

A Novel Approach to Find Analytical Solutions for Oldroyd 6-Constant Fluid

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Abstract

The study is concerned with the application of a novel idea used to find analytical solutions of a nonlinear boundary value problem that arises in channel flow problems. The velocity profile of a viscoelastic fluid between two planes is obtained with both slip and no-slip boundary conditions. We present analytical solutions on classical problems – the Poiseuille flow, and the generalized Couette flow of a viscoelastic fluid between two parallel planes. The viscoelastic fluid is modeled by an Oldroyd 6-constant fluid, giving rise to a highly nonlinear ordinary differential equation. This equation has been solved via a novel approach by reducing the differential equation to a cubic algebraic equation, giving rise to a non-recursive series solution to the equation. Finally, after the solution of the general differential equation is given Newton's Method is used for faster convergence to solve the equation in the case of three common boundary conditions.

Keywords: Analytical Solutions, viscoelastic fluid, spectral methods

1 Introduction

Viscoelastic fluids, characterized by their intricate rheological properties, defy description through the classical Navier-Stokes model employed for Newtonian

fluids due to their nonlinear stress-strain relationship [1, 2]. Unlike their Newtonian counterparts, which display a straightforward linear relationship between stress and strain rate, viscoelastic fluids exhibit a more complex behavior that necessitates alternative modeling approaches. In response to this challenge, various constitutive equations and non-Newtonian fluid models have been developed over recent decades [3].

One particularly notable viscoelastic fluid model is the Oldroyd 6-constant fluid. This model has gained substantial prominence in diverse fields, including polymer processing, food processing, and biological systems, thanks to its capacity to elucidate intricate flow phenomena. Unlike Newtonian fluids that merely follow a linear stress-strain response, Oldroyd 6-constant fluids showcase a distinctive behavior. They exhibit a memory effect, enabling them to recall their original shape after deformation, and they possess both viscous and elastic characteristics when subjected to shear forces [4]. Consequently, this unique behavior makes Oldroyd 6-constant fluids an invaluable tool for simulating and modeling materials that manifest both viscous and elastic responses during flow, paving the way for a deeper understanding of complex flow dynamics in various industrial and biological contexts.

When investigating the behavior of different types of Oldroyd fluids, the quest for analytical solutions to nonlinear differential equations stands as a pivotal endeavor in scientific research. However, this pursuit is often beset with formidable challenges stemming from the inherent complexity and nonlinearity of the equations under scrutiny. In response to these challenges, the scientific community has devised a multitude of innovative methodologies to approximate solutions for such intricate problems. One such formidable tool is perturbation theory, which has garnered significant attention for its utility in deducing approximate solutions in cases characterized by weak nonlinearity [5]. However, it is important to note that its applicability is often contingent upon the specific parameter ranges involved, as it may falter beyond certain limits. Beyond perturbation theory, a diverse array of alternative methodologies has been employed to tackle nonlinear differential equations in their various forms. Notably, the Adomian decomposition method, as introduced by Siddiqui et al. [6], offers a distinctive approach to resolving these equations. Similarly, several linearization techniques have been proposed, such as those delineated by Ramos [7]. These linearization methods aim to simplify complex nonlinear equations into more tractable forms. Furthermore, researchers have explored the effectiveness of modified Lindsedt-Poincaré methods and the optimal homotopy perturbation method [8, 9, 10, 11] as viable avenues for finding analytical solutions to nonlinear differential equations for Oldroyd type fluids.

This paper presents a novel approach to obtain analytical solutions for the Oldroyd 6-constant fluid; a task never previously attempted. We achieve this by transforming the nonlinear ordinary differential equation into a cubic algebraic equation, resulting in a non-recursive series solution. Additionally, to expedite convergence, we apply Newton's Method to solve the general differential equation. We apply this methodology to address two classical problems: Poiseuille flow, and generalized Couette flow of a viscoelastic fluid between two parallel planes under an applied

pressure field. To validate the accuracy of our method, we also employ the spectral Chebyshev method [12], a numerical technique known for its precision in solving differential equations. Remarkably, our spectral solutions align perfectly with our analytical solutions, affirming the effectiveness of our approach.

2 Mathematical Formulation

We consider the steady laminar flow of an incompressible Oldroyd 6-constant fluid between two parallel plates separated at distance d . As shown in figure 1-a,b

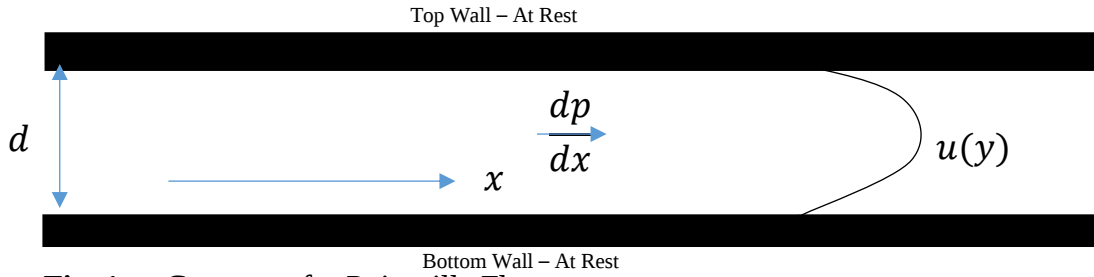


Fig. 1-a: Geometry for Poiseuille Flow

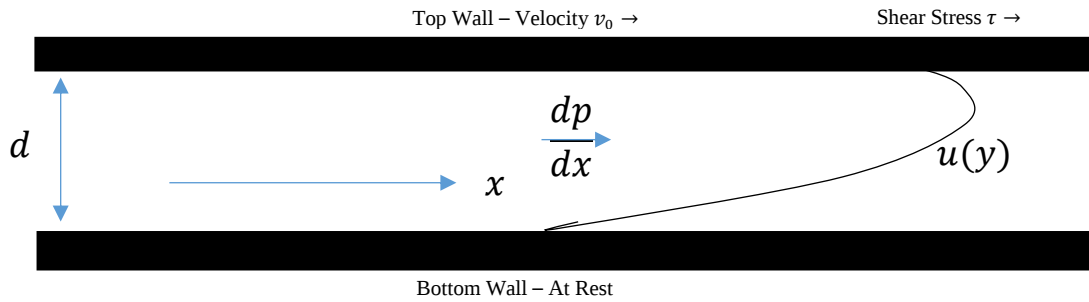


Fig. 1-b: Generalized Couette Flow

The channel flow of an Oldroyd 6-constant fluid is a classic problem in fluid mechanics. Oldroyd 6-constant fluid is a type of non-Newtonian fluid with a viscoelastic behavior, where the stress in the fluid depends not only on the rate of deformation but also on the deformation history. The governing equations for the flow of Oldroyd 6-constant fluid can be derived from the generalized Navier-Stokes equations.

For this geometry of flow, governing equations have been derived by many authors [12], without re-producing the derivation, we directly start with the most general governing equation for the flow under consideration.

$$\frac{d^2u}{dy^2} + \left[(3\alpha - \beta) + \alpha\beta \left(\frac{du}{dy} \right)^2 \right] \left(\frac{du}{dy} \right)^2 \frac{d^2u}{dy^2} - \frac{1}{\mu} \left(\frac{dp}{dx} \right) \left[1 + \beta \left(\frac{du}{dy} \right)^2 \right]^2 = 0 \quad (1)$$

In this equation, u is the axial component of velocity, and p is pressure. Most importantly, the viscoelastic parameters α and β are defined as:

$$\begin{aligned} \alpha &= \lambda_1 \lambda_4 - (\lambda_3 + \lambda_5)(\lambda_4 + \lambda_2), \\ \beta &= \lambda_1 \lambda_3 - (\lambda_3 + \lambda_5)(\lambda_3 + \lambda_1). \end{aligned} \quad (2)$$

Both α and β are two key parameters that play a crucial role in shaping the velocity profile of flow with Oldroyd 6-constant fluids. Parameter α represents the relaxation time of the fluid, while β quantifies the shear-thinning behavior. These parameters can significantly influence the flow characteristics and, consequently, have implications for various practical applications in industries ranging from polymer processing to biomedical engineering.

Furthermore, λ_i ($i = 1, 2, 3, 4, 5$) and μ are the six material constants of the Oldroyd fluid. Before we move forward, it is worth noting that the model exhibits different fluid behaviors based on the values of the six relaxation time parameters, λ_1 to λ_5 . When all these material parameters are zero, the model behaves as a classical viscous Newtonian fluid. The model also describes other non-Newtonian fluid behaviors, including a Maxwell model, second grade fluid, and Johnson-Segalman model, depending on the values of the relaxation time parameters.

We define following set of non-dimensional variables:

$$\bar{u} = \frac{u}{U}, \quad \bar{y} = \frac{y}{d}, \quad \bar{x} = \frac{x}{d}, \quad \bar{\alpha} = \frac{\alpha U^2}{d^2}, \quad \bar{\beta} = \frac{\beta U^2}{d^2}, \quad \bar{p} = \frac{p}{\mu U}, \quad c = \frac{1}{\mu} \frac{dp}{dx}.$$

Using these non-dimensional variables in Eq (1) and in Eq. (2) and after dropping bars for simplification, we arrive the following set of nonlinear differential equations and boundary conditions.

$$\frac{d^2u}{dy^2} + \left[(3\alpha - \beta) + \alpha\beta \left(\frac{du}{dy} \right)^2 \right] \left(\frac{du}{dy} \right)^2 \frac{d^2u}{dy^2} - c \left[1 + \beta \left(\frac{du}{dy} \right)^2 \right]^2 = 0 \quad (3)$$

3 Analytical Solution Technique

To find analytical solution to Eq. (3), we initiate the process by introducing a substitution: $z = y'$. This substitution transforms the equation into a separable ordinary differential equation. Upon integrating both sides, we obtain an algebraic equation that can be further manipulated into a cubic equation, solvable through the

cubic formula. We subsequently reverse our initial substitution, effectively performing indefinite integration. Following this step, we can incorporate specific boundary conditions to restrict the constants, resulting in a set of analytical solutions within a defined parameter space. Given below are the details.

Define $z = \frac{du}{dy}$ and $\frac{d^2u}{dy^2} = \frac{dz}{dy}$. This transform Eq.(3) to first order ODE in z :

$$\frac{dz}{dy} + (3\alpha - \beta)z^2 \frac{dz}{dy} + \alpha\beta z^4 \frac{dz}{dy} = c(1 + \beta z^2)^2 \quad (4)$$

We factor out the $\frac{dz}{dy}$ and separate the variables:

$$\frac{(1 + (3\alpha - \beta)z^2 + \alpha\beta z^4)}{(1 + \beta z^2)^2} dz = c dy \quad (5)$$

Integrate both sides:

$$\int \frac{(1 + (3\alpha - \beta)z^2 + \alpha\beta z^4)}{(1 + \beta z^2)^2} dz = \int c dy$$

We get

$$\frac{\alpha z^3 + z}{1 + \beta z^2} = cy + \gamma_0 \quad (7)$$

Where γ_0 is our first constant of integration. We make the substitution $\gamma_0 = -\gamma/c$ in order to make γ easier to solve for. Then (7) becomes

$$\frac{\alpha z^3 + z}{1 + \beta z^2} = c(y - \gamma) \quad (8)$$

Rearrange Eq. (8) and solve for z (which is u' in disguise) using a cubic formula.

To keep our calculation simple, we introduce the following functions:

$$p = \frac{\beta c(y-\gamma)}{3\alpha} \quad r = \frac{1}{3\alpha} \quad q = p^3 - \frac{\beta c(y-\gamma) - 3\alpha c(y-\gamma)}{6\alpha^2}.$$

Using these functions, we solve Eq. (8) for z , which is actually u' :

$$u' = f(y, \gamma) = p + \sqrt[3]{q + \sqrt[2]{q^2 + (r - p^2)^3}} + \sqrt[3]{q - \sqrt[2]{q^2 + (r - p^2)^3}}. \quad (9)$$

We will refer to (9) as $f(y, \gamma)$. Integrating it once gives, $u(y)$

$$u(y) = \int p + \sqrt[3]{q + \sqrt{q^2 + (r - p^2)^3}} + \sqrt[3]{q - \sqrt{q^2 + (r - p^2)^3}} dt$$

We may also write u as

$$u(y, \gamma, C) = \int_0^y f(\tau, \gamma) d\tau + C \quad (10)$$

In the form of (10), we can solve for C and γ using the appropriate boundary conditions. We approach these problems in Sections 3 and 4.

3 Analytical Solution for Plane Poiseuille Flow

Poiseuille flow is a type of laminar flow where a fluid flows between two parallel plates, driven by a pressure gradient as shown in figure 1-a. The flow is characterized by a parabolic velocity profile, where the velocity is maximum at the center of the channel and decreases towards the walls. Poiseuille flow occurs when both plates are stationary and there is no relative motion between the plates. We will solve the equation (3) with following boundary conditions:

$$u(0) = 0, \quad u(d) = 0, \quad (11)$$

where d is the width of the channel. Armed with general solution given in Eq. (10) we immediately note that $u(0) = 0$, iff $C = 0$. We also notice that if we set $\gamma = 0$, $f(y, 0)$ becomes an odd function, i.e., f can be shown to satisfy the relation $f(y) = -f(-y)$. This implies that the line $x = f(y)$ is symmetric about the x axis. Since γ only represents a translation in y , this means that $f(y, \gamma)$ is symmetric about the axis $x = \gamma$. In order to induce the B.C. $u(d) = 0$, we set $\gamma = d/2$. This means that the axis of symmetry is *half* the distance from one end of the pipe to the other. Therefore, since $u(0) = 0$, $u(d) = 0$ as well. Having obtained $C = 0$ and $\gamma = d/2$, we present the solution as

$$u(y) = \int_0^y f(\tau, \frac{d}{2}) d\tau. \quad (12)$$

We have plotted the solution for several parameter values in Figures 2 and 3..

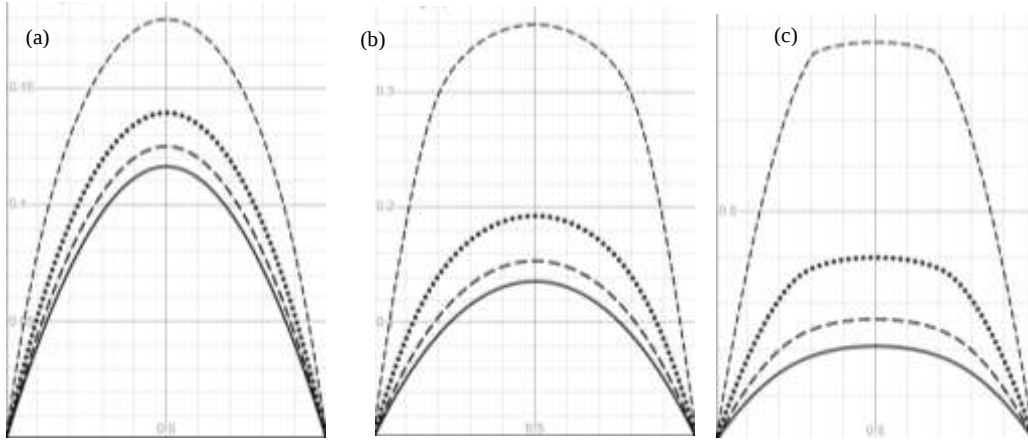


Figure 2: Poiseuille flow for various values of α keeping β constant. The solid black line in each figure is $\alpha = 4$.

(a) $c = 1, \beta = 3, \alpha = 1, 2, 3, 4$

(b) $c = 1, \beta = 5, \alpha = 1, 2, 3, 4$

(c) $c = 1, \beta = 9, \alpha = 1, 2, 3, 4$
(c)

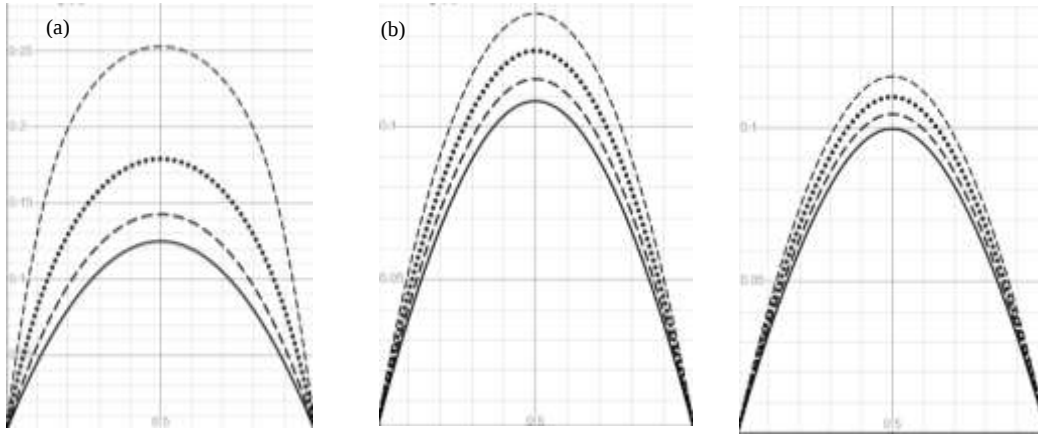


Figure 3: Poiseuille flow for various values of β keeping α constant. The solid black line is $\beta = 1$.

(a) $c = 1, \alpha = 1, \beta = 1, 2, 3, 4$

(b) $c = 1, \alpha = 3, \beta = 1, 2, 3, 4$

(c) $c = 1, \alpha = 5, \beta = 1, 2, 3, 4$

4 Analytical Solution for Generalized Couette Flow

Generalized Couette flow refers to a laminar flow pattern that occurs when a fluid flows between two parallel plates with one in motion, but in presence of pressure gradient. As the plates move, the fluid experiences a changing shear stress and a varying velocity profile, leading to a nonlinear velocity distribution. We will solve the equation (3) with following boundary modified conditions:

$$u(0) = 0, \quad u(d) = v_0. \quad (13)$$

where d is the width of the channel. Bottom plate stays stationary, whereas the upper wall moves at a certain velocity v_0 . We immediately can satisfy the first condition by noting that when we integrate from 0 this condition is satisfied. Thus $C = 0$. This and the second condition leaves us with

$$u(y, \gamma, 0) = u(d, \gamma) = \int_0^d f(\tau, \gamma) d\tau = v_0 \quad (14)$$

We solve this equation for γ using Newton's Method. The recursion involved is

$$\gamma_{n+1} = \gamma_n - \frac{u(d, \gamma) - v_0}{\frac{d}{d\gamma} u(d, \gamma)} \quad (15)$$

A reasonable guess for γ_0 is $\frac{1}{2}$. It and four iterations of Newton's method is used to create plots for this analysis. Plots are shown in Figs. 4 and 5.

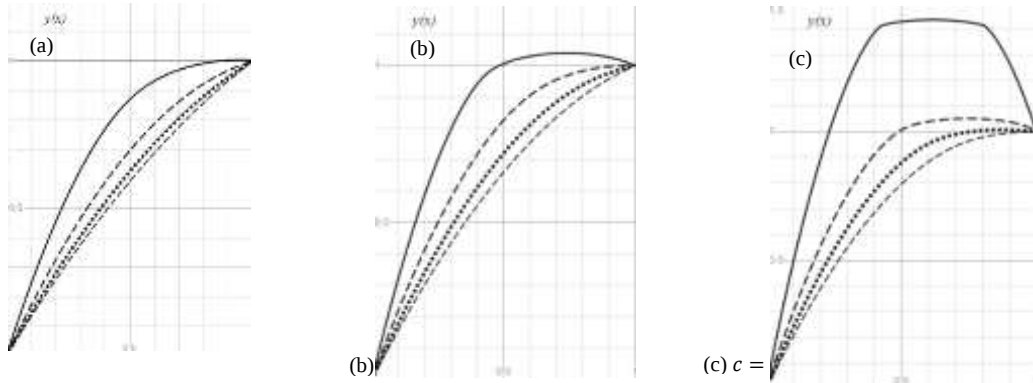


Figure 4: Generalized Couette Flow is plotted for $d = 1$ and $v_0 = 1$. α is held constant while β varies from 1 to 4. The solid line is $\beta = 1$.

(a) $c = 1, \beta = 3, \alpha = 1, 2, 3, 4$

(b) $c = 1, \beta = 5, \alpha = 1, 2, 3, 4$

(c) $c = 1, \beta = 9, \alpha = 1, 2, 3, 4$

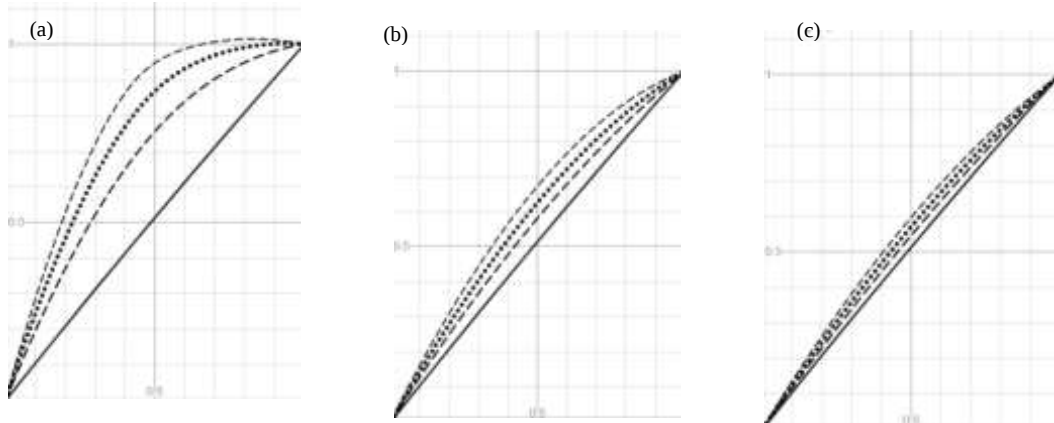


Figure 5: Generalized Couette Flow is plotted for $d = 1$ and $v_0 = 1$. β is held constant while α varies from 1 to 4. The solid line is $\alpha = 1$. (a) $c = 1, \alpha = 1, \beta = 1, 2, 3, 4$ (b) $c = 1, \alpha = 3, \beta = 1, 2, 3, 4$ (c) $c = 1, \alpha = 5, \beta = 1, 2, 3, 4$

4 Chebyshev Spectral Solutions

In this section, we employ Chebyshev spectral solutions to assess the accuracy of our analytical solutions. Specifically, we employ the Chebyshev spectral collocation method to obtain numerical solutions. It's noteworthy that Chebyshev polynomials have a well-established track record in effectively delivering numerical solutions across a broad spectrum of boundary value problems and have significantly contributed to the advancement of computational fluid dynamics [13].

When addressing ordinary differential equations, whether they exhibit linearity or nonlinearity, particularly within straightforward domains characterized by smooth data that define the problem, spectral methods consistently emerge as the preferred methodology. These methods exhibit an extraordinary capacity for achieving exceptionally high levels of precision, often attaining up to ten significant digits. This stands in sharp contrast to finite difference or finite element methods [14], which typically offer lower-order accuracy.

While we refrain from delving into the intricate details of implementation, we present a comparative analysis between spectral and analytical solutions for plane Poiseuille flow. Figure 6 clearly demonstrates a perfect match between the results of the analytical and numerical solutions, thereby affirming the validity and accuracy of our chosen methodology.

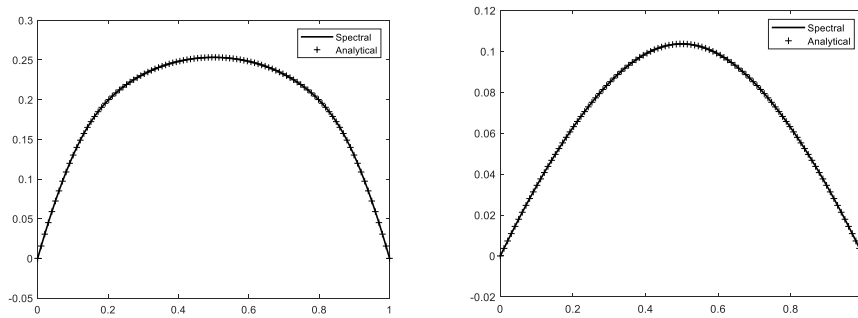


Figure 6: Comparison of the graphs of numerical and analytical solutions for the plane Poiseuille flow
 (a) $c = -1$, $\alpha = 1$, $\beta = 4$,
 (b) $c = -1$, $\alpha = 4$, $\beta = 1$,

5 Conclusions

In this study, we introduce a novel methodology for deducing analytical solutions to the nonlinear differential equations governing the flow of Oldroyd 6-constant fluid between two parallel planes. These solutions carry substantial significance within the realm of fluid dynamics and rheology. The availability of closed-form solutions plays a pivotal role in enhancing our theoretical comprehension of fluid behavior and opens up avenues for rigorous mathematical analysis.

This heightened understanding, in turn, has the potential to offer valuable insights into the fundamental physics at play, thus empowering researchers to refine and construct more precise constitutive models for industrial fluids of this nature.

This work presents analytical and numerical results across a broad spectrum of parameter values of the fluid. A comprehensive examination of the Oldroyd 6-constant model, encompassing viscoelastic and shear-thinning characteristics, is conducted to scrutinize the impact of α and β on the velocity profile within the context of parallel plate flow for an Oldroyd 6-constant fluid. This rigorous investigation provides valuable insights into the dynamic behavior of these fluids under diverse conditions.

The material parameters α and β , of paramount significance, exert a profound influence on the velocity profile. α represents the relaxation time of the fluid, while β quantifies its shear-thinning behavior. Their pivotal roles in shaping the flow characteristics bear significant implications for a myriad of practical applications, spanning industries from polymer processing to biomedical engineering.

Examining Figures 2 to 5 reveals that, as α assumes large values relative to the characteristic timescales of the flow, the fluid behaves akin to a purely elastic material, yielding a parabolic velocity profile reminiscent of Newtonian fluids. However, as α decreases or approaches the characteristic timescales, the viscoelastic effects become increasingly conspicuous, leading to deviations from the traditional Poiseuille flow profile (Fig. 2-c). The fluid exhibits a delayed response to alterations in applied shear stress, resulting in elongation and the emergence of "stress waves" within the fluid, particularly near the walls where the shear rate attains its zenith.

Conversely, β governs the shear-thinning propensity of the fluid. With higher β values, the fluid exhibits a more pronounced shear-thinning behavior, signifying a reduction in viscosity as the shear rate escalates. This phenomenon can induce variations in the velocity profile, with the region near the wall experiencing elevated shear rates and consequently lower viscosity relative to the central region. The intricate interplay between α and β , coupled with other factors such as boundary conditions and flow rates, engenders a diverse spectrum of velocity profiles within Oldroyd 6-constant fluids. Mastery of these profiles is indispensable for the optimization of processes and systems involving the transportation of these intricate fluids.

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