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Peakon Solutions for a Shallow Water System with Coriolis Effect

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Abstract

In this paper, we study whether a shallow water system with Coriolis effect admits peakon-delta weak solutions in distribution sense. Meanwhile, we find that the relationship between the coefficients of key nonlinear terms of the system plays an important role in the study of existence of peakon-delta weak solutions in distribution sense.

Mathematics Subject Classifications: 35D05, 35G25, 35L05, 35Q35

Keywords: Peakon weak solutions; The relationship between the coefficients; A shallow water system with Coriolis effect

1 Introduction

In this paper, we consider the following shallow water system with Coriolis effect

$$u_t - u_{txx} + \alpha u u_x + \beta u_x u_{xx} + \gamma u u_{xxx} + \eta_x + 2\Omega\eta_t = 0, t > 0, x \in \mathbb{R}, \quad (1)$$

$$\eta_t + ((1+\eta)u)_x = 0, t > 0, x \in \mathbb{R}, \quad (2)$$

where u is related to the average of horizontal velocity, η denotes free surface elevation from equilibrium with the boundary condition $u \to 0$ and $\eta \to 0$ as $|x| \to \infty$, α, β and γ are real constants, Ω is a dimensionless parameter describing the strength of the Coriolis effect ($\Omega > 0$). For $\alpha = 1$, $\beta = 4$ and $\gamma = 1$, system (1)-(2) reads to

$$u_t - u_{txx} + uu_x + 4u_x u_{xx} + uu_{xxx} + \eta_x + 2\Omega\eta_t = 0, t > 0, x \in \mathbb{R} \qquad (3)$$

$$\eta_t + ((1+\eta)u)_x = 0, t > 0, x \in \mathbb{R}, \qquad (4)$$

which are derived by Luo, Liu, Mi and Moon [1] from the rotation-Green-Naghdi equations by using the asymptotic expansion in the Camassa-Holm scaling [1].

The most interesting feature of system (3)-(4) is that it admits all kinds of travelling-wave solutions [1]. But it does not possess the Camassa-Holm-type peaked solution [1]. It is well known that peakon solution is a kind of the weak solution in the sense of distribution. It is shown that many equations and systems admit peakon solution, such as Camassa-Holm equation [2–4], Degasperis-Procesi equation [5, 6], modified Camassa-Holm equation [7, 8], Dullin-Gottwald-Holm equation [9], the two-component Camassa-Holm system [10, 11] and the rotation-two-component Camassa-Holm system [12] and so on.

The main objective of this paper is to investigate whether the shallow water system with Coriolis effect (1)-(2) admits peakon-delta weak solutions in distribution sense. Comparing with system (3)-(4), we find that in the study of existence of peakon-delta weak solutions in distribution sense, the relationship between the coefficients α, β and γ in equation (1) acts a key role. Namely, if and only if $\alpha + \frac{\beta}{\sigma^2} + \frac{\gamma}{\sigma^2} = 0$ (where σ is real number to be determined.), there exist peakon-delta weak solutions in distribution sense.

2 Construction of peakon solutions

Let us consider the traveling-wave solution of the system (1)-(2) by the setting

$$u(x,t) = u(\xi), \rho(x,t) = \rho(\xi), \xi = x - q(t), \qquad (5)$$

where $q(t)$ is a function to be determined. Substituting (5) into system (1)-(2), we get

$$-q'(t)u_\xi + q'(t)u_{\xi\xi\xi} + \alpha uu_\xi + \beta u_\xi u_{\xi\xi} + \gamma uu_{\xi\xi\xi} + \eta_\xi - 2\Omega q'(t)\eta_\xi = 0, \qquad (6)$$

$$-q'(t)\eta_\xi + u_\xi + u\eta_\xi + \eta u_\xi = 0. \qquad (7)$$

For (7), a distribution has zero derivative if and only if it is a constant. So there is an $\epsilon \in \mathbb{R}$ such that

$$\eta = \frac{u - \epsilon}{q'(t) - u}. \qquad (8)$$

Substituting (8) into (6) gives rise to

$$\begin{aligned}&(-q'(t)+\alpha u)(q'(t)-u)^2u_\xi+(q'(t)+\gamma u)(q'(t)-u)^2u_{\xi\xi\xi}\\&+\beta(q'(t)-u)^2u_\xi u_{\xi\xi}+(1-2\Omega q'(t))(q'(t)-\epsilon)u_\xi=0.\end{aligned}\tag{9}$$

We search for peakon solution of (9) in the form

$$u=r+pe^{\frac{-|\xi|}{\sigma}},\tag{10}$$

where the r and p are constants to be determined. With the help of distribution theory, we deduce that

$$u_\xi=-\frac{p}{\sigma}e^{\frac{-|\xi|}{\sigma}}\mathrm{sgn}(\xi),\tag{11}$$

$$u_{\xi\xi}=\frac{p}{\sigma^2}e^{\frac{-|\xi|}{\sigma}}-\frac{2p}{\sigma}\delta(\xi),\tag{12}$$

$$u_{\xi\xi\xi}=-\frac{p}{\sigma^3}e^{\frac{-|\xi|}{\sigma}}\mathrm{sgn}(\xi)-\frac{2p}{\sigma}\delta_\xi(\xi).\tag{13}$$

Substituting (11)-(13) into (9) arrives at

$$\begin{aligned}&-\frac{2p}{\sigma}(q'(t)+\gamma r+\gamma pe^{\frac{-|\xi|}{\sigma}})(q'(t)-r-pe^{\frac{-|\xi|}{\sigma}})^2\delta_\xi(\xi)\\&+\frac{2p^2}{\sigma^2}\beta(q'(t)-r-pe^{\frac{-|\xi|}{\sigma}})^2e^{\frac{-|\xi|}{\sigma}}\delta(\xi)\mathrm{sgn}(\xi)\\&+\frac{p}{\sigma}[(1-\frac{1}{\sigma^2})q'(t)-\alpha r-\gamma r\frac{1}{\sigma^2}](q'(t)-r-pe^{\frac{-|\xi|}{\sigma}})^2e^{\frac{-|\xi|}{\sigma}}\mathrm{sgn}(\xi)\\&-\frac{p^2}{\sigma}(\alpha+\frac{\beta}{\sigma^2}+\frac{\gamma}{\sigma^2})(q'(t)-r-pe^{\frac{-|\xi|}{\sigma}})^2e^{\frac{-2|\xi|}{\sigma}}\mathrm{sgn}(\xi)\\&-\frac{p}{\sigma}(q'(t)-\epsilon)(1-2\Omega q'(t))e^{\frac{-|\xi|}{\sigma}}\mathrm{sgn}(\xi)=0.\end{aligned}\tag{14}$$

Noticing that for any $\varphi(\xi)\in C_0^\infty(\mathbb{R})$, we claim

$$\begin{aligned}&\left\langle\frac{2p}{\sigma}(q'(t)+\gamma r+\gamma pe^{\frac{-|\xi|}{\sigma}})(q'(t)-r-pe^{\frac{-|\xi|}{\sigma}})^2\delta_\xi(\xi),\varphi(\xi)\right\rangle\\&=\int_{-\infty}^{\infty}\frac{2p}{\sigma}(q'(t)+\gamma r+\gamma pe^{\frac{-|\xi|}{\sigma}})(q'(t)-r-pe^{\frac{-|\xi|}{\sigma}})^2\delta_\xi(\xi)\varphi(\xi)d\xi\\&=\frac{d}{d\xi}[\frac{2p}{\sigma}(q'(t)+\gamma r+\gamma pe^{\frac{-|\xi|}{\sigma}})(q'(t)-r-pe^{\frac{-|\xi|}{\sigma}})^2\varphi(\xi)]\mid_{\xi=0}\\&=\frac{2p}{\sigma}(q'(t)+\gamma r+\gamma p)(q'(t)-r-p)^2\varphi_\xi(0),\end{aligned}\tag{15}$$

and

$$\left\langle \frac{2p^2}{\sigma^2}\beta(q'(t)-r-pe^{\frac{-|\xi|}{\sigma}})^2 e^{\frac{-|\xi|}{\sigma}}\delta(\xi)\mathrm{sgn}(\xi),\varphi(\xi)\right\rangle$$
$$=\int_{-\infty}^{\infty}\frac{2p^2}{\sigma^2}\beta(q'(t)-r-pe^{\frac{-|\xi|}{\sigma}})^2 e^{\frac{-|\xi|}{\sigma}}\delta(\xi)\mathrm{sgn}(\xi)\varphi(\xi)d\xi$$
$$=\frac{2p^2}{\sigma^2}\beta(q'(t)-r-pe^{\frac{-|\xi|}{\sigma}})^2 e^{\frac{-|\xi|}{\sigma}}\mathrm{sgn}(\xi)\varphi(\xi)\mid_{\xi=0}=0. \tag{16}$$

Making inner product with (14) by $\varphi(\xi)\in C_0^\infty(\mathbb{R})$ and using (15) and (16), we get

$$\frac{2p}{\sigma}(q'(t)+\gamma r+\gamma p)(q'(t)-r-p)^2\varphi_\xi(0)$$
$$+\frac{p}{\sigma}[(1-\frac{1}{\sigma^2})q'(t)-\alpha r-\gamma r\frac{1}{\sigma^2}]\int_{-\infty}^{\infty}(q'(t)-r-pe^{\frac{-|\xi|}{\sigma}})^2 e^{\frac{-|\xi|}{\sigma}}\mathrm{sgn}(\xi)\varphi(\xi)d\xi$$
$$-\frac{p^2}{\sigma}(\alpha+\frac{\beta}{\sigma^2}+\frac{\gamma}{\sigma^2})\int_{-\infty}^{\infty}(q'(t)-r-pe^{\frac{-|\xi|}{\sigma}})^2 e^{\frac{-2|\xi|}{\sigma}}\mathrm{sgn}(\xi)\varphi(\xi)d\xi$$
$$-\frac{p}{\sigma}(q'(t)-\epsilon)(1-2\Omega q'(t))\int_{-\infty}^{\infty}e^{\frac{-|\xi|}{\sigma}}\mathrm{sgn}(\xi)\varphi(\xi)d\xi=0. \tag{17}$$

Equation (17) holds in the sense of distribution, provided that

$$\frac{2p}{\sigma}(q'(t)+\gamma r+\gamma p)(q'(t)-r-p)^2=0, \tag{18}$$

$$\frac{p}{\sigma}[(1-\frac{1}{\sigma^2})q'(t)-\alpha r-\gamma r\frac{1}{\sigma^2}]=0, \tag{19}$$

$$\frac{p^2}{\sigma}(\alpha+\frac{\beta}{\sigma^2}+\frac{\gamma}{\sigma^2})=0 \tag{20}$$

and

$$\frac{p}{\sigma}(q'(t)-\epsilon)(1-2\Omega q'(t))=0. \tag{21}$$

Let us discuss the solutions of the system (1)-(2) from system (18)-(21) for different cases

2.1 $\epsilon=0$

In this case, the system (18)-(21) reduces to

$$(q'(t)+\gamma r+\gamma p)(q'(t)-r-p)^2=0, \tag{22}$$

$$[(1-\frac{1}{\sigma^2})q'(t)-\alpha r-\gamma r\frac{1}{\sigma^2}]=0, \tag{23}$$

$$(\alpha + \frac{\beta}{\sigma^2} + \frac{\gamma}{\sigma^2}) = 0 \tag{24}$$

and

$$q'(t)(1 - 2\Omega q'(t)) = 0. \tag{25}$$

Case 1

For $\sigma = 1$, then $r = 0$. Therefore under the condition $\alpha + \beta + \gamma = 0$, we deduce

$$q'(t) = \frac{1}{2\Omega} = p, \tag{26}$$

which leads to

$$q(t) = \frac{1}{2\Omega}t + x_0. \tag{27}$$

So we obtain

$$\begin{aligned} u &= \frac{1}{2\Omega}e^{-|x-\frac{1}{2\Omega}t-x_0|}, \\ \eta &= -1 + \frac{1}{1 - e^{-|x-\frac{1}{2\Omega}t-x_0|}}. \end{aligned} \tag{28}$$

Case 2

For $\sigma = 1$, then $r = 0$. Therefore under the condition $\alpha + \beta + \gamma = 0$, we deduce

$$q'(t) = \frac{1}{2\Omega} = -\gamma p, \tag{29}$$

which leads to

$$q(t) = \frac{1}{2\Omega}t + x_0, p = -\frac{1}{2\gamma\Omega}. \tag{30}$$

So we obtain

$$\begin{aligned} u &= -\frac{1}{2\gamma\Omega}e^{-|x-\frac{1}{2\Omega}t-x_0|}, \\ \eta &= -1 + \frac{\gamma}{\gamma - e^{-|x-\frac{1}{2\Omega}t-x_0|}}. \end{aligned} \tag{31}$$

Case 3

Assume that $0 < \sigma^2 = -\frac{\beta+\gamma}{\alpha} \neq 1$, then

$$q'(t) = \frac{1}{2\Omega} = r + p = \frac{\alpha\beta r}{\alpha + \beta + \gamma}, \tag{32}$$

which implies that

$$r = \frac{\alpha + \beta + \gamma}{2\Omega\alpha\beta}, \tag{33}$$

$$p = \frac{\alpha\beta - \alpha - \beta - \gamma}{2\Omega\alpha\beta}, \tag{34}$$

$$q(t) = \frac{1}{2\Omega}t + x_0. \tag{35}$$

So, we get

$$\begin{aligned} u &= \frac{\alpha+\beta+\gamma}{2\Omega\alpha\beta} + \frac{\alpha\beta-\alpha-\beta-\gamma}{2\Omega\alpha\beta} e^{\frac{-|x-\frac{1}{2\Omega}t-x_0|}{\sigma}}, \\ \eta &= -1 + \frac{\alpha\beta}{(\alpha\beta-\alpha-\beta-\gamma)(1-e^{\frac{-|x-\frac{1}{2\Omega}t-x_0|}{\sigma}})}, \end{aligned} \tag{36}$$

where $\sigma = \sqrt{-\frac{\beta+\gamma}{\alpha}}$.

Case 4

Assume that $0 < \sigma^2 = -\frac{\beta+\gamma}{\alpha} \neq 1$, then

$$q'(t) = \frac{1}{2\Omega} = -\gamma(r+p) = \frac{\alpha\beta r}{\alpha+\beta+\gamma}, \tag{37}$$

which implies that

$$r = \frac{\alpha+\beta+\gamma}{2\Omega\alpha\beta}, \tag{38}$$

$$p = -\frac{1}{2\Omega\gamma} - \frac{\alpha+\beta+\gamma}{2\Omega\alpha\beta}, \tag{39}$$

$$q(t) = \frac{1}{2\Omega}t + x_0. \tag{40}$$

So, we have

$$\begin{aligned} u &= \frac{\alpha+\beta+\gamma}{2\Omega\alpha\beta} - \left(\frac{1}{2\Omega\gamma} + \frac{\alpha+\beta+\gamma}{2\Omega\alpha\beta}\right) e^{\frac{-|x-\frac{1}{2\Omega}t-x_0|}{\sigma}}, \\ \eta &= -1 + \frac{\alpha\beta\gamma}{\gamma(\alpha\beta-\alpha-\beta-\gamma) + (\alpha\beta+\gamma(\alpha+\beta+\gamma))e^{\frac{-|x-\frac{1}{2\Omega}t-x_0|}{\sigma}}}, \end{aligned} \tag{41}$$

where $\sigma = \sqrt{-\frac{\beta+\gamma}{\alpha}}$.

Remark. Using same procedure with the above, we can obtain peakons under the condition $\epsilon \neq 0$. Here, we omit them.

3 discussion

In this paper, we construct single peakon weak solutions for the shallow-water system with the Coriolis effect in distribution sense. Comparing with (3)-(4), we find that in the study of existence of peakon-delta weak solutions in distribution sense, the relationship between the coefficients α, β and γ in equation (1) acts a key role. If and only if $\alpha + \frac{\beta}{\sigma^2} + \frac{\gamma}{\sigma^2} = 0$, there exist peakon-delta weak solutions in distribution sense.

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