

# A Blow-up Criterion of Strong Solutions to the 3D Incompressible Kelvin-Voigt-Cahn-Hilliard System

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## Abstract

In this paper, we study the initial-boundary value problem of 3D incompressible Kelvin-Voigt-Cahn-Hilliard system for the case of initial density away from zero. We construct a new blow-up mechanism of strong solution.

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**Keywords:** initial-boundary value problem; incompressible Kelvin-Voigt-Cahn-Hilliard system; blow-up criterion.

## 1 Introduction

In the present paper, we are interested in the study of incompressible diphasic nonhomogeneous mixtures flows. The model consists of a Cahn-Hilliard equation coupled with a nonhomogeneous Kelvin-Voigt equations derived by

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Oskolkov [10] in 1973, which can be written as follows:

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, \\ \rho \partial_t \mathbf{u} + \rho(\mathbf{u} \cdot \nabla) \mathbf{u} - \operatorname{div}(\nu(\phi) D\mathbf{u}) + \nabla P \\ = \chi \Delta \partial_t \mathbf{u} + \rho \mathbf{f} - \operatorname{div}(\nabla \phi \otimes \nabla \phi), \\ \operatorname{div} \mathbf{u} = 0, \\ \rho \partial_t \phi + \rho \mathbf{u} \cdot \nabla \phi = \Delta \mu, \\ \rho \mu = -\Delta \phi + \rho \Phi'(\phi), \end{cases} \quad (1)$$

in  $\Omega \times (0, T)$ , where  $\Omega$  is a bounded domain in  $\mathbb{R}^d (d = 3)$  with a regular boundary  $\partial\Omega$ ,  $T > 0$  is a given positive time,  $\rho = \rho(x, t)$  is the density of the mixture,  $\mathbf{u} = \mathbf{u}(x, t)$  is the velocity of the mixture,  $\mathbf{f} = \mathbf{f}(x, t)$  is the external force, and in this paper we consider  $\mathbf{f} = 0$ ,  $P = P(x, t)$  is the pressure of the mixture,  $\varphi = \varphi(x, t)$  is the difference of fluids concentrations,  $\mu = \mu(x, t)$  is the chemical potential,  $\chi > 0$  is assumed to be a positive constant,  $D\mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^t)$  stands for the deformation tensor, whereas the viscosity function  $\nu$ , the potential  $\Phi(s)$  and the boundary and initial conditions satisfy:

$$\nu = \nu(s) \in W^{1,\infty}(\mathbb{R}), \quad 0 < \nu_* \leq \nu(s) \leq \nu^* \quad \text{for all } s \in \mathbb{R}. \quad (2)$$

$$\Phi_0(s) = \frac{1}{4}(s^2 - 1)^2 \quad \forall s \in \mathbb{R}. \quad (3)$$

$$\begin{aligned} \mathbf{u} &= 0, \quad \partial_{\mathbf{n}} \mu = \partial_{\mathbf{n}} \phi = 0 \quad \text{on } \partial\Omega \times (0, T), \\ \rho(\cdot, 0) &= \rho_0, \quad \mathbf{u}(\cdot, 0) = \mathbf{u}_0, \quad \phi(\cdot, 0) = \phi_0 \quad \text{in } \Omega, \end{aligned} \quad (4)$$

It is well known that the application of the Diffuse Interface theory has become a fundamental method in fluid mechanics. We refer the reader to the review articles [4, 5, 6]. A paradigm model of the Diffuse Interface theory for two-phase flows is the homogeneous Kelvin-Voigt-Cahn-Hilliard (for short KVCH) system which is the system (1). Over the past years there have been important developments concerning the mathematical modeling and analysis of KVCH system for binary mixtures (see, e.g., [1, 12, 11, 2, 7]) and references therein. Regarding the mathematical analysis of the system (1) is very recent, we mention the work [9], in which the author proved the global existence of weak solutions and [3] where the authors proved the existence of weak solutions and density and the existence of a unique pressure.

Nevertheless, a finite-time blow-up phenomena for this model remains lacking. To address this, the present work focuses on the initial-boundary value problem of the three-dimensional nonhomogeneous incompressible KVCH system in a bounded domain. Under the condition that the initial density is bounded away from zero, we establish a novel blow-up criterion for strong solutions. We prove that if the maximum existence time  $T^*$  a strong solution

is finite, then the velocity field must diverge in a specific Lebesgue norm as  $t \rightarrow T^*$ . This result reveals a sufficient condition for the blow-up of solutions and lays a theoretical foundation for further studies on the global existence and regularity of solutions.

## 2 Preliminary Notes

Firstly, we review some basic inequalities and important lemmas which are used widely in this paper.

**Lemma 2.1.** *Let  $v \in W^{1,2}(\Omega)$ , and let  $\rho$  be a non-negative function such that*

$$0 < M \leq \int_{\Omega} \rho dx, \quad \int_{\Omega} \rho^{\gamma} dx \leq E_0,$$

where  $\Omega \subset \mathbb{R}^3$  is a bounded domain,  $M, E_0$  are positive constants, and  $\gamma > 1$ . Then there exists a constant  $c$  depending solely on  $M, E_0$  such that

$$\|v\|_{L^2(\Omega)}^2 \leq c(E_0, M) \left( \|\nabla_x v\|_{L^2(\Omega)}^2 + \left( \int_{\Omega} \rho |v| dx \right)^2 \right).$$

**Lemma 2.2.** *For  $1 \leq q < \infty$ , we make*

$$L_{(m)}^q(\Omega) = \{f \in L^q(\Omega) : m(f) = m\} \text{ with } m(f) = \frac{1}{|\Omega|} \int_{\Omega} f(x) dx.$$

Let  $u \in W^{2,q}(\Omega)$ ,  $\Delta u = f$  in a suitable bounded domain  $\Omega \subset \mathbb{R}^3$ , and  $\partial_n u|_{\partial\Omega} = 0$  in the trace sense. If  $f \in W^{1,q}(\Omega) \cap L_{(0)}^q(\Omega)$  and  $\partial\Omega \in C^3$ , then  $u \in W^{3,q}(\Omega)$ . Moreover,

$$\|u\|_{W^{k+2,q}(\Omega)} \leq C_q \|f\|_{W^{k,q}(\Omega)}$$

keeps for all any  $f \in W^{k,q}(\Omega) \cap L_{(0)}^q(\Omega)$  ( $k = 0, 1$ ), where the positive constant  $C_q$  depends only on  $1 < q < \infty$  and  $\Omega$ .

**Lemma 2.3.** *Let  $\Omega$  be a suitable bounded domain in  $\mathbb{R}^3$  with  $\partial\Omega \in C^3$ ,  $v \in C^2(\mathbb{R})$ ,  $\phi \in W^{1+j,r}(\Omega)$  ( $j = 0, 1$ ),  $r > 3$  and  $\|\phi\|_{W^{1+j,r}(\Omega)} \leq R$ . Assume that  $u \in V_2^{1+j}(\Omega)$  is the solution to the following equation*

$$(v(\phi)\mathbb{D}u, \mathbb{D}\psi)_{L^2(\Omega)} = (f, \psi)_{L^2(\Omega)} \quad (\forall \psi \in C_{0,\sigma}^{\infty}(\Omega)), \quad \text{where } f \in H^s(\Omega) \text{ and } s \in [0, j].$$

Then  $u \in H^{2+s}(\Omega)$  satisfies

$$\|u\|_{H^{2+s}(\Omega)} \leq C(R) \|f\|_{H^s(\Omega)},$$

where  $C(R)$  only depends on  $\Omega, v, r, R > 0$ , and the space  $V_2^{1+j}(\Omega)$  is defined as

$$V_2^{1+j}(\Omega) = H^{1+j}(\Omega) \cap H_0^1(\Omega) \cap H^1(\Omega) \cap L_{\sigma}^2(\Omega) \quad (j = 0, 1).$$

**Lemma 2.4.** *Let  $\Omega$  be a bounded domain of class  $C^3$  in  $\mathbb{R}^3$ . If  $\rho_0 \in L^\infty(\Omega)$ ,  $u_0 \in V_\sigma(\Omega)$  and  $\phi_0 \in H^2(\Omega)$  are given such that  $0 < \rho_* \leq \rho_0 \leq \rho^*$ ,  $\partial_n \phi_0 = 0$  on  $\partial\Omega$ , and*

$$\mu_0 = -\frac{\Delta \phi_0}{\rho_0} + \Psi'(\phi_0) \in H^1(\Omega).$$

*Then, there exist  $T^* > 0$ , depending on the norms of the initial data, and a strong solution  $(\rho, u, \phi, \mu)$  to the initial-boundary problem (1.1)-(1.4) on  $[0, T^*]$  satisfying*

$$\begin{cases} \rho \in C([0, T^*]; L^r(\Omega)) \cap L^\infty(\Omega \times (0, T^*)) \cap L^\infty(0, T^*; H^{-1}(\Omega)), \\ u \in C([0, T^*]; V_\sigma(\Omega)) \cap L^2(0, T^*; H^2(\Omega)) \cap H^1(0, T^*; H_\sigma(\Omega)), \\ \phi \in C([0, T^*]; (W^{2,6}(\Omega))_w) \cap H^1(0, T^*; H^1(\Omega)), \\ \mu \in L^\infty(0, T^*; H^1(\Omega)) \cap L^2(0, T^*; W^{2,6}(\Omega)) \end{cases} \quad \text{for any } r \in [2, \infty].$$

### 3 Main Results

**Theorem 3.1.** *Let  $\Omega$  be a bounded smooth domain in  $\mathbb{R}^3$  with boundary  $\partial\Omega$  uniformly of class  $C^3$ . If the condition (2) is satisfied and the initial value  $(\rho_0, \mathbf{u}_0, \phi_0, \mu_0)$  satisfies*

$$\begin{aligned} 0 < \rho_* \leq \rho_0 \leq \rho^* < \infty, \quad \mathbf{u}_0 \in V_\sigma(\Omega), \quad \phi_0 \in H^2(\Omega), \\ \mu_0 = -\frac{\Delta \phi_0}{\rho_0} + \Psi'(\phi_0) \in H^1(\Omega), \quad \partial_n \phi_0 = 0 \text{ on } \partial\Omega. \end{aligned} \quad (5)$$

*Let  $(\rho, u, \phi, \mu)$  be a strong solution to the initial-boundary problem (1)-(4) on  $[0, T]$  satisfying*

$$\begin{cases} \rho \in C([0, T]; L^q(\Omega)) \cap L^\infty(\Omega \times (0, T)) \cap L^\infty(0, T; H^{-1}(\Omega)), \\ \mathbf{u} \in C([0, T]; V_\sigma(\Omega)) \cap L^\infty(0, T; H^2(\Omega)) \cap H^1(0, T; V_\sigma(\Omega)), \\ \phi \in C([0, T]; (W^{2,6}(\Omega))_w) \cap H^1(0, T; H^1(\Omega)), \\ \mu \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; W^{2,6}(\Omega)) \end{cases} \quad (6)$$

*for any  $q \in [2, \infty]$ . When  $0 < T^* < +\infty$  is the maximum time of existence, then*

$$\lim_{T \rightarrow T^*} \|\mathbf{u}\|_{L^{\frac{2r}{r-3}}(0, T; L^r(\Omega))} = \infty \quad \text{for any fixed } r > 3. \quad (7)$$

For the rest of this section, we denote by  $C$  a generic constant depending only on  $\Omega$ ,  $v_*$ ,  $v^*$ ,  $\|\sqrt{\rho_0} \mathbf{u}_0\|_{L^2(\Omega)}$ ,  $\|\nabla \mathbf{u}_0\|_{L^2(\Omega)}$ ,  $\|\nabla \mu_0\|_{L^2(\Omega)}$ ,  $\|\nabla \phi_0\|_{L^2(\Omega)}$ . The positive constant  $C(\rho^*)$  depending on  $\Omega$ ,  $v_*$ ,  $v^*$ ,  $\rho^*$  and  $\|\nabla \mathbf{u}_0\|_{L^2(\Omega)}$ ,  $\|\nabla \mu_0\|_{L^2(\Omega)}$ ,  $\|\nabla \phi_0\|_{L^2(\Omega)}$ .

Let  $T^* \in (0, \infty)$  be the maximum time of existence of strong solution  $(\rho, u, \phi, \mu)$  to the initial-boundary problem (1)-(4). Namely,  $(\rho, u, \phi, \mu)$  is a strong solution to the problem (1)-(4) in  $\Omega \times [0, T]$  for any  $T \in (0, T^*)$ , but not a strong solution in  $\Omega \times [0, T^*]$ . Suppose that (7) were not right, i.e.

$$M := \|u\|_{L^{\frac{2r}{r-3}}(0, T^*; L^r(\Omega))} < \infty. \quad (8)$$

The goal is to show under (8), there is a bound  $C > 0$  such that

$$0 < \frac{\rho_*}{C} \leq \rho \leq C\rho_*, \quad (9)$$

and there is a bound  $C(\rho^*) > 0$  such that

$$\begin{aligned} & \sup_{t \in [0, T]} \left( \|\mathbf{u}\|_{H^2(\Omega)}^2 + \|\nabla \mu\|_{L^2(\Omega)}^2 + \|\phi\|_{W^{2,6}(\Omega)}^2 \right) \\ & + \int_0^T \left( \|\phi_t\|_{H^1(\Omega)}^2 + \|\mathbf{u}_t\|_{H^1(\Omega)}^2 \right) dt \leq C(\rho^*). \end{aligned} \quad (10)$$

Proposition 3.2 to 3.4 holds any  $T \in (0, T^*)$

**Proposition 3.2.** *Under the conditions of Theorem 1, there exists a positive constant  $C > 0$  such that*

$$0 < \frac{\rho_*}{C} \leq \rho(x, t) \leq C\rho^*.$$

*Proof.* Multiplying (1)<sub>2</sub>, (1)<sub>4</sub> by  $\mathbf{u}$ ,  $\mu$  respectively and integrating over  $\Omega$ , using (1)<sub>5</sub> and (2), summing and integrating above equation respect to time, finally we get

$$\begin{aligned} & \int_{\Omega} \left( \frac{1}{2} \rho \mathbf{u}^2 + \frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} |\nabla \mathbf{u}|^2 + \rho \Psi(\phi) \right) dx + \int_0^t \int_{\Omega} (v(\phi) |\mathbb{D}\mathbf{u}|^2 + |\nabla \mu|^2) dx d\tau \\ & = \int_{\Omega} \left( \frac{1}{2} \rho_0 |\mathbf{u}_0|^2 + \frac{1}{2} |\nabla \phi_0|^2 + \rho_0 \Psi(\phi_0) \right) dx, \end{aligned}$$

for all  $t \in [0, T]$ . The desired inequality Proposition 3.3 is obtained. The key estimates on  $\|\phi\|_{H^1(\Omega)}$ ,  $\|\mu\|_{H^1(\Omega)}$  and  $\|\mathbf{u}\|_{H^2(\Omega)}$  will be given in the following lemma.  $\square$

**Proposition 3.3.** [8] *Under the conditions of Theorem 3.1, the energy inequality*

$$\sup_{t \in [0, T]} \int_{\Omega} \left( \frac{\rho \mathbf{u}^2}{2} + \frac{\rho(\phi^2 - 1)^2}{4} + \frac{|\nabla \phi|^2}{2} \right) dx + \int_0^T \int_{\Omega} (v(\phi) |\mathbb{D}\mathbf{u}|^2 + |\nabla \mu|^2) dx dt \leq C.$$

**Proposition 3.4.** [8] *Under the conditions of Theorem 3.1 and (8), we can obtain*

$$\sup_{t \in [0, T]} \|\phi\|_{H^1(\Omega)} \leq C, \quad \|\mu\|_{H^1(\Omega)} \leq C(1 + \|\nabla\mu\|_{L^2(\Omega)}), \quad \|\phi\|_{W^{2,6}(\Omega)} \leq C(1 + \|\nabla\mu\|_{L^2(\Omega)}).$$

**Proposition 3.5.** *Under the conditions of Theorem 3.1 and (8), it holds that*

$$\sup_{t \in [0, T]} \left( \|\mathbf{u}\|_{H^2(\Omega)}^2 + \|\nabla\mu\|_{L^2(\Omega)}^2 + \|\phi\|_{W^{2,6}(\Omega)}^2 \right) + \int_0^T \left( \|\phi_t\|_{H^1(\Omega)}^2 + \|\mathbf{u}_t\|_{H^1(\Omega)}^2 \right) dt \leq C.$$

*Proof.* Multiplying by  $\Delta\mathbf{u}$  on both sides of equation (1)<sub>2</sub> and integrating the resulting equation over  $\Omega$ , using  $\operatorname{div}(\nabla\phi \otimes \nabla\phi) = \Delta\phi\nabla\phi + \nabla\phi \cdot \nabla^2\phi$ , Holder's inequality,  $\|\phi\|_{W^{2,6}} \leq C(1 + \|\nabla\mu\|_{L^2})$ , developing the viscous contribution and applying elliptic regularity, and Lemma 2.3, then choosing sufficiently small  $\eta$ , finally we obtain the following differential inequality for  $\|\Delta\mathbf{u}\|_{L^2}^2$ :

$$\frac{\chi}{2} \frac{d}{dt} \|\Delta\mathbf{u}\|_{L^2}^2 + c_0 \|\mathbf{u}\|_{H^2}^2 \leq \frac{\eta}{2} \|\nabla\partial_t\mathbf{u}\|_{L^2}^2 + C \left( (1 + \|\nabla\mu\|_{L^2}^2)(1 + \|\nabla\mu\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2}^2 + 1) \right).$$

Multiplying by  $\mathbf{u}_t$  on both sides of equation (1)<sub>2</sub> and integrating the resulting equation over  $\Omega$ , multiplying by  $\mu_t$  and  $\phi_t$  respectively on both sides of equation (1)<sub>4</sub> and integrating over  $\Omega$ , then using (1)<sub>1</sub> and (1)<sub>5</sub>, finally we obtain

$$\begin{aligned} & \frac{d}{dt} \left( \int_{\Omega} \frac{v(\phi)}{2} |\nabla\mu|^2 dx + \frac{1}{2} \|\nabla\mathbf{u}\|_{L^2(\Omega)}^2 + \frac{\chi}{2} \|\Delta\mathbf{u}\|_{L^2}^2 + \int_{\Omega} \rho\mu u \cdot \nabla\phi dx \right) + \int_{\Omega} |\nabla\phi_t|^2 dx \\ & + \int_{\Omega} \rho|\phi_t|^2 dx + \int_{\Omega} |\nabla\mathbf{u}_t|^2 dx + c_0 \|\mathbf{u}\|_{H^2}^2 + \int_{\Omega} \rho|u_t|^2 dx + 3 \int_{\Omega} \rho\phi^2|\phi_t|^2 dx \\ & = - \int_{\Omega} \rho(u \cdot \nabla)u \cdot u_t dx + \int_{\Omega} -\rho\nabla\Psi(\phi) \cdot u_t dx + 2 \int_{\Omega} \rho\mu\nabla\phi \cdot u_t dx \\ & + \int_{\Omega} v'(\phi)\phi_t|Du|^2 dx + \int_{\Omega} -\rho\Psi''(\phi)\phi_t u \cdot \nabla\phi dx + \int_{\Omega} -\rho\Psi'(\phi)u \cdot \nabla\phi_t dx \\ & + \int_{\Omega} \rho\phi_t u \cdot \nabla\mu dx + \int_{\Omega} \rho(u \cdot \nabla\mu)(u \cdot \nabla\phi) dx + \int_{\Omega} \rho\mu u \cdot \nabla(u \cdot \nabla\phi) dx \\ & + 2 \int_{\Omega} \rho\mu u \cdot \nabla\phi_t dx + 2 \int_{\Omega} -\rho u \cdot \nabla\phi_t dx + 2 \int_{\Omega} -\nabla\mu \cdot \nabla\phi_t dx \triangleq \sum_{i=1}^{12} L_i. \end{aligned}$$

By Proposition 3.5, Sobolev inequality, Holder's inequality and Young's inequality, each term on the right hand of the above equation is estimated as follows.

$$|L_1| \leq \eta \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \epsilon \|\mathbf{u}\|_{H^2}^2 + C \|\mathbf{u}\|_{L^r}^{\frac{2r}{r-3}}, \quad |L_2| \leq \eta \|\sqrt{\rho}\mathbf{u}_t\|_{L^2(\Omega)}^2 + C(\rho^*, \eta)(1 + \|\nabla\mu\|_{L^2(\Omega)}^{\frac{5}{2}}),$$

$$\begin{aligned}
|L_3| &\leq \eta \|\sqrt{\rho} \mathbf{u}_t\|_{L^2(\Omega)}^2 + C(\rho^*, \eta)(1 + \|\nabla \mu\|_{L^2(\Omega)}^3), \\
|L_4| &\leq \eta (\|\phi_t\|_{L^6(\Omega)}^2 + \|\mathbf{u}\|_{H^2(\Omega)}^2) + C(\rho^*, \eta) \|\nabla \mathbf{u}\|_{L^2(\Omega)}^6, \\
|L_5| &\leq \eta \|\phi_t\|_{L^6(\Omega)}^2 + C(\rho^*, \eta) \|\mathbf{u}\|_{L^r(\Omega)}^2 (1 + \|\nabla \mu\|_{L^2(\Omega)}^{\frac{5}{3}}), \\
|L_6| &\leq \eta \|\nabla \phi\|_{L^2(\Omega)}^2 + C(\rho^*, \eta) \|\mathbf{u}\|_{L^r(\Omega)}^2 (1 + \|\phi\|_{L^2(\Omega)}^{\frac{3}{2}}) \|\phi\|_{W^{2,6}(\Omega)}^{\frac{3}{2}}, \\
|L_7| &\leq \eta \|\phi\|_{L^6(\Omega)}^2 + C(\rho^*, \eta) \|\mathbf{u}\|_{L^r(\Omega)}^2 \|\nabla \mu\|_{L^2(\Omega)}^2 \\
|L_8| &\leq C(\rho^*) \|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 (1 + \|\nabla \mathbf{u}\|_{L^2(\Omega)}^2) \\
|L_9| &\leq C(\rho^*) (1 + \|\nabla \mu\|_{L^2(\Omega)}^2) (\|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 + 1) \\
|L_{10}| &\leq \eta \|\nabla \phi_t\|_{L^2(\Omega)}^2 + C(\rho^*, \eta) \|\mathbf{u}\|_{L^r(\Omega)}^2 (1 + \|\nabla \mu\|_{L^2(\Omega)}^2), \\
|L_{11}| &\leq \eta \|\sqrt{\rho} \phi_t\|_{L^2(\Omega)}^2 + C(\rho^*, \eta) \|\mathbf{u}\|_{L^r(\Omega)}^2 (1 + \|\nabla \mathbf{u}\|_{L^2(\Omega)}^2), \\
|L_{12}| &\leq \eta \|\nabla \phi_t\|_{L^2(\Omega)}^2 + C(\eta) \|\nabla \mu\|_{L^2(\Omega)}^2
\end{aligned}$$

Combining the above estimates, according to Proposition 3.2, Proposition 3.4, we get Proposition 3.5.

By Proposition 3.5, we can take  $t' \in (0, T^*)$  and find the pair  $(\rho, u, \phi, \mu)(x, t') \triangleq \lim_{t \rightarrow t'} (\rho, u, \phi, \mu)(x, t)$  satisfies the conditions with the initial data (5) at the time  $t = t'$ . Next, we make  $(\rho, u, \phi, \mu)(x, t')$  as the initial data and apply Lemma 2.4 to extend the local strong solution beyond  $T^*$  (the maximum time of existence). A contradiction appears and the proof of Theorem 3.1 is completed.  $\square$

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