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Evolution of Interfaces for the Nonlinear Double Degenerate Parabolic Equation of Turbulent Filtration with Fast Diffusion and Strong Absorption

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Abstract

We prove the short-time asymptotic formula for the interfaces and local solutions near the interfaces for the nonlinear double degenerate reaction-diffusion equation of turbulent filtration with fast diffusion and strong absorption

$$u_t = (|(u^m)_x|^{p-1}(u^m)_x)_x - bu^\beta,\ 0 < mp < 1, \beta > 0.$$

A complete classification in terms of the nonlinearity parameters m, p, β and asymptotics of the initial function near its support is given. In the case of an infinite speed of propagation of the interface (no interface), the asymptotic behavior of the local solution is classified at infinity.

Mathematics Subject Classification: 35A01; 35C07; 35K55

Keywords: Nonlinear degenerate parabolic PDE; Reaction-diffusion equation; Fast diffusion; Nonlinear scaling laws

1 Introduction

This work is a sequel to the work presented in [7]. We consider the Cauchy problem (CP) for the nonlinear double degenerate parabolic equation

$$Lu \equiv u_t - (|(u^m)_x|^{p-1}(u^m)_x)_x + bu^\beta = 0,\ x \in \mathbb{R},\ 0 < t < T, \tag{1}$$

$$u(x,0) = u_0(x),\ \ x \in \mathbb{R}. \tag{2}$$

where $u = u(x,t), m, p, \beta > 0, b \in \mathbb{R}$, with $0 < mp < 1$, and $T \le +\infty$ and u_0 is a continuous and nonnegative function. We assume that either $b \ge 0$ or $b < 0$ and $\beta \ge 1$ Equation (1) models turbulent polytropic filtration of a gas in a porous medium [10, 11]. The condition $0 < mp < 1$ corresponds to *fast* diffusion – when the equation (1) with $b = 0$ possesses an infinite speed of propagation property [10]. The main constituent of the equation (1) is to model competition between the double degenerate fast diffusion with infinite speed of propagation property and the absorption or reaction term. We define the interface function as

$$\eta(t) := \sup\{x : u(x,t) > 0\},$$

with $\eta(0) = 0$. Additionally, we assume that

$$u_0(x) \sim C(-x)_+^\alpha,\ \text{as}\ \ x \to 0^-,\ \text{for some}\ C > 0,\ \alpha > 0, \tag{3}$$

unless stated otherwise, where $(\cdot)_+ = \max(\cdot; 0)$. Solution of the CP is understood in the weak sense.

A full classification of the short-time behavior of $\eta(t)$ and of the local solution near $\eta(t)$ depending on the parameters m, p, b, β, C, and α in the case of slow diffusion ($mp > 1$) is presented in [7]. The aim of the paper is to classify short-time behavior of the interfaces and local solutions near the interfaces and at infinity in the CP with a compactly supported initial function. In all cases when $\eta(t) < +\infty$ we classify the short-time asymptotic behavior of the interface $\eta(\cdot)$, and local solution near $\eta(\cdot)$, while in all cases with $\eta(t) = +\infty$ we classify the short-time asymptotic behavior of the solution as $x \to +\infty$.

Most of the results of the paper are local. Therefore, we assume that u_0 is either bounded or unbounded with growth condition as $x \to -\infty$, which is suitable for the existence of the solution. In some cases we will consider

$$u_0(x) = C(-x)_+^\alpha,\ x \in \mathbb{R}, \tag{4}$$

specifically cases when the solution to (1), (4) is of self-similar form. In these cases the estimations will be global in time.

A full classification of the short-time behavior of interfaces for the reaction-diffusion equation (1) with $p = 1$ is presented in [9] for the slow diffusion case ($m > 1$) and in [3] for the fast diffusion case ($0 < m < 1$). The analogous classifications for the nonlinear degenerate multidimensional reactiondiffusion equation (multidimensional version of (1) with $p = 1$) was presented in [5] for the slow diffusion case and in [6] for the fast diffusion case.

The methods of the proof developed in [9, 3] are based on nonlinear scaling laws, and a barrier technique using special comparison theorems in irregular domains with characteristic boundary curves [1, 2, 4]. Full classification of interfaces and local solutions near the interfaces and at infinity for the p-Laplacian type reaction-diffusion equation ((1) with $m = 1$) are presented in [8]. The semilinear case ($m = p = 1$ in (1)) was analyzed in [12, 13].

We refer to [7] for the definition of the weak solution to CP (1), (2) (see Definition 1 from [7]) and main results on the general theory of the PDE (1).

We also make use of the standard comparison result from [7] as well as the notion of the minimal solution to prove our main results (see Lemma 1, Definition 2, and Lemma 2 from [7]).

The paper is organized in the following way. In Section 2 the main results are outlined, with further details in Section 3. Essential lemmas are formulated and proven using nonlinear scaling in Section 4. Finally, in Section 5, the results of Sections 2 are proved. To improve readability, the explicit values of all constants that appear in Sections 2, 3, and 5 are relegated to the Appendix.

2 The Main Result

Throughout this section we assume that u is a unique weak solution of the CP (1), (3). The main results are classified according to regions I-V, respectively, in Figure 1, the (α,β) parameter space diagram, below.

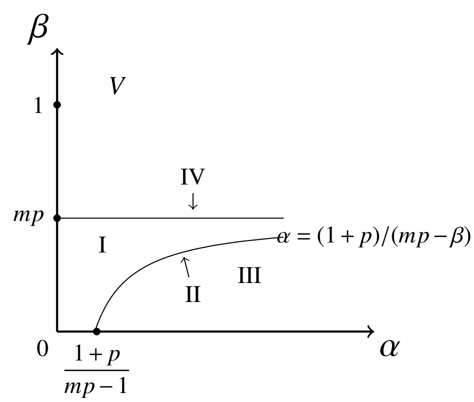

FIGURE 1. (α,β) parameter space diagram for interface development for the CP (1), (3).

- **Region I**: $b > 0, 0 < \beta < mp$ and $0 < \alpha < (1+p)/(mp-\beta)$.
The interface initially expands and there exists a number $\delta > 0$ such that
$$z_1 t^{\frac{mp-\beta}{(1-\beta)(1+p)}} \le \eta(t) \le z_2 t^{\frac{mp-\beta}{(1-\beta)(1+p)}}, \ 0 \le t \le \delta, \tag{5}$$
(see the Appendix for the explicit values of z_1 and z_2).
Further, for any $\sigma \in \mathbb{R}$, there is a number $f(\sigma) > 0$ (depending on C, m, and p) such that
$$u(\chi_\sigma(t), t) \sim f(\sigma) t^{\frac{\alpha}{1+p-\alpha(mp-1)}}, \text{ as } t \to 0^+, \tag{6}$$
where $\chi_\sigma(t) = \sigma t^{\frac{1}{1+p-\alpha(mp-1)}}$.

- **Region II**: $b > 0, 0 < \beta < mp, \alpha = (1+p)/(mp-\beta)$, and
$$C_* = \left[\frac{b(mp-\beta)^{1+p}}{(m(1+p))^p p(m+\beta)}\right]^{\frac{1}{mp-\beta}}.$$
Then the interface shrinks or expands accordingly as $C < C_*$ or $C > C_*$ and we have that
$$\eta(t) \sim z_* t^{\frac{mp-\beta}{(1+p)(1-\beta)}}, \text{ as } t \to 0^+, \tag{7}$$
where $z_* \lesseqgtr 0$ if $C \lesseqgtr C_*$, and for arbitrary $\sigma < z_*$ there exists $f_1(\sigma) > 0$ such that
$$u(z_\sigma(t), t) \sim t^{1/(1-\beta)} f_1(\sigma), \text{ as } t \to 0^+, \tag{8}$$
where $z_\sigma(t) = \sigma t^{\frac{mp-\beta}{(1-\beta)(1+p)}}$.

- **Region III**: $b > 0, 0 < \beta < mp$ and $\alpha > (1+p)/(mp-\beta)$.
Then the interface initially shrinks and we have that
$$\eta(t) \sim -\tau_* t^{\frac{1}{\alpha(1-\beta)}}, \text{ as } t \to 0^+, \tag{9}$$
where $\tau_* = C^{-1/\alpha}(b(1-\beta))^{\frac{1}{\alpha(1-\beta)}}$ and, for any $\tau > \tau_*$, we have
$$u(\eta_\tau(t), t) \sim \left[C^{1-\beta}\tau^{\alpha(1-\beta)} - b(1-\beta)\right]^{\frac{1}{1-\beta}} t^{\frac{1}{1-\beta}}, \text{ as } t \to 0^+, \tag{10}$$
where $\eta_\tau(t) = -\tau t^{\frac{1}{\alpha(1-\beta)}}$.

- **Region IV**: $b > 0$, $\beta = mp$ and $\alpha > 0$.
In this case there is an infinite speed of propagation. For arbitary $\epsilon > 0$, there exists a number $\delta = \delta(\epsilon) > 0$ such that
$$t^{\frac{1}{1-mp}} \varphi(x) \le u(x,t) \le (t+\epsilon)^{\frac{1}{1-mp}} \varphi(x), \ x > 0, \ 0 \le t \le \delta, \tag{11}$$
where $\varphi = \varphi(x) > 0$ is a solution of the stationary problem
$$\begin{cases} (|(\varphi^m)'|^{p-1}(\varphi^m)')' = \frac{1}{1-mp}\varphi + b\varphi^{mp}, \ x > 0, \\ \varphi(0) = 1, \ \varphi(+\infty) = 0. \end{cases} \tag{12}$$

Moreover, we have

$$\ln u(x,t) \sim -\frac{1}{m}\Big(\frac{b}{p}\Big)^{1/(1+p)} x, \text{ as } x \to +\infty, 0 \le t \le \delta. \tag{13}$$

- **Region V**: This region is divided into cases V(a), V(b) and V(b).
  - **V(a)**: Either $b > 0, \beta > mp$ or $b < 0, \beta \ge 1$ or $b = 0$, and

    $$\mathcal{D} = \Big[\frac{(m(1+p))^p(m+1)}{(1-mp)^p}\Big]^{\frac{1}{1-mp}}.$$

    Then there is an infinite speed of propagation of the interface and (6) holds. If $b > 0, \beta \ge \frac{p(1-m)+2}{1+p}$ or $b < 0, \beta \ge 1$ or $b = 0$, then there exists a number $\delta > 0$ such that

    $$u(x,t) \sim \mathcal{D} t^{\frac{1}{1-mp}} x^{\frac{1+p}{1-mp}}, \text{ as } x \to +\infty, t \in (0,\delta], \tag{14}$$

  - **V(b)**: $b > 0$ and $1 \le \beta < \frac{p(1-m)+2}{1+p}$.
    Then,

    $$\lim_{t\to 0^+} \lim_{x\to+\infty} \frac{u(x,t)}{t^{\frac{1}{1-mp}} x^{\frac{1+p}{mp-1}}} = \mathcal{D}. \tag{15}$$

  - **V(c)**: $b > 0$ and $0 < mp < \beta < 1$.
    Then there exists a number $\delta > 0$ such that

    $$u(x,t) \sim C_* x^{\frac{1+p}{mp-\beta}}, \text{ as } x \to +\infty, t \in (0,\delta]. \tag{16}$$

3 Additional details of the Results

In this section we outline some additional, essential, details of Results I-V. We refer to the Appendix for the explicit values of relevant constants that appear throughout this section.

- *Region I*. The solution u satisfies the estimation

  $$C_1 t^{\frac{1}{1-\beta}} (z_1 - z)_+^{\frac{1+p}{mp-\beta}} \le u(x,t) \le C_* t^{\frac{1}{1-\beta}} (z_2 - z)_+^{\frac{1+p}{mp-\beta}}, 0 < t \le \delta, \tag{17}$$

  where $z = xt^{\frac{\beta-mp}{(1-\beta)(1+p)}}$. The left-hand side of (17) is valid for $0 \le x < +\infty$, while the right-hand side is valid for $x \ge \tau_0 t^{(mp-\beta)/(1-\beta)(1+p)}$. C_1, z_1, z_2, and τ_0, are positive constants depending on m, p, β, and b. Moreover,

  $$f(\sigma) = C^{\frac{1+p}{1+p-\alpha(mp-1)}} f_0(C^{\frac{mp-1}{1+p-\alpha(mp-1)}}\sigma), \; f_0(\sigma) = w(\sigma,1), \sigma \in \mathbb{R}, \tag{18}$$

where w is a minimal solution of the CP (1), (4) with $C = 1, b = 0$. If u_0 is given by (4), then the right-hand sides of (17) and (5) are valid for all $t > 0$.

- *Region II*. Assume u solves the CP (1), (4). If $C = C_*$, then u_0 is the stationary solution to the CP. If $C \neq C_*$, then the minimal solution of the CP is given by

$$u(x,t) = t^{\frac{1}{1-\beta}} f_1(z),\ z = xt^{\frac{\beta-mp}{(1-\beta)(1+p)}}, \tag{19}$$

and

$$\eta(t) = z_* t^{\frac{mp-\beta}{(1-\beta)(1+p)}},\ t \geq 0, \tag{20}$$

If $C > C_*$, the interface initially expands and we have

$$C'\Big(z' t^{\frac{mp-\beta}{(1-\beta)(1+p)}} - x\Big)_+^{\frac{1+p}{mp-\beta}} \leq u \leq C''\Big(z'' t^{\frac{mp-\beta}{(1-\beta)(1+p)}} - x\Big)_+^{\frac{1+p}{mp-\beta}}, \tag{21}$$

$$z' \leq z_* \leq z'',\ 0 \leq x < +\infty,\ t > 0, \tag{22}$$

where $C' = C_2, C'' = C_*, z' = z_3,$ and $z'' = z_4$. If $0 < C < C_*$, then the interface shrinks and there exists $\tau_1 > 0$ such that for all $\tau \leq \tau_1$ there exists a number $\varrho > 0$ such that

$$u\Big(\tau t^{\frac{mp-\beta}{(1+p)(1-\beta)}}, t\Big) = \varrho t^{\frac{1}{1-\beta}},\ t \geq 0, \tag{23}$$

and u and z_* satisfy the estimates (21), (22) with $C' = C_*, C'' = C_3, z' = -z_5,$ and $z'' = -z_6$.

- *Region IV*. The explicit solution of the problem (12) is given by

$$\varphi(x) = \mathcal{F}^{-1}(x),\ 0 \leq x < +\infty, \tag{24}$$

where $\mathcal{F}^{-1}(\cdot)$ is the inverse of the function

$$\mathcal{F}(z) = \int_z^1 ms^{-1}\left[\frac{b}{p} + \frac{m(1+p)}{p(1-mp)(1+m)} s^{1-mp}\right]^{-\frac{1}{1+p}} ds,\ 0 < z \leq 1. \tag{25}$$

The function $\varphi(x)$ satisfies

$$\ln\varphi(x) \sim -\frac{1}{m}\Big(\frac{b}{p}\Big)^{1/(1+p)} x,\ \text{as } x \to +\infty, \tag{26}$$

and the global estimation

$$0 < \varphi(x) \leq \exp\left(-\frac{1}{m}\left(\frac{b}{p}\right)^{\frac{1}{1+p}} x\right),\ x > 0, \tag{27}$$

and therefore

$$\frac{\varphi(x)}{e^{-\gamma x}} \to +\infty,\ \text{as } x \to +\infty,\ \text{if } \gamma > \frac{1}{m}\left(\frac{b}{p}\right)^{\frac{1}{1+p}}. \tag{28}$$

From (11) and (28), it follows that

$$\lim_{t\to 0^+}\lim_{x\to+\infty} u(x,t)\exp\left(-\frac{1}{m}\left(\frac{b}{p}\right)^{\frac{1}{1+p}} x\right) = 0, \tag{29}$$

and respectively

$$\frac{u(x,t)}{e^{-\gamma x}} \to +\infty, \text{ as } x \to +\infty, 0 \le t \le \delta(\epsilon), \text{ if } \gamma > \frac{1}{m}\left(\frac{b}{p}\right)^{\frac{1}{1+p}}. \tag{30}$$

- *Region V*. If $\beta \ge 1$, then for arbitrary $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$C_5 t^{\frac{\alpha}{1+p-\alpha(mp-1)}} (\chi_1 + \chi)^{\frac{1+p}{mp-1}} \le u(x,t) \le C_6 t^{\frac{\alpha}{1+p-\alpha(mp-1)}} (\chi_2 + \chi)^{\frac{1+p}{mp-1}}, \tag{31}$$

where $\chi = xt^{\frac{-1}{1+p-\alpha(mp-1)}}$, for all $x \in [0,\infty)$ and $0 \le t \le \delta(\epsilon)$. C_5, C_6, χ_1, and χ_2, are positive constants depending on m, p, β, b, and ϵ. If $b > 0$ and $\beta \ge 1$, we have the upper estimation

$$u(x,t) \le \mathcal{D} t^{\frac{1}{1-mp}} x^{\frac{1+p}{mp-1}}, 0 < x < +\infty, 0 < t < +\infty. \tag{32}$$

While if $b < 0$ and $\beta \ge 1$, then for small $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$u(x,t) \le \mathcal{D}(1-\epsilon)^{\frac{1}{mp-1}} t^{\frac{1}{1-mp}} x^{\frac{1+p}{mp-1}}, \text{ for } \varkappa t^{\frac{1}{1+p+\alpha(1-mp)}} < x < +\infty, 0 < t \le \delta, \tag{33}$$

where

$$\varkappa = \left[\frac{A_0 + \epsilon}{\mathcal{D}(1-\epsilon)^{\frac{1}{mp-1}}}\right]^{\frac{mp-1}{1+p}}. \tag{34}$$

If $b > 0$ and $mp < \beta < 1$, then there exists $\delta > 0$ such that

$$t^{\frac{1}{1-\beta}} C_*(1-\epsilon)(z_8 + z)^{\frac{1+p}{mp-\beta}} \le u(x,t) \le C_* x^{\frac{1+p}{mp-\beta}}, 0 < x < +\infty, 0 < t \le \delta, \tag{35}$$

where $z = xt^{\frac{\beta-mp}{(1-\beta)(1+p)}}$, $\epsilon > 0$ is an arbitrary sufficiently small number, and z_8 is a positive constant depending on m, p, β, b, and ϵ.

If $b = 0$ and $\alpha > 0$, then the minimal solution to the CP (1), (4) has the self-similar form

$$u(x,t) = t^{\frac{\alpha}{1+p+\alpha(1-mp)}} f(\chi), \chi = xt^{\frac{-1}{1+p+\alpha(1-mp)}}. \tag{36}$$

where f satisfies (18). Moreover the following global estimation is valid:

$$\mathcal{D} t^{\frac{\alpha}{1+p+\alpha(1-mp)}} (\chi_3 + \chi)^{\frac{1+p}{mp-1}} \le u(x,t) \le C_7 t^{\frac{\alpha}{1+p+\alpha(1-mp)}} (\chi_4 + \chi)^{\frac{1+p}{mp-1}}, \tag{37}$$
$$0 \le x < +\infty, 0 < t < +\infty,$$

where C_7, χ_3, and χ_4, are positive constants depending on m and p.

Explicit solution (33) provides sharper upper bound than (37) as $x \to +\infty$. From (37) and (33), asymptotic result (14) easily follows. In a similar way asymptotic result (14) follows in the local case (3).

4 Preliminary Results

In this section we establish a series of lemmas that describe preliminary estimations for the CP. The proof of these results is based on nonlinear scaling.

Lemma 1. *If $b = 0$ and $\alpha > 0$, then the minimal solution u of the CP* (1), (4) *has the self-similar form* (36)*, where the self-similarity function f satisfies* (18)*. If u_0 satisfies* (3)*, and u is the unique weak solution to CP* (1), (2)*, then u satisfies* (6).

The proof coincides with that given for Lemma 3 from [7].

Lemma 2. *Let u be a weak solution to the CP* (1), (2)*, with u_0 satisfying the condition* (3)*. Let one of the following cases be valid*

$$\begin{cases} b > 0, 0 < \beta < mp, 0 < \alpha < (1+p)/(mp-\beta) & \textit{Case 1},\\ b > 0, \beta \geq mp, \alpha > 0 & \textit{Case 2},\\ b < 0, \beta \geq 1, \alpha > 0 & \textit{Case 3}. \end{cases}$$

Then, for any $\sigma \in \mathbb{R}$, u satisfies (6) *with the same function f as in Lemma 1.*

The proof of Cases 1 and 2 coincides with the proof of Lemma 4 from [7]. The proof of Case 3 coincides with the proof of Case (c) of Lemma 3.2 of [3], the only difference being when $\beta > 1$, we choose the function g as the following

$$g(x,t) = (C+1)(1+|x|^{\mu})^{\frac{\alpha}{\mu}}(1-\nu t)^{\gamma},\ x \in \mathbb{R}, 0 \leq t \leq t_0 = \nu^{-1}/2,$$

where

$$\gamma < 0, \mu > \frac{p+1}{p},\ \nu = -h_* + 1,\ h_* = \min_{\mathbb{R}} h(x) > -\infty,$$

$$h(x) = p(\alpha m)^p (C+1)^{p-1}\gamma^{-1}(1-\nu t)^{\gamma(mp-1)+1}(1+|x|^{\mu})^{\frac{\alpha(mp-1)-\mu(p+1)}{\mu}}|x|^{(\mu-1)p-1}\times$$
$$\times[(\mu-1)(1+|x|^{\mu}) + (\alpha m - \mu)\mu|x|^{\mu}].$$

Lemma 3. *If $b > 0, 0 < \beta < mp < 1$, and $\alpha = (1+p)/(mp-\beta)$, then the minimal solution u to the CP* (1), (3) *has the self-similar form* (19)*, where the self-similarity function f_1 satisfies*

$$\begin{cases} \mathcal{L}^0 f_1 \equiv (|(f_1^m)'|^{p-1}(f_1^m)')' + \frac{mp-\beta}{(1+p)(1-\beta)} z f_1' - \frac{1}{1-\beta} f_1 - b f_1^{\beta} = 0,\ z \in \mathbb{R},\\ f_1(z) \sim C(-z)^{(1+p)/(mp-\beta)},\ \textit{as } z \downarrow -\infty,\ \textit{and } f_1(z) \to 0,\ \textit{as } z \uparrow +\infty. \end{cases} \tag{38}$$

There exists $\tau_1, \varrho > 0$ such that for any $\tau \in (-\infty, -\tau_1)$ we have

$$u\left(\tau t^{\frac{mp-\beta}{(1+p)(1-\beta)}}, t\right) = \varrho t^{\frac{1}{1-\beta}},\ t \geq 0. \tag{39}$$

If $0 < C < C_*$, then we have

$$0 < \varrho < C_*(-\tau)^{\frac{1+p}{mp-\beta}}, \tag{40}$$

while if $C > C_*$, then $f_1(0) = A_1(m, p, \beta, C, b) = A_1 > 0$.

Proof of Lemma 3. We define

$$u_k(x,t) = ku(k^{\frac{\beta-mp}{1+p}}x, k^{\beta-1}t),\ k > 0, \tag{41}$$

(see Lemma 6 of [7]). The function (41) satisfies the CP (1), (4). We consider u to be a unique minimal solution of CP (1), (4) such that

$$u(x,t) \le ku(k^{\frac{\beta-mp}{1+p}}x, k^{\beta-1}t),\ k > 0. \tag{42}$$

By changing the variables in (42) as

$$y = k^{\frac{\beta-mp}{1+p}}x,\ \ell = k^{\beta-1}t, \tag{43}$$

we derive (42) with opposite inequality and with k replaced with k^{-1}. Since $k > 0$ is arbitrary, (42) follows with "=". Taking $k = t^{1/(1-\beta)}$, (41) implies (19) with $f_1(z) = u(z,1)$, where f_1 is a solution to problem (38). This proves the first part of the lemma. The second part of the lemma is proved in the same way as the second part of Lemma 3.3 of [3]. □

Lemma 4. *Let $b > 0$, $0 < \beta < mp < 1$, and $\alpha = (1+p)/(mp-\beta)$, and let u be the minimal solution to the CP* (1), (3). *Then u satisfies*

$$u\Big(\tau t^{\frac{mp-\beta}{(1+p)(1-\beta)}}, t\Big) \sim \varrho t^{\frac{1}{1-\beta}},\ as\ t \to 0^+, \tag{44}$$

where $\tau_1, \varrho > 0$ are the same as in Lemma 3. Furthermore, if $0 < C < C_*$, then

$$0 < \varrho < C_*(-\tau)^{\frac{1+p}{mp-\beta}}. \tag{45}$$

If $C > C_*$, then

$$u(0,t) \sim A_1 t^{\frac{1}{1-\beta}},\ as\ t \to 0^+;\ f_1(0) = A_1 > 0. \tag{46}$$

The proof of Lemma 4 follows as a localization of the proof of Lemma 3, exactly as local results were proven for Lemma 4 of [7].

Lemma 5. *If $b > 0$, $0 < \beta < mp < 1$, and $\alpha > (1+p)/(mp-\beta)$, then the unique weak solution u to the CP* (1), (3) *satisfies* (10).

The proof of Lemma 5 coincides with the proof of Lemma 7 of [7].

5 Proof of the Main Result

In this section we prove the results classified by region in the (α,β) parameter space diagram (Figure 1) described in Section 2.

Region I. From Lemma 2, the asymptotic formulas (6) and (18) follow. For any $\epsilon > 0$, from (6), there exists a number $\delta_1 = \delta_1(\epsilon) > 0$ such that

$$(A_0-\epsilon)t^{\alpha/(1+p-\alpha(mp-1))} \le u(0,t) \le (A_0+\epsilon)t^{\alpha/(1+p-\alpha(mp-1))},\ 0 \le t \le \delta_1(\epsilon), \tag{47}$$

where $A_0 = f(0) > 0$. The proof of results for Region I follows exactly as the proof of result (1) from [3] for $b \neq 0$, by choosing

$$g(x,t) = t^{\frac{1}{1-\beta}} f_1(z),\ z = xt^{\frac{\beta-mp}{(1-\beta)(1+p)}}, \tag{48}$$

$$f_1 = C_0(z_0 - z)_+^{\frac{1+p}{mp-\beta}}, 0 < z < +\infty \tag{49}$$

with $C_0, z_0 > 0$ to be determined. To prove the left-hand sides of estimates (17) and (5), we choose $C_0 = C_1$ and $z_0 = z_1$ (see the Appendix) and apply the comparison theorem. We prove the right-hand sides of the estimations (17) and (5) by choosing $C_0 = C_*$, $z_0 = z_2$ and τ_0 and the applying comparison theorem in the curved region $G_{\tau_0,\delta}$, where

$$G_{\tau,\delta} = \{(x,t) : z_\tau(t) = \tau t^{\frac{mp-\beta}{(1+p)(1-\beta)}} < x < +\infty, 0 < t \le \delta\}.$$

□

Region II. Assume that u_0 is defined as (4). The self-similar solution (19) follows from Lemma 3. The proof of estimation (21) when $C > C_*$ (also when u_0 is given through (3)) coincides with the proof given in [7]. Let $0 < C < C_*$. The formula (23) follows from Lemma 3. The proof of the right-hand side of (21) (also when u_0 is given through (3)) coincides with the proof given in [7] and the proof of the left-hand side of (21) follows in the same way as the proof of the analogous estimate from result (3) of [3]. □

Region III. The asymptotic estimate (10) follows from Lemma 4. The proof of the asymptotic estimate (9) coincides with the proof given in [7]. □

Region IV. The asymptotic estimation (6) is proved in Lemma 2. From (6), (47) follows. The proof of estimate (11) follows in the same way as the analogous estimate in result (4) from [3].

Intergration of (12) implies (24). By rescaling x with $\varepsilon^{-1}x, \varepsilon > 0$ from (24) we have

$$\frac{x}{\varepsilon} = \int_{\varphi(\frac{x}{\varepsilon})}^{1} \frac{m}{s}\left[\frac{b}{p} + \frac{m(1+p)}{p(1-mp)(1+m)} s^{1-mp}\right]^{-\frac{1}{1+p}} ds,\ s > 0.$$

Letting $r = -\varepsilon \ln s$ implies

$$x = \mathcal{F}[\Lambda_\varepsilon(x)], \tag{50}$$

where

$$\mathcal{F}(s) = \int_0^s m\left[\frac{b}{p} + \frac{m(1+p)}{p(1-mp)(1+m)} e^{r(mp-1)/\varepsilon}\right]^{-\frac{1}{1+p}} dr,$$

and

$$\Lambda_\varepsilon(x) = -\varepsilon \ln \varphi\Big(\frac{x}{\varepsilon}\Big).$$

From (50) it follows that

$$\Lambda_\varepsilon(x) = \mathcal{F}^{-1}(x), \tag{51}$$

where $\mathcal{F}^{-1}$ is an inverse function of $\mathcal{F}$.
Since $0 < mp < 1$ it follows that

$$\lim_{\varepsilon\to 0} \mathcal{F}(y) = m(b/p)^{-\frac{1}{1+p}} y,\ \lim_{\varepsilon\to 0} \mathcal{F}^{-1}(y) = m^{-1}(b/p)^{\frac{1}{1+p}} y, \tag{52}$$

for $y \geq 0$, uniformly, on bounded subsets. From (51) and (52) it follows that

$$-\lim_{\varepsilon\to 0^+} \varepsilon \ln \varphi\left(\frac{x}{\varepsilon}\right) = m^{-1}(b/p)^{\frac{1}{1+p}} x. \tag{53}$$

The asymptotic formula (26) follows by choosing $y = x/\varepsilon$. Inequality (27), as well as estimations (28),(29),(30) follow from (24) and (25). □

Region V. Let either $b > 0, \beta > mp$ or $b < 0, \beta \geq 1$. The proof of this case follows in the same way as the analogous case of result (5) of [3] The asymptotic estimation (6) follows from Lemma 2. Take arbitrary small $\epsilon > 0$. From (6), there exists a number $\delta_1 = \delta_1(\epsilon) > 0$ such that (47) holds. Let $\beta \geq 1$, and consider a function

$$g(x,t) = t^{\frac{\alpha}{1+p+\alpha(1-mp)}} f(\chi), \chi = xt^{\frac{-1}{1+p+\alpha(1-mp)}}. \tag{54}$$

We have

$$Lg = t^{\frac{\alpha mp-1-p}{1+p+\alpha(1-mp)}} L_1 f, \tag{55}$$

where

$$\begin{aligned} L_1 f = \frac{\alpha}{1+p+\alpha(1-mp)} f - \frac{1}{1+p+\alpha(1-mp)} \chi f' - \\ -(|(f^m)'|^{p-1}(f^m)')' + bt^{\frac{1+p-\alpha(mp-\beta)}{1+p+\alpha(1-mp)}} f^\beta. \end{aligned} \tag{56}$$

As a function f we select

$$f(\chi) = C_0(\chi_0 + \chi)^{\frac{1+p}{mp-1}}, \chi \geq 0, \tag{57}$$

where C_0 and χ_0 are positive constants.

From (47) and Lemma 1 of [7], the right-hand side of (31) follows with $\delta = \delta_2$, where where

$$\delta_2 = \delta_1, \text{ if } b > 0; \delta_2 = \min(\delta_1, \delta_3), \text{ if } b < 0,$$

and

$$\delta_3 = \left[\frac{\alpha\epsilon(A_0+\epsilon)^{1-\beta}}{(1+\epsilon)(-b(1+p+\alpha(1-mp)))}\right]^{\frac{1+p+\alpha(1-mp)}{1+p+\alpha(\beta-mp)}}.$$

To prove a lower bound in this case we take $C_0 = C_5$ and $\chi_0 = \chi_1$.

The proof of the left-hand side of (31) if either $b > 0, \beta < \frac{p(1-m)+2}{1+p}$ or $b < 0, \beta \geq 1$ or $b > 0, \beta \geq \frac{p(1-m)+2}{1+p}$, follows in the same way as the analogous estimate in result (5) of [3].

If $b > 0$ and $\beta \geq 1$, then the proof of estimates (14), (15), and (33) follow as the analogous proof from result (5) of [3]. While if $b > 0$ and $0 < mp < \beta < 1$ the left-hand side of (35) may be proved as left-hand side of (17) was previously proved. The only difference is that $f_1(z) = C_*(1-\epsilon)(z_8+z)_+^{\frac{1+p}{mp-\beta}}$ is chosen in (48). The proof of (16) follows in the same way as the proof of the analogous result from result (5) of [3].

Now, let $b = 0$. First assume that u_0 is defined by (4). The self-similar form (36) and the formula (18) follow from Lemma 1. To prove (37), again consider the function g as in (54), which satisfies (55) with $b = 0$. As a function f take (57). The proof of estimate (37) follows in the same way as the proof of the analogous result from result (5) of [3]. To prove an upper estimate we choose $C_0 = C_7$ and $\chi_0 = \chi_4$ and to prove a lower estimation we choose $C_0 = \mathcal{D}$ and $\chi_0 = \chi_3$. □

6 Conclusions

The following is a summary of the main results

- If $b > 0$, $0 < \beta < mp$, and $0 < \alpha < (1+p)/(mp-\beta)$, then diffusion weakly dominates over the absorption and the interface expands with asymptotic formula given by
$$\eta(t) \sim \psi(C,m,p,\alpha)t^{(mp-\beta)/(1-\beta)(1+p)}, \text{ as } t \to 0^+,$$
where, $\psi(C,m,p,\alpha) > 0$.
- If $b > 0$, $0 < \beta < mp$, and $\alpha = (1+p)/(mp-\beta)$, then diffusion and absorption are in balance, and there is a critical value C_* such that the interface expands or shrinks accordingly as $C > C_*$ or $C < C_*$ and
$$\eta(t) \sim z_*(C,m,p)t^{(mp-\beta)/(1-\beta)(1+p)}, \text{ as } t \to 0^+,$$
where $z_* \lesseqgtr 0$ if $C \lesseqgtr C_*$.
- If $b > 0$, $0 < \beta < mp$, and $\alpha > (1+p)/(mp-\beta)$, then absorption strongly dominates over diffusion and the interface shrinks with asymptotic formula given by
$$\eta(t) \sim -\tau_*(C,m,p,\alpha,\beta)t^{1/\alpha(1-\beta)}, \text{ as } t \to 0^+,$$
where, $\tau_*(C,m,p,\alpha,\beta) > 0$.
- If $b > 0$, $0 < \beta = mp < 1$, and $\alpha > 0$, then domination of the diffusion over absorption is moderate, there is an infinite speed of propagation, and the solution has exponential decay at infinity.
- If either $b > 0, \beta > mp$ or $b < 0, \beta \geq 1$, then diffusion strongly dominates over the absorption, and the solution has power type decay at infinity independent of $\alpha > 0$, which coincides with the asymptotics of the fast diffusion equation $(b = 0)$.

7 Appendix

Below are the explicit values of the constants used in Sections 2, 3, and 5.

I. $0 < \beta < mp$ and $0 < \alpha < (1+p)/(mp-\beta)$

$$z_1 = (b(1-\beta))^{\frac{mp-1}{(1+p)(1-\beta)}}(m(1+p))^{\frac{p}{1+p}}(p(m+\beta))^{\frac{1}{1+p}}(mp-\beta)^{\frac{p(m+\beta-1)-1}{(1+p)(1-\beta)}}(1-mp)^{\frac{1-mp}{(1+p)(1-\beta)}},$$

$$C_1 = \left(\frac{1-\beta}{1-mp}\right)^{\frac{1}{mp-\beta}}C_*,$$

$$\tau_0 = \left(\frac{1-mp}{mp-\beta}\right)^{\frac{mp-1}{1-\beta}}\left(\frac{\mathcal{D}}{C_*}\right)^{\frac{(mp-1)(\beta-mp)}{(1+p)(1-\beta)}}, z_2 = \tau_0\frac{1-\beta}{mp-\beta}.$$

II. $0<\beta<mp$ and $\alpha=(1+p)/(mp-\beta)$

$$C_2=A_1 z_3^{\frac{1+p}{\beta-mp}},\, A_1=f_1(0)>0,$$

$$z_3=(m(1+p))^{\frac{p}{p+1}}(m+\beta)^{\frac{1}{1+p}}p^{\frac{1}{1+p}}(mp-\beta)^{-1}(1-\beta)^{\frac{1}{1+p}}A_1^{\frac{m-1}{1+p}}\Big[b(1-\beta)A_1^{\beta-1}+1\Big]^{\frac{-1}{1+p}},$$

$$z_4=\left(\frac{A_1}{C_*}\right)^{\frac{mp-\beta}{1+p}},\, z_5=\tau_1-\left(\frac{\varrho}{C_*}\right)^{\frac{1+p}{mp-\beta}},\, C_3=C\left(\frac{1}{1-\delta_*\Gamma}\right)^{\frac{1+p}{mp-\beta}},$$

$$\Gamma=1-\left(\frac{C}{C_*}\right)^{\frac{mp-\beta}{1+p}},\, z_6=\delta_*\Gamma\tau_2,\ \text{with } \delta_* \text{ such that } g(\delta_*)=\max_{\delta\in(0,1)}g(\delta),$$

$$g(\delta)=(\delta\Gamma)^{\frac{1+p-p(m+\beta)}{(1+p)(1-\beta)}}\left[1-\delta\Gamma-\left(\frac{C}{C_*}\right)^{mp-\beta}\left(\frac{1}{1-\delta\Gamma}\right)^{p}\right]^{\frac{mp-\beta}{(1+p)(1-\beta)}},$$

$$\tau_2=C^{\frac{\beta-mp}{1+p}}\left[\frac{b(1-\beta)}{\delta_*\Gamma}\left(1-\delta_*\Gamma-\left(\frac{C}{C_*}\right)^{mp-\beta}\left(\frac{1}{1-\delta_*\Gamma}\right)^{p}\right)\right]^{\frac{mp-\beta}{(1+p)(1-\beta)}}.$$

V. $\beta>mp$

$$C_5=(1-\epsilon)^{\frac{1}{1-mp}}\mathcal{D},$$

$$C_6=\Big(\frac{\alpha(1-mp)^{p+1}}{\varkappa_b(1+p+\alpha(1-mp))(m(1+p))^p p(m+1)}\Big)^{\frac{1}{mp-1}},$$

$$\chi_1=(A_0-\epsilon)^{(mp-1)/(1+p)}(1-\epsilon)^{1/(1+p)}\mathcal{D}^{(1-mp)/(1+p)},\ \text{if } b>0,\ 1\le\beta<(p(1-m)+2)/(1+p),$$

$$\chi_1=(A_0-\epsilon)^{(mp-1)/(1+p)}\mathcal{D}^{(1-mp)/(1+p)},\ \text{if } b>0, \beta\ge(p(1-m)+2)/(1+p) \text{ or } b<0,\beta\ge1,$$

$$\chi_2=\Big(\frac{A_0+\epsilon}{C_6}\Big)^{\frac{mp-1}{1+p}},\ A_0=f(0)>0,\ \ \varkappa_b=\begin{cases}1, \text{ if } b>0,\\ 1+\epsilon, \text{ if } b<0,\end{cases}$$

$$z_8=\Big[b(1-\beta)C_*^{\beta-1}(1-\epsilon)^{mp-1}(1-(1-\epsilon)^{\beta-mp})\Big]^{\frac{mp-\beta}{(1+p)(1-\beta)}},$$

$$\chi_3=(A_0/\mathcal{D})^{(mp-1)/(1+p)},$$

$$\chi_4=\chi_3(1+(1+p)/\alpha(1-mp))^{1/(1+p)},$$

$$C_7=\mathcal{D}(1+(1+p)/\alpha(1-mp))^{1/(1+mp)}.$$

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References

[1] U. G. Abdulla, Reaction-diffusion in irregular domains, *Journal of Differential Equations,* **164** (2000), 321-354. http://doi.org/10.1006/jdeq.2000.3761

[2] U. G. Abdulla, On the Dirichlet problem for the nonlinear diffusion equation in non-smooth domains, *Journal of Mathematical Analysis and Applications,* **260** (2001), 384-403. http://doi.org/10.1006/jmaa.2001.7458

[3] U. G. Abdulla, Evolution of interfaces and explicit asymptotics at infinity for the fast diffusion equation with absorption, *Nonlinear Analysis,* **50** (2002), 541-560. http://doi.org/10.1016/S0362-546X(01)00764-7

[4] U. G. Abdulla, Well-posedness of the Dirichlet problem for the nonlinear diffusion equation in non-smooth domains, *Transactions of the American Mathematical Society,* **357** (2005), 247-265. http://doi.org/10.1090/S0002-9947-04-03464-6

[5] U. G. Abdulla, A. Abu Weden, Interface development for the nonlinear degenerate multidimensional reactiondiffusion equations, *Nonlinear Differential Equations and Applications,* **27** (2020). https://doi.org/10.1007/s00030-019-0606-2

[6] U. G. Abdulla, A. Abu Weden, Interface development for the nonlinear degenerate multidimensional reactiondiffusion equations II: fast diffusion versus absorption, *Nonlinear Differential Equations and Applications,* **30** (2023). https://doi.org/10.1007/s00030-023-00847-x

[7] U. G. Abdulla, J. Du, A. Prinkey, C. Ondracek, S. Parimoo, Evolution of interfaces for the nonlinear double degenerate parabolic equation of turbulent filtration with absorption, *Mathematics and Computers in Simulation,* **153** (2018), 59-82. http://doi.org/10.1016/j.matcom.2018.05.017

[8] U. G. Abdulla, R. Jeli, Evolution of interfaces for the non-linear parabolic p-Laplacian type reaction-diffusion equations, *European Journal of Applied Mathematics,* **28** (2017), 827-853. http://doi.org/10.1017/S0956792516000528

[9] U. G. Abdulla, J. R. King, Interface development and local solutions to reaction-diffusion equations, *SIAM Journal on Mathematical Analysis,* **32** (2000), 235-260. http://doi.org/10.1137/S003614109732986X

[10] G. I. Barenblatt, On some unsteady motions of a liquid or a gas in a porous medium, *Prikladnaya Matematika i Mekhanika (Applied Mathematics and Mechanics (PMM)),* **16** (1952), 67-78. (Russian)

[11] J. R. Esteban, J. L. Vazquez, On the equation of turbulent filtration in one-dimensional porous media, *Nonlinear Analysis: Theory, Methods & Applications,* **10** (1986), 1303-1325. http://doi.org/10.1016/0362-546X(86)90068-4

[12] R. E. Grundy, L. A. Peletier, Short time behaviour of a singular solution to the heat equation with absorption, *Proceedings of the Royal Society of Edinburgh Section A: Mathematics,***107** (1987), 271-288 https://doi.org/10.1017/S0308210500031152

[13] R. E. Grundy, L. A. Peletier, The initial interface development for a reaction-diffusion equation with power-law initial data, *Quarterly Journal of Mechanics and Applied Mathematics,***43** (1990), 535-559 https://doi.org/10.1093/qjmam/43.4.535

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