,1}}T_2^{m_{2,2}}...T_n^{m_{n,n}}...x \in (V \bigcap T_1^{m_{1,1}-m_{1,0}}T_2^{m_{2,2}-m_{2,0}}...T_n^{m_{n,n}-m_{n,0}}...)$$

Here it can be concluded that $\mathcal{T}$ is hypercyclic $\infty$-tuple.

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