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Necessary and Sufficient Conditions for Arbitrary Linear Combinations of Solutions of a Class of Nonlinear Partial Differential Equations to Satisfy the Differential Equation

Namik Yener

Electrical-Electronics Engineering Department
Istanbul Gedik University
Istanbul, Turkey

Abstract

The class of nonlinear partial differential equations that can be decomposed into sums of terms that are products of not necessarily the same number of linear differential operators is considered. Necessary and sufficient conditions for an arbitrary linear combination of a finite and an infinite number of solutions to satisfy the equation are derived.

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1. Introduction

Unlike linear differential equations, the linear combination of solutions of the nonlinear equation needs not satisfy the differential equation. However with some classes of equations such as equations of Navier-Stokes, Burgers, Hamilton-Jacobi type, pseudo-linear combination of solutions (in which the operations of addition and multiplication are replaced by pseudo-operations) is again a solution [4, 5].

A brief review of the past work related to the nonlinear superposition principles can be found in [3] whereas reference [2] also provides some physically motivated examples for them.

In [6], the author considered classes of nonlinear partial differential equations that can be decomposed into sums of terms that are products of linear partial differential operators. In this paper a necessary and sufficient condition for the linear combinations of the two solutions of the equation that is a sum of terms that are products of the same number of linear partial differential operators to satisfy the equation was obtained.

Equations that have a structure made up of sum of terms each of which is composed of the products of the results of the application of not necessarily the same number of linear partial differential operators on the dependent variable u were taken up in [7]. In this paper necessary and sufficient conditions were obtained for the linear combination of two solutions to satisfy the equation.

In this paper again we consider equations that have a structure made up of sums of terms each of which is composed of the products of the results of the application of not necessarily the same number of linear partial differential operators on the dependent variable u. We derive necessary and sufficient conditions for the linear combination of an arbitrary number of solutions to satisfy the equation.

2. Obtaining the necessary and sufficient conditions

In the paper as a convention, we shall denote a linear partial differential operator by the symbol L with proper subscripts. Then an equation from the class of nonlinear partial differential equations investigated in this paper can be represented as;

$$\sum_{k=1}^{n_1}\prod_{j=1}^{m_1}L_{kj}(u)+\sum_{k=n_1+1}^{n_2}\prod_{j=1}^{m_2}L_{kj}(u)+\cdots+\sum_{k=n_{q-1}+1}^{n_q}\prod_{j=1}^{m_q}L_{kj}(u)=0 \quad (1)$$

In other words in the sum, each term with $n_{p-1}+1\le k\le n_p$, is made up of a product of m_p linear operators L_{kj} where $1\le j\le m_p$ and $1\le p\le q$ with $n_0=0$. In this section we shall obtain necessary and sufficient conditions for linear combinations of solutions of (1) to satisfy (1). We shall first obtain the conditions for linear combinations of N solutions to satisfy the equation and next we shall extend this result to an infinite set of solutions.

2.1 Necessary and sufficient conditions for linear combinations of N solutions

Because different than [7] now we are concerned with linear combinations of more than two solutions, we will use a representation of (1) different than Equations (2) and (5) of [7]. Also, our approach of derivation and the resulting formulation of the necessary and sufficient conditions aimed at will be different than those in this reference.

The linear combination of N solutions of (1) will be

$$u=a_1u_1+a_2u_2+\cdots+a_Nu_N, \quad (2)$$

where the set $S_{aN} = \{a_1, a_2, \cdots, a_N\}$ is a set of arbitrary constants, while the set $S_{uN} = \{u_1, u_2, \cdots, u_N\}$ is the set of solutions of (1). When (2) is substituted in (1), the resulting equation will be;

$$\sum_{p=1}^{q}\{\sum_{k=n_{p-1}+1}^{n_p}[\sum_{i=1}^{C(N,m_p)} \Lambda_{pki} h_{pki}(u_1, u_2, \cdots, u_N)]\} = 0. \quad (3)$$

To explain the quantities $C(N, m_p), \Lambda_{pki}$ and h_{pki} that appear in (3), we note that when (2) is substituted in (1), for a particular p, each $\prod_{j=1}^{m_p} L_{kj}(a_1u_1 + a_2u_2 + \cdots + a_Nu_N) = \sum_{i=1}^{C(N,m_p)} \Lambda_{pki}h_{pki}(u_1, u_2, \cdots, u_N)$ term in the resulting sum will have a composition such that;

1) The quantity Λ_{pki} is the product $a_{r1} \cdot a_{r2} \cdot \cdots \cdot a_{rm_p}$ where the set $S_{am_p} = (a_{r_1}, a_{r_2}, \cdots, a_{r_{m_p}})$ is a combination from the set S_{aN} with repetition. $C(N, m_p) = \frac{(m_p+N-1)!}{m_p!(N-1)!}$ is the number of all such combinations.

2) The quantity $h_{pki}(u_1, u_2, \cdots, u_N)$ is the sum of *all* terms $\prod_{j=1}^{m_p} L_{kj}(u_{rj})$ where for such a product $L_{k1}(u_{r_1})L_{k2}(u_{r_2}) \cdots L_{km_p}(u_{r_{m_p}})$, the elements of the set $S_{um_p} = (u_{r_1}, u_{r_2}, \cdots, u_{r_{m_p}})$ have the same indices as the elements of the set S_{am_p}in the definition of Λ_{pki} above, but now the order of the elements of this set appearing in $L_{k1}(u_{r_1})L_{k2}(u_{r_2}) \cdots L_{km_p}(u_{r_{m_p}})$ is important. The *overall* number of these permutations with repetition when i runs from 1 to $C(N, m_p)$ is N^{m_p} .

3)

To illustrate the ideas with an example, let us suppose $N = 5$ and $m_p = 3$. We shall have $S_{aN} = \{a_1, a_2, \cdots, a_5\}, S_{uN} = \{u_1, u_2, \cdots, u_5\}\ S_{am_p} = (a_{r_1}, a_{r_2}, \cdots, a_{r_3})$. Then $C(N, m_p) = \frac{(m_p+N-1)!}{m_p!(N-1)!} = 35$. Λ_{pki} will be constituted of the following combinations:

$a_1^2a_2, a_1^2a_3, a_1^2a_4, a_1^2a_5, a_2^2a_3, a_3^2a_4, a_4^2a_5, a_2^2a_5, a_3^2a_5, a_2^2a_4, a_1a_2^2, a_1a_3^2, a_1a_4^2,$
$a_1a_5^2, a_2a_3^2, a_3a_4^2, a_4a_5^2, a_2a_5^2, a_3a_5^2, a_2a_4^2, a_1^3, a_2^3, a_3^3, a_4^3, a_5^3, a_1a_2a_3, a_1a_2a_4,$
$a_1a_2a_5, a_2a_3a_4, a_2a_3a_5, a_1a_3a_4, a_1a_3a_5, a_2a_4a_5, a_3a_4a_5, a_1a_4a_5.$

Corresponding to $\Lambda_{pki} = a_2a_3^2$, the coefficient $h_{pki}(u_1, u_2, \cdots, u_N)$ will be the sum;

$$L_{k1}(u_2)L_{k2}(u_3)L_{km_p}(u_3) + L_{k1}(u_3)L_{k2}(u_2)L_{km_p}(u_3) + L_{k1}(u_3)L_{k2}(u_3)L_{km_p}(u_2).$$

Notice that for given p, the Λ_{pki} are common for all $n_{p-1} + 1 \leq k \leq n_p$.

Now we observe that each Λ_{pki} in (3) is a unique composition of products of the form $a_{r1} \cdot a_{r2} \cdot \cdots \cdot a_{rm_p}$. Notice that the conditions derived in this work are not for a specific linear combination of solutions but for an arbitrary linear combination.

Then because each element of the set S_{aN} is arbitrary, the Λ_{pki} can be considered as linearly independent functions. Therefore (3) can be true for arbitrary elements of the set S_{aN} only if coefficients of Λ_{pki} vanish. Hence a necessary condition for the linear combination (2) to satisfy (1) is that

$$\sum_{k=n_{p-1}+1}^{n_p} h_{pki}(u_1, u_2, \cdots, u_N) = 0, \text{ for all } 1 \le i \le C(N, m_p) \text{ and } 1 \le p \le q \;. \quad (4)$$

It follows that to determine whether a linear combination of N solutions will satisfy an equation with the present approach, $\sum_{p=1}^{q} C(N, m_p)$ conditions will have to be checked.

This necessary condition is at the same time a sufficient condition for (2) to satisfy (1). Indeed in (3) the order of the inner two summations with respect to k and i can be interchanged. Then if (4) holds, the left side of (3) will vanish and (1) will have been satisfied.

2.2 Necessary and sufficient conditions for linear combinations of an infinite number of solutions

We will extend the truth of the above necessary and sufficient condition for N solutions to the case of an infinite number of solutions by induction.

For $N = 2$, from Section 2.1 we know that Λ_{pki} will be constituted of terms $a_1^x a_2^y$ where the integer powers (x, y) will vary as $0 \le x \le m_p$, $0 \le y \le m_p$ while $x + y = m_p$ is maintained. With the argument of the linear independence of the functions $a_1^x a_2^y$ and hence of Λ_{pki} for all p and i, and the requirement of (3) to vanish, we set the coefficients of Λ_{pki} equal to zero and write (4) for $N = 2$ as;

$$\sum_{k=n_{p-1}+1}^{n_p} h_{pki}(u_1, u_2) = 0, \text{ for all } 1 \le i \le C(N, m_p) \text{ and } 1 \le p \le q \;. \quad (5)$$

This necessary condition is also sufficient, since as in Section 2.1, its imposition will give zero result for the left side of (3) when $N = 2$, and (1) will have been satisfied.

Next, we assume the truth of the necessary and sufficient condition (4) which is for the linear combination of N solutions and we try to prove the truth of the condition for $N + 1$ solutions. When $u = a_1 u_1 + a_2 u_2 + \cdots + a_N u_N + a_{N+1} u_{N+1}$, we must have $S_{aN+1} = \{a_1, a_2, \cdots, a_N, a_{N+1}\}$ for the coefficients and $S_{uN+1} = \{u_1, u_2, \cdots, u_N, u_{N+1}\}$ for the solutions. We write (3) for this new case as;

$$\sum_{p=1}^{q} \{\sum_{k=n_{p-1}+1}^{np} [\sum_{i=1}^{C(N+1, m_p)} \Lambda_{pki} h_{pki}(u_1, u_2, \cdots, u_{N+1})]\} = 0. \quad (6)$$

Since we have the truth of the necessary and sufficient conditions for N solutions, in (6) for Λ_{pki} which is the product $a_{r1} \cdot a_{r2} \cdot \cdots \cdot a_{rm_p}$ where the $S_{am_p} = (a_{r_1}, a_{r_2}, \cdots, a_{r_{m_p}})$ is a combination from the set S_{aN} with repetition, all corresponding coefficients $h_{pki}(u_1, u_2, \cdots, u_N)$ will satisfy (4) and thus will cease to exist in (6). The terms left over in (6) will be only those terms corresponding to coefficients of Λ_{pki} which is again the product $a_{r1} \cdot a_{r2} \cdot \cdots \cdot a_{rm_p}$, but where the set $S_{am_p} = (a_{r_1}, a_{r_2}, \cdots, a_{r_{m_p}})$ which is a combination from S_{aN+1} with repetition, will now necessarily have a_{N+1} as an element. Thus only terms with $\Lambda_{pki} = a_{r_1}^{x_1} \cdot a_{r_2}^{x_2} \cdot \cdots \cdot a_{r_{m_p}}^{x_{m_p}}$ where the exponents satisfy $x_1 + x_2 + \cdots + x_{m_p} = m_p$ with $0 \leq x_j \leq m_p$, $1 \leq j \leq m_p$ and one of factors is necessarily a nonzero power of a_{N+1}, will appear in (6). But then each such Λ_{pki} is a linearly independent function again, and therefore to satisfy (6), its coefficients have to vanish. This brings about vanishing of the full coefficients of Λ_{pki} in (6), or

$$\sum_{k=n_{p-1}+1}^{np} h_{pki}(u_1, u_2, \cdots, u_{N+1}) = 0,$$
$$\text{for all } 1 \leq i \leq C(N+1, m_p) \text{ and } 1 \leq p \leq q \,. \qquad (7)$$

This necessary condition is also sufficient for the linear combinations of $N+1$ solutions to satisfy (1). Indeed, interchanging orders of the inner two summations in (6) and substitution of (7) makes the left side of (6) zero, QED.

3. Application to initial-boundary value problems

When a solution set of the differential equation is complete and passes the proved necessary and sufficient conditions for a linear combination to satisfy the equation, then a solution can be constructed that will also satisfy an arbitrary initial-boundary value function by way of expanding the unknown solution as series of linear combinations of the elements of the complete set of solutions. However, it should be remarked that completeness of the solution set is necessary but not sufficient for this series with coefficients as the inner products of the initial-boundary value function and the orthonormalized solutions of the differential equation to converge to this function. A sufficient condition for this expansion to converge to the function is the uniform convergence of the series involved [1]. When complete systems of functions of one variable are known one method of constructing complete systems of functions of several variables is given in [1].

4. Two examples

4.1 Example 1

Consider the following differential equation,

$$\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial t^2}=0, \quad (8)$$

which has the solutions

$u_1=\exp(k_{y1}t)\exp(k_{x1}x)\sin(k_{y1}y)$, $u_2=\exp(k_{y2}t)\exp(k_{x2}x)\sin(k_{y2}y)$,
$u_3=\exp(k_{y3}t)\exp(k_{x3}x)\sin(k_{y3}y)$.

We shall check conditions (4) to see whether arbitrary linear combinations of these three solutions satisfy (8). In this case,

$N=3, S_{a3}=\{a_1,a_2,a_3\}, S_{u3}=\{u_1,u_2,u_3\}$.
$p=1$:
$n_1=2, \quad 1\le k\le 2, \quad m_1=2, \quad L_{11}=\frac{\partial}{\partial x}, L_{12}=\frac{\partial^2}{\partial t^2}, L_{21}=\frac{\partial}{\partial x}, L_{22}=\frac{\partial^2}{\partial y^2},$
$C(N,m_p)=C(3,2)=6$
$\Lambda_{1k1}=a_1a_2, \Lambda_{1k2}=a_1a_3, \Lambda_{1k3}=a_2a_3, \Lambda_{1k4}=a_1^2, \Lambda_{1k5}=a_2^2, \Lambda_{1k6}=a_3^2,$
$h_{111}=L_{11}(u_1)L_{12}(u_2)+L_{11}(u_2)L_{12}(u_1)=(k_{x1}k_{y2}^2+k_{x2}k_{y1}^2)\exp[(k_{y1}+k_{y2})t]\exp[(k_{x1}+k_{x2})x]\sin(k_{y1}y)\sin(k_{y2}y),$
$h_{112}=L_{11}(u_1)L_{12}(u_3)+L_{11}(u_3)L_{12}(u_1)=(k_{x1}k_{y3}^2+k_{x3}k_{y1}^2)\exp[(k_{y1}+k_{y3})t]\exp[(k_{x1}+k_{x3})x]\sin(k_{y1}y)\sin(k_{y3}y),$
$h_{113}=L_{11}(u_2)L_{12}(u_3)+L_{11}(u_3)L_{12}(u_2)=(k_{x3}k_{y2}^2+k_{x2}k_{y3}^2)\exp[(k_{y3}+k_{y2})t]\exp[(k_{x3}+k_{x2})x]\sin(k_{y3}y)\sin(k_{y2}y),$
$h_{114}=L_{11}(u_1)L_{12}(u_1)=k_{x1}k_{y1}^2\exp[2k_{y1}t]\exp[2k_{x1}x]\sin^2(k_{y1}y),$
$h_{115}=L_{11}(u_2)L_{12}(u_2)=k_{x2}k_{y2}^2\exp[2k_{y2}t]\exp[2k_{x2}x]\sin^2(k_{y2}y),$
$h_{116}=L_{11}(u_3)L_{12}(u_3)=k_{x3}k_{y3}^2\exp[2k_{y3}t]\exp[2k_{x3}x]\sin^2(k_{y3}y),$
$h_{121}=L_{21}(u_1)L_{22}(u_2)+L_{21}(u_2)L_{22}(u_1)=-(k_{x1}k_{y2}^2+k_{x2}k_{y1}^2)\exp[(k_{y1}+k_{y2})t]\exp[(k_{x1}+k_{x2})x]\sin(k_{y1}y)\sin(k_{y2}y),$
$h_{122}=L_{21}(u_1)L_{22}(u_3)+L_{21}(u_3)L_{22}(u_1))=-(k_{x1}k_{y3}^2+k_{x3}k_{y1}^2)\exp[(k_{y1}+k_{y3})t]\exp[(k_{x1}+k_{x3})x]\sin(k_{y1}y)\sin(k_{y3}y),$
$h_{123}=L_{21}(u_2)L_{22}(u_3)+L_{21}(u_3)L_{22}(u_2)=-(k_{x3}k_{y2}^2+k_{x2}k_{y3}^2)\exp[(k_{y3}+k_{y2})t]\exp[(k_{x3}+k_{x2})x]\sin(k_{y3}y)\sin(k_{y2}y),$
$h_{124}=L_{21}(u_1)L_{22}(u_1)=-k_{x1}k_{y1}^2\exp[2k_{y1}t]\exp[2k_{x1}x]\sin^2(k_{y1}y),$
$h_{125}=L_{21}(u_2)L_{22}(u_2)=-k_{x2}k_{y2}^2\exp[2k_{y2}t]\exp[2k_{x2}x]\sin^2(k_{y2}y),$
$h_{126}=L_{21}(u_3)L_{22}(u_3)=-k_{x3}k_{y3}^2\exp[2k_{y3}t]\exp[2k_{x3}x]\sin^2(k_{y3}y).$
$p=2$:
$n_2=4, \quad 3\le k\le 4, \quad m_2=1, L_{31}=-\frac{\partial^2}{\partial y^2}, L_{41}=-\frac{\partial^2}{\partial t^2}, \quad C(N,m_p)=C(3,1)=3$,
$\Lambda_{2k1}=a_1, \quad \Lambda_{2k2}=a_2, \quad \Lambda_{2k3}=a_3$,
$h_{231}=L_{31}(u_1)=k_{y1}^2\exp(k_{y1}t)\exp(k_{x1}x)\sin(k_{y1}y),$
$h_{232}=L_{31}(u_2)=k_{y2}^2\exp(k_{y2}t)\exp(k_{x2}x)\sin(k_{y2}y),$

$h_{233} = L_{31}(u_3) = k_{y3}^2 \exp(k_{y3}t) \exp(k_{x3}x) \sin(k_{y3}y),$
$h_{241} = L_{41}(u_1) = -k_{y1}^2 \exp(k_{y1}t) \exp(k_{x1}x) \sin(k_{y1}y),$
$h_{242} = L_{41}(u_2) = -k_{y2}^2 \exp(k_{y2}t) \exp(k_{x2}x) \sin(k_{y2}y),$
$h_{243} = L_{41}(u_3) = -k_{y3}^2 \exp(k_{y3}t) \exp(k_{x3}x) \sin(k_{y3}y).$

It will be found that $\sum_{k=1}^{n_1} h_{1ki}(u_1, u_2, u_3) = 0$ holds for $1 \le i \le 6$ and $\sum_{k=3}^{n_2} h_{2ki}(u_1, u_2, u_3) = 0$ holds for $1 \le i \le 3$ as stipulated by (4). Indeed, it can be checked by direct substitution that an arbitrary linear combination of elements of S_{u3} satisfies (8).

Note that the above conditions for the linear combination of any number of solutions can be expressed as

$$\frac{\partial^2 u_i}{\partial t^2} + \frac{\partial^2 u_i}{\partial y^2} = 0\ , \tag{9a}$$
$$\frac{\partial u_i}{\partial x}\left(\frac{\partial^2 u_i}{\partial t^2} + \frac{\partial^2 u_i}{\partial y^2}\right) = 0, \tag{9b}$$
$$\frac{\partial u_i}{\partial x}\left(\frac{\partial^2 u_j}{\partial t^2} + \frac{\partial^2 u_j}{\partial y^2}\right) + \frac{\partial u_j}{\partial x}\left(\frac{\partial^2 u_i}{\partial t^2} + \frac{\partial^2 u_i}{\partial y^2}\right) = 0. \tag{9c}$$

From these conditions we can infer that the condition

$$\left(\frac{\partial^2 u_i}{\partial t^2} + \frac{\partial^2 u_i}{\partial y^2}\right) = 0, \text{ for all } i, \tag{9d}$$

is necessary and sufficient for linear combinations of solutions to satisfy the equation. Hence as another outcome of the necessary and sufficient condition (4), we can deduce that the linear combinations of an infinite number of solutions of (8) that satisfy (9d) will satisfy (8).

4.2 Example 2

As a second example consider the differential equation

$$\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} + u^2 = 0, \tag{10}$$

which has the solutions $u_1 = \exp(x)\sin(y)$, $u_2 = \exp(x)\cos(y)$, $u_3 = \exp(-x)\exp(y)$. By the procedure similar to the above one can see that linear combinations of u_1 and u_2 satisfy (10). We shall test condition (4) to see whether linear combinations of u_1 and u_3 satisfy (10). Therefore, we have:

$N = 2, S_{a2} = \{a_1, a_3\}, S_{u2} = \{u_1, u_3\}.$
$p = 1:$

$n_1 = 2,\ 1 \le k \le 2,\ m_1 = 2,\ L_{11} = \frac{\partial}{\partial x}, L_{12} = \frac{\partial^2}{\partial y^2}, L_{21} = 1, L_{22} = 1,\ C(N, m_p) = C(2,2) = 3,$
$\Lambda_{1k1} = a_1 a_3,\ \Lambda_{1k2} = a_1^2,\ \Lambda_{1k3} = a_3^2,$
$h_{111} = L_{11}(u_1)L_{12}(u_3) + L_{11}(u_3)L_{12}(u_1) = 2\sin(y)\exp(y),$
$h_{112} = L_{11}(u_1)L_{12}(u_1) = -\exp(2x)\sin^2(y),$
$h_{113} = L_{11}(u_3)L_{12}(u_3) = -\exp(-2x)\exp(2y),$
$h_{121} = L_{21}(u_1)L_{22}(u_3) + L_{21}(u_3)L_{22}(u_1) = 2\sin(y)\exp(y),$
$h_{122} = L_{21}(u_1)L_{22}(u_1) = \exp(2x)\sin^2(y),$
$h_{123} = L_{21}(u_3)L_{22}(u_3) = \exp(-2x)\exp(2y),$

It can be observed that $\sum_{k=1}^{n_1} h_{1k2}(u_1, u_3) = h_{112} + h_{122} = 0,$ and $\sum_{k=1}^{n_1} h_{1k3}(u_1, u_3) = h_{113} + h_{123} = 0,$ as per the requirement stated by (4). However $\sum_{k=1}^{n_1} h_{1k1}(u_1, u_3) = h_{111} + h_{121} = 4\sin(y)\exp(y) \neq 0$ and the necessary condition fails. Indeed, it can be found by direct substitution that a linear combination of u_1 and u_3 does not satisfy (10).

5. Conclusion

Conditions obtained for the linear combinations of the solutions to satisfy the equation are free of the coefficients of the solutions in the linear combination. This will enable a check for the satisfaction of the equation by the linear combinations solely by using the solutions set $u_1, u_2, \cdots$ and the conditions (4). Another application could be use of the presented results when the solutions set is given as a data set and whether an arbitrary and not a particular linear combination will satisfy the equation is desired to be determined. Also, the derived conditions can help in a design problem while trying to construct a differential equation whose linear combinations of solutions are required to satisfy the equation.

When checking whether an infinite linear combination will satisfy the equation, the feature of independence of the obtained conditions from the coefficients $a_1, a_2, \cdots$ can be exploited and based on only the properties of the solutions set $u_1, u_2, \cdots$ and of the conditions (4) for infinite N, whether an infinite linear combination will satisfy the equation can be determined. In the first example in Section 4 this application of the results is demonstrated.

If by means of the conditions derived, it is determined that an infinite linear combination of a complete set of solutions of the equation satisfies the equation, then this finding can be applied to solve initial-boundary value problems with arbitrary initial-boundary value functions.

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