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Global Well-Posedness in Besov-Morrey Space for a Two-Species Chemotaxis Model with Two Chemicals

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Abstract

In this paper, we study the Cauchy problem for a two-species chemotaxis model in $\mathbb{R}^N$ for $N \geq 2$. We prove the global well-posedness with small initial data in Besov-Morrey spaces.

Keywords: Chemotaxis models; Global well-posedness; Besov-Morrey spaces

1 Introduction

In the present paper, we study the Keller-Segel system of a two-species chemotaxis model with two chemicals:

$$\begin{cases} u_t = \nabla \cdot (\nabla u - \chi_1 u \nabla v), \\ \tau_1 v_t = \Delta v - v + \omega, \\ \omega_t = \nabla \cdot (\nabla \omega - \chi_2 \omega \nabla z), \\ \tau_2 z_t = \Delta z - z + u, \\ (u, \tau_1 v, \omega, \tau_2 z)(x,t)|_{t=0} = (u_0, \tau_1 v_0, \omega_0, \tau_2 z_0)(x), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^N, N \geq 2$, $u(x,t)$ and $w(x,t)$ respectively denote the unknown density of macrophages and tumor cells, while $v(x,t)$ and $z(x,t)$ represent the concentrations of chemical signals secreted by w and u respectively. The two species are coupled in such a way that they move towards higher concentrations of the chemicals v and w respectively. The modelling parameters $\chi_i > 0$, $\tau_i \geq 0$ $(i = 1, 2)$ are given constants.

When $\tau_1 = \tau_2 = 0$, Tao and Winkler [4] systematically studied the boundedness vs blow-up, wherein χ_1 and χ_2 are allowed to be real: for either $\chi_1 < 0$ or $\chi_2 < 0$, boundedness for large initial data is guaranteed in $\leq$ 3D. When both $\chi_1 > 0$ and $\chi_2 > 0$, boundedness vs blow-up is characterized by the total mass of both species: writing

$$m_1 = \int_\Omega u_0, \quad m_2 = \int_\Omega \omega_0,$$

then boundedness is ensured for $\max\{m_1, m_2\} < C_0$ with some $C_0 > 0$, whereas, for $\chi_1 = \chi_2 = 1$, finite time blow-up in 2D may occur for $\min\{m_1, m_2\} > 4\pi$. These results were improved by Yu et al. in [5] by showing that $C_0 = 4\pi$ and a blow-up criterion that

$$\frac{1}{m_2\chi_1} + \frac{1}{m_1\chi_2} < \frac{1}{2\pi}.$$

When $\tau_1 > 0$, $\tau_2 > 0$, Li and Wang [3] obtained boundedness for (1.1) under an implicit samllness condition. Lin and Xiang [2] obtained global well-posedness and finite time blow-up for system (1.1) in a 2D bounded and smooth domain. However, the problem whether system (1.1) is well-posedness in critical Besov-Morrey spaces is still unknown. Motivated by the arguments in [1], we generalize their method to the chemotaxis system (1.2) and prove the global well-posedness with small initial data in a large caritical framework based on Besov-Morrey spaces.

For simplicity, we take $\chi_i = \tau_i = 1$ $(i = 1, 2)$. Then, Model (1.1) can be rewritten as

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + w, \\ w_t = \Delta w - \nabla \cdot (w \nabla z), \\ z_t = \Delta z - z + u, \\ (u, v, \omega, z)(x,t)|_{t=0} = (u_0, v_0, \omega_0, z_0)(x), \end{cases} \tag{1.2}$$

and model (1.2) has the scaling

$$u_\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t), \quad v_\lambda(x,t) = v(\lambda x, \lambda^2 t),$$

$$\omega_\lambda(x,t) = \lambda^2\omega(\lambda x, \lambda^2 t), \quad z_\lambda(x,t) = z(\lambda x, \lambda^2 t),$$

which, by taking $t = 0$, induces the initial data scaling

$$u_{0\lambda} = \lambda^2 u_0(\lambda x), \quad v_{0\lambda} = v_0(\lambda x), \quad \omega_{0\lambda} = \lambda^2\omega_0(\lambda x), \quad z_{0\lambda}(x,t) = z(\lambda x).$$

This paper is organized as follows. In Section 2, we recall the definitions of Morrey and Besov-Morrey spaces. In Section 3, we stated our results on well-posedness. Section 4 is devoted to the proof of the Theorem 3.1.

2 Preliminaries

This section is devoted to some preliminaries about Morrey and Besov-Morry spaces.

Definition 2.1 *For $1 \le p_1 \le p < \infty$, the Morrey space $\mathcal{M}^p_{p_1} = \mathcal{M}^p_{p_1}(\mathbb{R}^N)$ is defined as the set of all measurable functions u such that*

$$\|u\|_{\mathcal{M}^p_{p_1}} = \sup_{x_0\in\mathbb{R}^N}\sup_{R>0} R^{\frac{N}{p}-\frac{N}{p_1}}\|u\|_{L^{p_1}(D(x_0,R))} < \infty, \tag{2.1}$$

where $D(x_0, R)$ denotes the closed ball in $\mathbb{R}^N$ with center x_0 and radius R.

The space $\mathcal{M}^p_{p_1}$ endowed with $\|\cdot\|_{\mathcal{M}^p_{p_1}}$ is a Banach space. In the case $p_1 = 1$, $\mathcal{M}^p_{p_1}$ is a space of signed Radon measures and $\|u\|_{L^1(D(x_0,R))}$ is meant as the total variation of the measure u in the ball $D(x_0, R)$. For $1 < p < \infty$ we have that $\mathcal{M}^p_{p_1} = L^p$ and $\mathcal{M}^1_1 = \mathcal{M}$ where $\mathcal{M}$ stands for the space of signed Radon measures with finite total variation. In the case $p = p_1 = \infty$, we consider $\mathcal{M}^\infty_\infty = L^\infty$.

Let us denote by $\mathcal{S}$ and $\mathcal{S}'$ the Schwartz class and the space of tempered distributions, respectively. For $u \in \mathcal{S}'$, we denote the Fourier transform of u by $\widehat{u}$ and its inverse by $u^\vee$. Let $\chi(z)$ be a C^∞-function on $[0,\infty)$ such that $0 \le \chi(z) \le 1, \chi(z) \equiv 1$ for $z \le 3/2$ and supp $\chi \subset [0, 5/3)$. Then, for all $j \in \mathbb{Z}$, put $\varphi_j(\xi) = \chi(2^{-j}|\xi|) - \chi(2^{1-j}|\xi|)$. It follows that $\varphi_j(\xi) \in C_0^\infty(\mathbb{R}^N)$ and we have the dyadic decomposition

$$\sum_{j=-\infty}^{\infty}\varphi_j(\xi) = 1, \text{ for all } \xi \neq 0.$$

Definition 2.2 *The homogeneous Besov-Morrey space* $\mathcal{N}^s_{p,p_1,r}=\mathcal{N}^s_{p,p_1,r}\left(\mathbb{R}^N\right)$ *is the set of all* $u\in\mathcal{S}'/\mathcal{P}$ *such that* $\varphi_j^\vee * u\in\mathcal{M}^p_{p_1}$ *for all* j*, and*

$$\|u\|_{\mathcal{N}^s_{p,p_1,r}}=\begin{cases}\Big(\sum\limits_{j\in\mathbb{Z}}(2^{sj}\|\varphi_j^\vee * u\|_{\mathcal{M}^p_{p_1}})^r\Big)^{\frac{1}{r}}<\infty, & \text{for } 1\le p_1\le p\le\infty, 1\le r<\infty, s\in\mathbb{R},\\ \sup\limits_{j\in\mathbb{Z}}(2^{sj}\|\varphi_j^\vee * u\|_{\mathcal{M}^p_{p_1}})<\infty, & \text{for } 1\le p_1\le p\le\infty, r=\infty, s\in\mathbb{R},\end{cases}\tag{2.2}$$

where $\mathcal{P}$ *denotes the set of polynomials with* N *variables.*

3 Results

In this section we present our global well-posedness for the system (1.2).

We consider the following critical initial data class

$$\begin{aligned} u_0\in\mathcal{N}^{\frac{N}{q}-2}_{q,q_1,\infty}(\mathbb{R}^N),\quad v_0\in\mathcal{S}'/\mathcal{P}\text{ with }\nabla\cdot v_0\in\mathcal{N}^{\frac{N}{r}-1}_{r,r_1,\infty}(\mathbb{R}^N),\\ w_0\in\mathcal{N}^{\frac{N}{q}-2}_{q,q_1,\infty}(\mathbb{R}^N),\quad z_0\in\mathcal{S}'/\mathcal{P}\text{ with }\nabla\cdot z_0\in\mathcal{N}^{\frac{N}{r}-1}_{r,r_1,\infty}(\mathbb{R}^N),\end{aligned}\tag{3.1}$$

where the exponents q,q_1,r,r_1 and N_1 are as in the Remark 3.2.

Let $\mathcal{Z}$ be a Banach space continuously included in $\mathcal{S}'$ and denote by $BC_w\big((0,\infty);\mathcal{Z}\big)$ the class of bounded functions from $(0,\infty)$ to $\mathcal{Z}$ that are weakly time continuous in the sense of $\mathcal{S}'$. We define the functional spaces

$$X_1=\{u:t^{-\frac{N}{2q}+1}u\in BC_w\big((0,\infty);\mathcal{M}^q_{q_1}\big)\},\tag{3.2}$$

$$X_2=\{v:v(\cdot,t)\in\mathcal{S}'/\mathcal{P}\text{ for }t>0\quad\text{and}\quad t^{-\frac{N}{2r}+\frac{1}{2}}\nabla v\in BC_w\big((0,\infty);\mathcal{M}^r_{r_1}\big)\},\tag{3.3}$$

$$X_3=\{w:t^{-\frac{N}{2q}+1}w\in BC_w\big((0,\infty);\mathcal{M}^q_{q_1}\big)\},\tag{3.4}$$

$$X_4=\{z:z(\cdot,t)\in\mathcal{S}'/\mathcal{P}\text{ for }t>0\quad\text{and}\quad t^{-\frac{N}{2r}+\frac{1}{2}}\nabla z\in BC_w\big((0,\infty);\mathcal{M}^r_{r_1}\big)\},\tag{3.5}$$

which are Banach spaces endowed with the respective norms

$$\begin{aligned}\|u\|_{X_1}&=\sup_{t>0}t^{-\frac{N}{2q}+1}\|u(t)\|_{\mathcal{M}^q_{q_1}},\\ \|v\|_{X_2}&=\sup_{t>0}t^{-\frac{N}{2r}+\frac{1}{2}}\|\nabla v(t)\|_{\mathcal{M}^r_{r_1}},\\ \|w\|_{X_3}&=\sup_{t>0}t^{-\frac{N}{2q}+1}\|w(t)\|_{\mathcal{M}^q_{q_1}},\end{aligned}$$

$$\|z\|_{X_4} = \sup_{t>0} t^{-\frac{N}{2r}+\frac{1}{2}} \|\nabla z(t)\|_{\mathcal{M}^r_{r_1}}.$$

Let us introduce the spaces $\mathcal{X}$ and $\mathcal{I}$ as

$$\mathcal{X} := \{(u, v, w, z) : u \in X_1, v \in X_2, w \in X_3, z \in X_4\} \tag{3.6}$$

with the norm

$$\|(u, v, w, z)\|_{\mathcal{X}} := \|u\|_{X_1} + \|v\|_{X_2} + \|w\|_{X_3} + \|z\|_{X_4},$$

and

$$\mathcal{I} := \{(u_0, v_0, w_0, z_0) : u_0, v_0, w_0, z_0 \quad \text{are as in (3.1)}\}$$

with the norm

$$\|(u_0, v_0, w_0, z_0)\|_{\mathcal{I}} := \|u_0\|_{\mathcal{N}^{\frac{N}{q}-2}_{q,q_1,\infty}} + \|\nabla v_0\|_{\mathcal{N}^{\frac{N}{r}-1}_{r,r_1,\infty}} + \|w_0\|_{\mathcal{N}^{\frac{N}{q}-2}_{q,q_1,\infty}} + \|\nabla z_0\|_{\mathcal{N}^{\frac{N}{r}-1}_{r,r_1,\infty}}.$$

Note the $\mathcal{X}$ and $\mathcal{I}$ are Banach spaces equipped with the norm $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{I}}$, respectively.

Using Duhamel's principle, system (1.2) can be formally converted to the following integral formulation

$$\begin{cases} u(t) = e^{t\Delta} u_0 - \int_0^t \nabla e^{(t-\tau)\Delta}(u \cdot \nabla v)(\tau) d\tau, \\ v(t) = e^{-t} e^{t\Delta} v_0 + \int_0^t e^{-(t-\tau)} e^{(t-\tau)\Delta} w(\tau) d\tau, \\ w(t) = e^{t\Delta} w_0 - \int_0^t \nabla e^{(t-\tau)\Delta}(w \cdot \nabla z)(\tau) d\tau, \\ z(t) = e^{-t} e^{t\Delta} z_0 + \int_0^t e^{-(t-\tau)} e^{(t-\tau)\Delta} u(\tau) d\tau. \end{cases} \tag{3.7}$$

A 4-tuple (u, v, w, z) satisfying (3.7) is called a mild solution of (1.2).

Theorem 3.1 *Let $N \geq 2$, and let the exponents q, q_1, r, r_1 and N_1 be as in Remark* 3.2. *Suppose that the initial data $(u_0, v_0, w_0, z_0) \in \mathcal{I}$. There exist positive constants ε, δ ($\delta = C\varepsilon$) and K_1 such that the system* (3.7) *has a unique global mild solution $(u, v, w, z) \in \mathcal{X}$ satisfying $\|(u, v, w, z)\|_{\mathcal{X}} \leq 2K_1\varepsilon$ provide that $\|(u_0, v_0, w_0, z_0)\|_{\mathcal{I}} \leq \delta$.*

Remark 3.2 *Assume that $N \geq 2$ and $\gamma \geq 0$. For $N \geq 3$, suppose that the exponents q, r satisfy either (a), (b) or (c) where*

$$
\begin{aligned}
&(a) \quad \frac{N}{2} < q < N, \quad N < r < \frac{Nq}{N-q};\\
&(b) \quad q = N, \quad N < r < \infty;\\
&(c) \quad N < q < 2N, \quad q < r < \frac{Nq}{q-N}.
\end{aligned}
$$

In the case $N = 2$ we assume that q, r satisfy the condition (c) above. Moreover, suppose also that q_1, r_1, N_1 satisfy the following conditions

$$
\begin{aligned}
&(A) \quad 1 \leq q_1 \leq q, \quad 1 \leq r_1 \leq r, \quad 1 \leq N_1 \leq N;\\
&(B) \quad \frac{1}{r_1} + \frac{1}{q_1} \leq 1, \quad \frac{1}{N_1} + \frac{1}{q_1} \leq 1;
\end{aligned}
$$

It is always possible to find indexes q_1, r_1, N_1 sufficiently close to q, r, N, respectively, satisfying either (a), (b) or (c), and such that $(A), (B)$ hold true. In other words, Remark 3.2 is not empty.

4 Proofs

In this section, we present the proofs of results stated in Section 3. First we prove a fixed point lemma which will be useful for our ends.

Lemma 4.1 *For $1 \leq i \leq 4$, let X_i be a Banach space with the norm $\|\cdot\|_{X_i}$. Consider the Banach space $\mathcal{X} = X_1 \times X_2 \times X_3 \times X_4$ endowed with the norm*

$$\|x\|_{\mathcal{X}} = \|x_1\|_{X_1} + \|x_2\|_{X_2} + \|x_3\|_{X_3} + \|x_4\|_{X_4},$$

where $x = (x_1, x_2, x_3, x_4) \in \mathcal{X}$. For $1 \leq i, j, k \leq 4$, assume that $B^k_{i,j} : X_i \times X_j \longrightarrow X_k$ is a continuous bilinear map, that is, there is a constant $C^k_{i,j} > 0$ such that

$$\left\|B^k_{i,j}(x_i, x_j)\right\|_{X_k} \leq C^k_{i,j}\|x_i\|_{X_i}\|x_j\|_{X_j}, \qquad \text{for all} \quad (x_i, x_j) \in X_i \times X_j. \tag{4.1}$$

Assume also that: $L_2 : X_2 \to X_3$ and $L_4 : X_1 \to X_4$ are continuous linear maps such that $\|L_2\|_{X_2 \to X_3} \leq \alpha$ and $\|L_4\|_{X_1 \to X_4} \leq \beta$. Set the constants

$$K_1 := 1 + \alpha + \beta \quad \text{and} \quad K_2 := \sum_{i,j=1}^{4} \left(C^1_{i,j}(\beta + 1) + C^3_{i,j}(\alpha + 1)\right),$$

and let $0 < \varepsilon < \frac{1}{4K_1K_2}$ *and* $\mathcal{B}_\varepsilon = \{x \in \mathcal{X} : \|x\|_{\mathcal{X}} \le 2K_1\varepsilon\}$. *If* $\|y\|_{\mathcal{X}} \le \varepsilon$ *then there exists a unique solution* $x \in \mathcal{B}_\varepsilon$ *for the equation* $x = y + B(x)$, *where* $y = (y_1, y_2, y_3, y_4)$, $B(x) = \big(B_1(x), B_2(x), B_3(x), B_4(x)\big)$ *and*

$$B_1(x) = \sum_{i,j=1}^{4} B^1_{i,j}(x_i, x_j), \quad B_2(x) = \big(L_2 \circ (y_3 + B_3)\big)(x),$$
$$B_3(x) = \sum_{i,j=1}^{4} B^3_{i,j}(x_i, x_j), \quad B_4(x) = \big(L_4 \circ (y_1 + B_1)\big)(x).$$

Proof. For all $x \in \mathcal{X}$, it follows from (4.1) that

$$\begin{aligned} \|B_1(x)\|_{X_1} &\le \sum_{i,j=1}^{4} \|B^1_{i,j}(x_i, x_j)\|_{X_1} \\ &\le \sum_{i,j=1}^{4} C^1_{i,j} \|x_i\|_{X_i} \|x_j\|_{X_j} \\ &\le \big(\sum_{i,j=1}^{4} C^1_{i,j}\big) \|x\|^2_{\mathcal{X}}. \end{aligned} \tag{4.2}$$

Similarly, we have

$$\|B_3(x)\|_{X_3} \le \big(\sum_{i,j=1}^{4} C^3_{i,j}\big) \|x\|^2_{\mathcal{X}}. \tag{4.3}$$

Next, using (4.1) and (4.3), we estimate B_2 as follows:

$$\begin{aligned} \|B_2(x)\|_{X_2} &\le \big\|L_2 \circ (y_3 + B_3)(x)\big\|_{X_3} \\ &\le \alpha\big(\|y\|_{\mathcal{X}} + (\sum_{i,j=1}^{4} C^3_{i,j}) \|x\|^2_{\mathcal{X}}\big) \\ &\le \alpha \sum_{i,j=1}^{4} C^3_{i,j} \|x\|^2_{\mathcal{X}} + \alpha \|y\|_{\mathcal{X}}. \end{aligned} \tag{4.4}$$

Similarly, it follows that

$$\|B_4(x)\|_{X_4} \le \beta \sum_{i,j=1}^{4} C^1_{i,j} \|x\|^2_{\mathcal{X}} + \beta \|y\|_{\mathcal{X}}. \tag{4.5}$$

Now, consider the mapping $\mathcal{F}:\mathcal{X}\to\mathcal{X}$ given by $F(x)=y+B(x)$. For $x\in\mathcal{B}_\varepsilon$, from (4.2)-(4.5), we obtain that

$$\begin{aligned}
\|\mathcal{F}(x)\|_{\mathcal{X}} &\le \|y\|_{\mathcal{X}}+\sum_{k=1}^{4}\|B_k(x)\|_{X_k}\\
&\le (1+\alpha+\beta)\|y\|_{\mathcal{X}}+\sum_{i,j=1}^{4}\big(C^1_{i,j}(\beta+1)+C^3_{i,j}(\alpha+1)\big)\|x\|^2_{\mathcal{X}}\\
&\le K_1\varepsilon+K_2(2K_1\varepsilon)^2\\
&\le 2K_1\varepsilon,
\end{aligned}$$

and then $\mathcal{F}(\mathcal{B}_\varepsilon)\subset\mathcal{B}_\varepsilon$. Next, we take $x,z\in\mathcal{B}_\varepsilon$ and estimate

$$\begin{aligned}
&\|\mathcal{F}(x)-\mathcal{F}(z)\|_{\mathcal{X}}\\
&=\|B(x)-B(z)\|_{\mathcal{X}}=\sum_{k=1}^{4}\|B_k(x)-B_k(z)\|_{X_k}\\
&\le \sum_{i,j=1}^{4}\|B^1_{i,j}(x_i-z_i,x_j)+B^1_{i,j}(z_i,x_j-z_j))\|_{X_1}+\sum_{i,j=1}^{4}\|B^3_{i,j}(x_i-z_i,x_j)+B^3_{i,j}(z_i,x_j-z_j))\|_{X_3}\\
&\quad+\|L_2\circ(y_3+B_3)(x)\|_{X_2}+\|L_4\circ(y_1+B_1)(x)\|_{X_4}\\
&\le \sum_{i,j}^{4}(C^1_{i,j}+C^3_{i,j})\big(\|x_i-z_i\|_{X_i}\|x_j\|_{X_j}+\|z_i\|_{X_i}\|x_j-z_j\|_{X_j}\big)\\
&\quad+\sum_{i,j}^{4}(\beta C^1_{i,j}+\alpha C^3_{i,j})\big(\|x_i-z_i\|_{X_i}\|x_j\|_{X_j}+\|z_i\|_{X_i}\|x_j-z_j\|_{X_j}\big)\\
&\le \sum_{i,j=1}^{4}\big(C^1_{i,j}(\beta+1)+C^3_{i,j}(\alpha+1)\big)\|x-z\|_{\mathcal{X}}\big(\|x\|_{\mathcal{X}}+\|z\|_{\mathcal{X}}\big)\\
&\le K_2 4K_1\varepsilon\|x-z\|_{\mathcal{X}}.
\end{aligned}$$

Since $4K_1K_2\varepsilon<1$, $\mathcal{F}$ is a contraction in $\mathcal{B}_\varepsilon$, and the Banach fixed point theorem concludes the proof.

Now, for each initial data tuple (u_0,v_0,w_0,z_0), we consider $\mathcal{F}(u,v,w,z)=$

(U, V, W, Z), we have

$$\begin{cases} U(t) = e^{t\Delta}u_0 - \displaystyle\int_0^t \nabla e^{(t-\tau)\Delta}(u\nabla v)(\tau)d\tau, \\ \qquad =: e^{t\Delta}u_0 + B^1_{1,2}(t), \\ V(t) = e^{-t}e^{t\Delta}v_0 + \displaystyle\int_0^t e^{-(t-\tau)}e^{(t-\tau)\Delta}w(\tau)d\tau, \\ \qquad =: e^{-t}e^{t\Delta}v_0 + L_2(t), \\ W(t) = e^{t\Delta}w_0 - \displaystyle\int_0^t \nabla e^{(t-\tau)\Delta}(w\nabla z)(\tau)d\tau, \\ \qquad =: e^{t\Delta}w_0 + B^3_{3,4}(t), \\ Z(t) = e^{-t}e^{t\Delta}z_0 + \displaystyle\int_0^t e^{-(t-\tau)}e^{(t-\tau)\Delta}u(\tau)d\tau, \\ \qquad =: e^{-t}e^{t\Delta}z_0 + L_4(t). \end{cases} \tag{4.6}$$

Lemma 4.2 *Under the hypotheses of Theorem* 3.1*. There exist positive constants* C_1, C_2 *such that*

$$\|B^1_{1,2}\|_{X_1} \le C_1\|u\|_{X_1}\|v\|_{X_2}, \tag{4.7}$$
$$\|B^3_{3,4}\|_{X_3} \le C_2\|w\|_{X_3}\|z\|_{X_4}, \tag{4.8}$$
$$\|L_2(w)\|_{X_2} \le \alpha\|w\|_{X_3}, \tag{4.9}$$
$$\|L_4(u)\|_{X_4} \le \beta\|u\|_{X_1}, \tag{4.10}$$

for all $u \in X_1$*,* $v \in X_2$*,* $w \in X_3$*, and* $z \in X_4$*.*

Proof. From the conditons $(a), (b), (c)$ in Remark 3.2, we have

$$\frac{1}{2} - \frac{N}{2r} > 0, \quad -\frac{1}{2} + \frac{N}{2q} + \frac{N}{2r} > 0.$$

Taking $s_1 = \frac{r_1 q_1}{r_1 + q_1}$, from $(A), (B)$ in Remark 3.2, it follows that

$$1 \le s_1 \le \frac{rq}{r+q} \quad \textit{and} \quad \frac{q}{q_1} \ge \frac{rq}{r+q}\frac{1}{s_1},$$

and we can estimate

$$
\begin{aligned}
&\|B^1_{1,2}\|_{\mathcal{M}^q_{q_1}} \\
&= \Big\| \int_0^t \nabla\cdot e^{(t-\tau)\Delta}(u\nabla v)(\tau)d\tau \Big\|_{\mathcal{M}^q_{q_1}} \le \int_0^t \big\|\nabla\cdot e^{(t-\tau)\Delta}(u\nabla v)(\tau)\big\|_{\mathcal{M}^q_{q_1}} d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{1}{q}+\frac{1}{r}-\frac{1}{q})-\frac{1}{2}} \|(u\nabla v)(\tau)\|_{\mathcal{M}^{\frac{rq}{r+q}}_{s_1}} d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{N}{2r}-\frac{1}{2}} \|u(\tau)\|_{\mathcal{M}^q_{q_1}} \|\nabla v(\tau)\|_{\mathcal{M}^r_{r_1}} d\tau \\
&\le C\|u\|_{X_1}\|v\|_{X_2} \int_0^t (t-\tau)^{-\frac{N}{2r}-\frac{1}{2}} \tau^{\frac{N}{2q}-1} \tau^{\frac{N}{2r}-\frac{1}{2}} d\tau \\
&= Ct^{\frac{N}{2q}-1}\|u\|_{X_1}\|v\|_{X_2} b\Big(\frac{1}{2}-\frac{N}{2r}, -\frac{1}{2}+\frac{N}{2q}+\frac{N}{2r}\Big) \\
&= C_1 t^{\frac{N}{2q}-1}\|u\|_{X_1}\|v\|_{X_2},
\end{aligned}
\tag{4.11}
$$

for all $t > 0$, where $C_1 = (N, q, q_1, r, r_1)$, and b denotes the beta function.

$$
\begin{aligned}
&\|\nabla L_2(w)\|_{\mathcal{M}^r_{r_1}} \\
&= \Big\|\nabla \int_0^t e^{-(t-\tau)} e^{(t-\tau)\Delta} w(\tau)d\tau \Big\|_{\mathcal{M}^r_{r_1}} \le \int_0^t \big\|\nabla e^{(t-\tau)\Delta} w(\tau)\big\|_{\mathcal{M}^r_{r_1}} d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}} \|w(\tau)\|_{\mathcal{M}^q_{q_1}} d\tau \\
&\le Ct^{\frac{N}{2r}-\frac{1}{2}} b\Big(\frac{1}{2}-\frac{N}{2q}+\frac{N}{2r}, \frac{N}{2q}\Big)\|w\|_{X_3} \\
&= \alpha t^{\frac{N}{2r}-\frac{1}{2}}\|w\|_{X_3},
\end{aligned}
\tag{4.12}
$$

for all $t > 0$, where $\alpha = \alpha(N, q, q_1, r, r_1)$, which gives (4.9). Similarly, we have

$$\|B^3_{3,4}(w,z)\|_{\mathcal{M}^q_{q_1}} = C_2 t^{\frac{N}{2q}-1}\|w\|_{X_3}\|z\|_{X_4}, \tag{4.13}$$

$$\|\nabla L_4(u)\|_{\mathcal{M}^r_{r_1}} = \beta t^{\frac{N}{2r}-\frac{1}{2}}\|u\|_{X_1}, \tag{4.14}$$

for all $t > 0$, where $C_2 = C(N, q, q_1, r, r_1)$, and $\beta = \beta(N, q, q_1, r, r_1)$, which gives (4.10).

Now, we deal with the global existence of solutions to prove Theorem 3.1.

Proof of Theorem 3.1

Consider X_1, X_2, X_3 and X_4 as in (3.2)-(3.5) and let $y = (e^{t\Delta}u_0, e^{-t}e^{t\Delta}v_0, e^{t\Delta}w_0, e^{-t}e^{t\Delta}z_0)$. For $\mathcal{X} = X_1 \times X_2 \times X_3 \times X_4$ and $x = (u, v, w, z) \in \mathcal{X}$, we denote

$$B_1(x) := B^1_{1,2}(u, v), \tag{4.15}$$

$$B_2(x) := L_2 \circ (e^{t\Delta} w_0 + B_3)(x), \tag{4.16}$$

$$B_3(x) := B^3_{3,4}(w, z), \tag{4.17}$$

$$B_4(x) := L_4 \circ (e^{t\Delta} u_0 + B_1)(x). \tag{4.18}$$

From Lemma 4.2, the operators $B^k_{i,j}$, L_2 and L_4 are continuous maps.

Next, we set

$$K_1 = 1 + \alpha + \beta \quad and \quad K_2 = C_1(\beta + 1) + C_2(\alpha + 1). \tag{4.19}$$

We have that

$$\begin{aligned} \|y\|_{\mathcal{X}} &= \|e^{t\Delta} u_0\|_{X_1} + \|e^{-t} e^{t\Delta} v_0\|_{X_2} + \|e^{t\Delta} w_0\|_{X_3} + \|e^{-t} e^{t\Delta} z_0\|_{X_3} \\ &= \sup_{t>0} t^{-\frac{N}{2q}+1} \|e^{t\Delta} u_0\|_{\mathcal{M}^q_{q_1}} + \sup_{t>0} t^{-\frac{N}{2r}+\frac{1}{2}} \|\nabla e^{-t} e^{t\Delta} v_0\|_{\mathcal{M}^r_{r_1}} \\ &\quad + \sup_{t>0} t^{-\frac{N}{2q}+1} \|e^{t\Delta} w_0\|_{\mathcal{M}^q_{q_1}} + \sup_{t>0} t^{-\frac{N}{2r}+\frac{1}{2}} \|\nabla e^{-t} e^{t\Delta} z_0\|_{\mathcal{M}^r_{r_1}} \\ &\le C_0 \big(\|u_0\|_{\mathcal{N}^{\frac{N}{q}-2}_{q,q_1,\infty}} + \|\nabla v_0\|_{\mathcal{N}^{\frac{N}{r}-1}_{r,r_1,\infty}} + \|u_0\|_{\mathcal{N}^{\frac{N}{q}-2}_{q,q_1,\infty}} + \|\nabla z_0\|_{\mathcal{N}^{\frac{N}{r}-1}_{r,r_1,\infty}} \big) \\ &\le C_0 \|(u_0, v_0, w_0, z_0)\|_{\mathcal{I}} \le \varepsilon, \end{aligned} \tag{4.20}$$

provided that $\|(u_0, v_0, w_0, z_0)\|_{\mathcal{I}} \le \delta = \frac{\varepsilon}{C_0}$. If $0 < \varepsilon < \frac{1}{4K_1K_2}$, then Lemma 4.1 implies that there exists a unique solution $(u, v, w, z) \in \mathcal{X}$ of (3.7) such that $\|(u, v, w, z)\|_{\mathcal{X}} \le 2K_1\varepsilon$.

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