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Nonexistence of the CH-type Peakon to a Generalized Camassa-Holm Equation

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Abstract

In this paper, we study nonexistence of the CH-type peakon for a generalized Camassa-Holm equation proposed by Novikov.

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Keywords: Weak solution; Nonexistence; A generalized Camassa-Holm equation

1 Introduction

In this paper, we consider the Cauchy problem of integrable dispersive wave equation

$$\begin{cases} u_t - u_{txx} - 4uu_x + 6u_xu_{xx} + 2uu_{xxx} - 2u_x^2 - 2uu_{xx} = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

which is presented in Novikov [1]. It is shown in [1] that Eq.(1) admits a hierarchy of local higher symmetries. Eq.(1) is regarded as a generalized Camassa-Holm equation (or a genenralized Degasperis-Procesi equation [2]) because it has similar structure with them. In [2], Li and Yin established the local existence and uniqueness of strong solutions for the problem (1) in nonhomogeneous Besov spaces by using the Littlewood-Paley theory. The well-posedness of (1) was studied in [3] for the periodic and the nonperiodic cases in the sense of Hadamard. In addition, nonuniform dependence was proved by using the method of approximate solutions and well-posedness estimates. However, to our best knowledge, weak solutions [4] for the Cauchy problem (1) have not been investigated yet.

The main objective of this paper is to investigate whether or not equations (1) with nonlocal nonlinearities has similar remarkable properties as Camassa-Holm equation [5]. By the study of weak solutions, we find that the problem (1) does not poss CH-type peakon [6], which is different from Camassa-Holm equation.

2 Preliminary

We write the equivalent form of the problem (1) as follows

$$\begin{cases} u_t - 2uu_x = \partial_x(1-\partial_x^2)^{-1}(u^2+(u^2)_x), \\ u(0,x) = u_0(x) \end{cases} \tag{2}$$

The characteristics $q(t,x)$ relating to (2) is governed by

$$\begin{cases} q_t(t,x) = -2u(t,q(t,x)), \quad t\in[0,T), \\ q(0,x) = x, \quad x\in\mathbb{R}. \end{cases}$$

Applying the classical results in the theory of ordinary differential equations, one can obtain that the characteristics $q(t,x)\in C^1([0,T)\times\mathbb{R})$ with $q_x(t,x)>0$ for all $(t,x)\in[0,T)\times\mathbb{R}$. Furthermore, it is shown from [2] that the potential $m=u-u_{xx}$ satisfies

$$m(t,q(t,x))q_x^2(t,x) \geq m_0(x)e^{-\int_0^t(2u_x-2u)(\tau,q(\tau,x))d\tau}. \tag{3}$$

2.1 Notation

We firstly give some notations.
Let $\mathbb{R}$ denote real number set. The space of all infinitely differentiable functions $\phi(t,x)$ with compact support in $[0,+\infty)\times\mathbb{R}$ is denoted by C_0^∞. Let $L^p = L^p(\mathbb{R})(1\leq p<+\infty)$ be the space of all measurable functions h such that $\| h \|_{L^P}^P = \int_{\mathbb{R}}|h(t,x)|^p dx < \infty$. We define $L^\infty = L^\infty(\mathbb{R})$ with the standard norm $\| h \|_{L^\infty} = \inf_{m(e)=0}\sup_{x\in R\setminus e}|h(t,x)|$. For any real number s, $H^s=H^s(\mathbb{R})$ denotes the Sobolev space with the norm defined by

$$\| h \|_{H^s} = \left(\int_{\mathbb{R}}(1+|\xi|^2)^s|\hat{h}(t,\xi)|^2 d\xi\right)^{\frac{1}{2}} < \infty,$$

where $\hat{h}(t,\xi)=\int_{\mathbb{R}}e^{-ix\xi}h(t,x)dx$.
We denote by $*$ the convolution, and the convolution product on $\mathbb{R}$ is defined by

$$(f*g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy. \tag{4}$$

Using the Green function $g(x) = \frac{1}{2}e^{-|x|}$, we have $(1-\partial_x^2)^{-1}f = g(x) * f$ for all $f \in L^2$, and $g * (u - u_{xx}) = u$. For $T > 0$ and nonnegative number s, $C([0,T); H^s(\mathbb{R}))$ denotes the Frechet space of all continuous H^s-valued functions on $[0,T)$. For simplicity, throughout this article, we let C denote any positive constant

3 Nonexistence of the CH-type peakon

In this section, we will study the weak solutions for problem (1).

Definition 3.1. *Given initial data $u_0 \in H^s$, $s > \frac{3}{2}$, the function u is said to be a weak solution to the initial-value problem (2) if it satisfies the following identity*

$$\int_0^T \int_{\mathbb{R}} u\varphi_t - u^2\varphi_x - p * (u^2 + 2uu_x)\varphi_x dxdt + \int_{\mathbb{R}} u_0(x)\varphi(0,x)dx = 0 \qquad (5)$$

for any smooth test function $\varphi(t,x) \in C_c^\infty([0,T) \times \mathbb{R})$. If u is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global weak solution. In this section, we will discuss the existence of weak solutions.

Theorem 3.2. *The peakon function of the form*

$$u(t,x) = a(c,t)e^{-|x-ct|}, \quad c \in \mathbb{R}, a(c,t) \neq 0, \qquad (6)$$

is not a global weak solution to (1) in the sense of Definition 1.1.

Proof.

We firstly claim that

$$u_t = \partial_t a e^{-|x-ct|} + csign(x-ct)u, \quad u_x = -sign(x-ct)u. \qquad (7)$$

Hence, using (4), (7) and integration by parts, we derive that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} u\varphi_t - u^2\varphi_x dxdt + \int_R u_0(x)\varphi(0,x)dx \\
&= -\int_0^T \int_{\mathbb{R}} \varphi(u_t - 2uu_x)dxdt \\
&= -\int_0^T \int_{\mathbb{R}} \varphi[\partial_t a e^{-|x-ct|} + csign(x-ct)u + 2sign(x-ct)u^2]dxdt. \qquad (8)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} -p * (u^2 + 2uu_x)\varphi_x dxdt \\
&= \int_0^T \int_{\mathbb{R}} \varphi p_x * [(1 - 2sign(x-ct))u^2]dxdt \qquad (9)
\end{aligned}$$

Note that $p_x = -\frac{1}{2}sign(x)e^{-|x|}$. For $x \leq ct$,

$$
\begin{aligned}
& p_x * (u^2 + 2uu_x) = -\frac{1}{2}\int_{\mathbb{R}} sign(x-y)e^{-|x-y|}(1-2sign(y-ct))e^{-2|y-ct|}dy \\
& = -\frac{1}{2}\Big(\int_{-\infty}^{x} + \int_{x}^{ct} + \int_{ct}^{\infty}\Big)sign(x-y)e^{-|x-y|} \\
& \qquad\qquad \times(1-2sign(y-ct))e^{-2|y-ct|}dy \\
& = I_1 + I_2 + I_3. \qquad (10)
\end{aligned}
$$

We directly compute I_1 as follows

$$
\begin{aligned}
I_1 &= -\frac{1}{2}\int_{-\infty}^{x} sign(x-y)e^{-|x-y|}(1-2sign(y-ct))e^{-2|y-ct|}dy \\
&= -\frac{3}{2}\int_{-\infty}^{x} e^{-x-2ct+3y}dy \\
&= -\frac{3}{2}e^{-x-2ct}\int_{-\infty}^{x} e^{3y}dy \\
&= -\frac{1}{2}e^{2x-2ct}. \qquad (11)
\end{aligned}
$$

In a similar procedure,

$$
\begin{aligned}
I_2 &= -\frac{1}{2}\int_{x}^{ct} sign(x-y)e^{-|x-y|}(1-2sign(y-ct))e^{-2|y-ct|}dy \\
&= \frac{3}{2}\int_{x}^{ct} e^{x-2ct+y}dy \\
&= \frac{3}{2}e^{x+\frac{8}{3}t}\int_{x}^{ct} e^{y}dy \\
&= \frac{3}{2}e^{x-ct} - \frac{3}{2}e^{2x-2ct}. \qquad (12)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= -\frac{1}{2}\int_{ct}^{\infty} sign(x-y)e^{-|x-y|}(1-2sign(y-ct))e^{-2|y-ct|}dy \\
&= -\frac{1}{2}\int_{ct}^{\infty} e^{x+2ct-3y}dy \\
&= -\frac{1}{2}e^{x+2ct}\int_{ct}^{\infty} e^{-3y}dy \\
&= -\frac{1}{6}e^{x-ct}. \qquad (13)
\end{aligned}
$$

Therefore, from (11)-(13), we deduce that for $x \leq ct$

$$
p_x * (u^2 + 2uu_x)(t,x) = -2a^2e^{2x-2ct} + \frac{4}{3}a^2e^{x-ct}. \qquad (14)
$$

For $x > ct$,

$$p_x * (u^2 + 2uu_x) = -\frac{1}{2}\int_{\mathbb{R}} sign(x-y)e^{-|x-y|}(1-2sign(y-ct))e^{-2|y-ct|}dy$$
$$= -\frac{1}{2}(\int_{-\infty}^{ct} + \int_{ct}^{x} + \int_{x}^{\infty})sign(x-y)e^{-|x-y|}$$
$$\times(1-2sign(y-ct))e^{-2|y-ct|}dy$$
$$= II_1 + II_2 + II_3. \tag{15}$$

We directly compute I_1 as follows

$$\begin{aligned} II_1 &= -\frac{1}{2}\int_{-\infty}^{ct} sign(x-y)e^{-|x-y|}(1-2sign(y-ct))e^{-2|y-ct|}dy \\ &= -\frac{3}{2}\int_{-\infty}^{ct} e^{-x-2ct+3y}dy \\ &= -\frac{3}{2}e^{-x-2ct}\int_{-\infty}^{x} e^{3y}dy \\ &= -\frac{1}{2}e^{-x+ct}. \end{aligned} \tag{16}$$

In a similar procedure,

$$\begin{aligned} II_2 &= -\frac{1}{2}\int_{ct}^{x} sign(x-y)e^{-|x-y|}(1-2sign(y-ct))e^{-2|y-ct|}dy \\ &= \frac{1}{2}\int_{ct}^{x} e^{-x+2ct-y}dy \\ &= \frac{1}{2}e^{-x+2ct}\int_{ct}^{x} e^{-y}dy \\ &= -\frac{1}{2}e^{-2x+2ct} + \frac{1}{2}e^{-x+ct}. \end{aligned} \tag{17}$$

and

$$\begin{aligned} II_3 &= -\frac{1}{2}\int_{x}^{\infty} sign(x-y)e^{-|x-y|}(1-2sign(y-ct))e^{-2|y-ct|}dy \\ &= -\frac{1}{2}\int_{x}^{\infty} e^{x+2ct-3y}dy \\ &= -\frac{1}{2}e^{x+2ct}\int_{x}^{\infty} e^{-3y}dy \\ &= -\frac{1}{6}e^{-2x+2ct}. \end{aligned} \tag{18}$$

Therefore, from (16)-(18), we deduce that for $x > ct$

$$p_x * (u^2 + 2uu_x)(t,x) = -\frac{2}{3}a^2e^{-2x+2ct}. \tag{19}$$

According to (14) and (19), we obtain from two cases mentioned above that

$$p_x * (u^2 + 2uu_x)(t,x) = \begin{cases} -\frac{2}{3}a^2e^{-2x+2ct}, & if \quad x > ct, \\ -2a^2e^{2x-2ct} + \frac{4}{3}a^2e^{x-ct}, & if \quad x \leq ct. \end{cases} \tag{20}$$

Due to $u = ae^{-|x-ct|}$,

$$\partial_t ae^{-|x-ct|} + csign(x-ct)u + 2sign(x-ct)u^2 =$$

$$\begin{cases} \partial_t ae^{-x+ct} + cae^{-x+ct} + 2a^2e^{-2x+2ct}, & if x > ct, \\ \partial_t ae^{x-ct} - cae^{x-ct} - 2a^2e^{2x-2ct}, & if x \leq ct. \end{cases} \tag{21}$$

If the function in the form of (6) is a weak solution of equation, then combining (20) and (21) yields that

$$\begin{cases} \partial_t ae^{-x+ct} + cae^{-x+ct} + \frac{8}{3}a^2e^{-2x+2ct} = 0, & if x > ct, \\ (\partial_t a - ca - \frac{4}{3}a^2)e^{x-ct} = 0, & if x \leq ct. \end{cases} \tag{22}$$

By the linear independence of the function e^{-x+ct}, e^{x-ct},$e^{-2x+2ct}$ and e^{2x-2ct}, the above condition holds if and only if

$$a(c,t) = 0, \tag{23}$$

which provides a trivial solution of equation, $u(t,x) = 0$. Therefore it completes the proof of Theorem 3.2.

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References

[1] V. Novikov, Generalizations of the Camassa-Holm equation, *J. Phys. A*, **42** (2009),342002. https://doi.org/10.1088/1751-8113/42/34/342002

[2] J. Li, Z. Yin, Well-poseness and global existence for a generalized Degasperis-Procesi equation, *Nonlinear Analysis: Real World Applications*, **28** (2016), 72-92. https://doi.org/10.1016/j.nonrwa.2015.09.003

[3] Y.S. Mi, Y. Liu, B.L. Guo and T. Luo, The Cauchy problem for a generalized Camassa-Holm equation, *J. Differential Equations*, **266** (2019), no. 5, 6739-6770. https://doi.org/10.1016/j.jde.2018.11.019

[4] A. Constantin, L. Molinet, Global Weak Solutions for a Shallow Water Equation, *Commun. Math. Phys.*, **211** (2000), 45-61. https://doi.org/10.1007/s002200050801

[5] R. Camassa, D. Holm, An integrable shallow water wave equation with peaked solitons, *Phys. Rev. Lett.*, **71** (1993), 1661-1664. https://doi.org/10.1103/physrevlett.71.1661

[6] A. Constantin, W.A. Strauss, Stability of peakons, *Commun. Pure Appl. Math.*, **53** (2000), no. 5, 603-610. https://doi.org/10.1002/(sici)1097-0312(200005)53:5<603::aid-cpa3>3.0.co;2-l

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