

# Stability of Switched Nonlinear Systems with Stable and Unstable Subsystems via Minimum Dwell Time Approach

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## **Abstract**

This paper discusses the stability of nonlinear switched systems in three subsystems, where two subsystems are stable, and one is unstable or vice versa. Using the method of multiple Lyapunov functions and other conditions, we obtain an estimate for the minimum dwell time (MDT), which assures the exponential stability of switched nonlinear systems. An illustrative example is presented to show the validity of the results.

**Mathematics Subject Classification:** 37C75, 93D05, 93C30

**Keywords:** Switched systems, minimum dwell time, multiple Lyapunov functions, stability

## 1 Introduction

Switched systems have received much attention in recent years because of their applications in many fields such as aircraft, automotive industry, biology, control systems, robotics, epidemic disease models; see [4, 8, 9, 11, 13, 15, 17]. A switched system is a private class of hybrid systems that consists of a family of continuous-time or discrete-time dynamical subsystems, and a switching signal that controls the switchings among them according to some environmental factors or seeking a specific performance.

The stability of switched systems is very important and has attracted many researchers, and has been studied using either the common or the multiple Lyapunov function method [14, 2, 20, 10]. It is more convenient to use multiple Lyapunov functions than the common Lyapunov function since having only one Lyapunov function for all the modes under study is not practical and is difficult to construct.

A peculiar property of a switched system is that the stability of the individual subsystems does not guarantee the stability of the switched system under an arbitrary switching [8]. However, it has been shown that such stability can be achieved if the dwell time ( $\tau_k$ ), the time between any two consecutive switchings, is sufficiently large [12, 3, 5]. From a practical perspective, the dwell-time condition may not hold in some subsystems due to their behaviour. In such cases, one can get the same stability result if the average dwell time  $\tau_a$  is satisfied [20, 21]. The minimum dwell time (MDT) concept has been introduced in [18], and then it was developed in [6]. Motivated by the above reasons, we extend previous results in [6] to investigate the stability of nonlinear switched system in three subsystems, two of which are stable, and one is unstable or vice versa.

The paper is regulated as follows. In Section 2, problem formulation is presented. In section 3, proposes the new analysis method and the main results of this paper. In Section 4, we study an illustrative example by using the results obtained in this paper. Finally, a conclusion is given in Section 5.

## 2 Preliminary Notes

Throughout this paper,  $\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean space;  $\mathbb{R}_+$  refers to the nonnegative real numbers. A symmetric matrix  $P$  is said to be positive(negative) definite if all its eigenvalues are positive(negative). Moreover, if  $P \in \mathbb{R}^{n \times n}$ , denote by  $\lambda_{\max}(P)$ ( $\lambda_{\min}(P)$ ) the maximum (minimum)

eigenvalue of  $P$ . If  $V(x) = x^T P x$ , the following inequalities are true

$$\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2. \quad (1)$$

If  $x \in \mathbb{R}^n$ , then  $\|x\|$  refers to the Euclidean vector norm of  $x$ .

Consider the switched nonlinear system:

$$\dot{x} = f_{\sigma(t)}(x) \quad (2)$$

where  $x \in \mathbb{R}^n$  is the system state,  $\sigma(t) : [t_0, \infty) \rightarrow I = \{1, 2, 3, \dots, n\}$ ,  $n \in \mathbb{N}$  is a piecewise right-continuous function, called the switching signals. The switching time sequence  $\{t_k\}_{k=0}^{\infty}$  satisfy  $t_0 < t_1 < t_2 < \dots < t_k < \dots < \infty$  and  $\sigma(t) = i$  means that the  $i$ -th subsystem is active when  $t \in [t_{k-1}, t_k)$ .  $f_i(x)$  is a smooth vector function,  $i \in I$ . we consider that each subsystem can be either stable or unstable. Let  $\tau_k = t_k - t_{k-1}$  denote the dwell time, where  $k = 1, 2, 3, \dots$ . For the sake of convenience, we let

$$\{x_k\}_{k=0}^{+\infty} : x_0, x_1, x_2, \dots, x_k, \dots$$

denote the switching state sequence corresponding to the switching time sequence  $\{t_k\}_{k=0}^{+\infty}$ , i.e.,  $x_k = x(t_k)$ . Now we discuss the stability of system (2) using minimum dwell time (MDT) method.

**Definition 2.1** (MDT). *If there exists a constant  $\tau_0$  such that a class of admitted switching signals satisfies the property that the switching time achieve the inequality  $\tau_k \leq \tau_0$  for all  $k \in \mathbb{N}$ , then we say that the switched system is with the minimal dwell time  $\tau_0$ . [19]*

### 3 Main Results

In this section, we present the main results for this paper. In theorems (3.1, 3.2 and 3.3) we assume that subsystem 1 is exponentially asymptotically stable and both subsystems 2 and 3 are unstable, while in theorems (3.4 and 3.5) subsystems 1 and 2 are exponentially asymptotically stable and subsystem 3 is unstable.

The following assumption will be needed in all the theorems:

- (A1)  $\exists$  a Lyapunov function  $V_i(x)$  such that

$$\dot{V}_i(x)|_{(i)} = \frac{\partial V_i}{\partial x} f_i(x) \leq -\lambda_i V_i(x), \quad \lambda_i > 0$$

where

$$a_i \|x\|^2 \leq V_i(x) \leq b_i \|x\|^2, \quad 0 < a_i < b_i.$$

- (A2)  $\exists$  a constant  $C_i > 0$  such that

$$\|x_{k+1}\| \leq C_i \|x_k\| \quad k = 0, 1, 2, \dots$$

- (A3)  $\exists$  a Lyapunov function  $V_i(x)$  such that

$$\dot{V}_i(x)|_{(i)} = \frac{\partial V_i}{\partial x} f_i(x) \leq \lambda_i V_i(x), \quad \lambda_i > 0$$

where

$$a_i \|x\|^2 \leq V_i(x) \leq b_i \|x\|^2, \quad 0 < a_i < b_i.$$

**Theorem 3.1.** *Consider the nonlinear switched system (2) with  $n = 3$ , and the three subsystems are active in turn. If the subsystem  $\dot{x} = f_1(x)$  is exponentially asymptotically stable, and (A1) is satisfied for  $i = 1$ , and the subsystems  $\dot{x} = f_2(x)$  and  $\dot{x} = f_3(x)$  are unstable, and (A2) is satisfied for  $i = 2, 3$ . Then system (2) is asymptotically stable under arbitrary switching signals with minimum dwell time MDT  $\tau_0$ , where*

$$\tau_0 = \frac{\ln(\mu_1 C_2 C_3)}{\lambda_1}, \quad \mu_1 = \frac{b_1}{a_1}, \quad (3)$$

**proof.** Assume that the 1<sup>st</sup> subsystem  $\dot{x} = f_1(x)$  is active on  $[t_{k+1}, t_{k+2})$ , the 2<sup>nd</sup> subsystem  $\dot{x} = f_2(x)$  is active on  $[t_k, t_{k+1})$ , and the 3<sup>rd</sup> subsystem  $\dot{x} = f_3(x)$  is active on  $[t_{k-1}, t_k)$ . According to the inequality (A1), we obtain that

$$\frac{\dot{V}_i(x)}{V_i(x)} \leq -\lambda_i \quad \text{solve by integration}$$

$$\int_{t_{k+1}}^{t_{k+2}} \frac{\dot{V}_i(x)}{V_i(x)} dx \leq - \int_{t_{k+1}}^{t_{k+2}} \lambda_i dx$$

$$\ln \left( \frac{V_i(x_{k+2})}{V_i(x_{k+1})} \right) \leq -\lambda_i (t_{k+2} - t_{k+1})$$

$$V_i(x_{k+2}) \leq e^{-\lambda_i \tau_{k+2}} V_i(x_{k+1}) \quad (4)$$

From (A1), (4) and (A2) it is easy to show that

$$\begin{aligned} \|x_{k+2}\|^2 &\leq \frac{1}{a_1} V_1(x_{k+2}) \leq \frac{1}{a_1} e^{-\lambda_1 \tau_{k+2}} V_1(x_{k+1}) \leq \frac{b_1}{a_1} e^{-\lambda_1 \tau_{k+2}} \|x_{k+1}\|^2 \\ &\leq \mu_1 e^{-\lambda_1 \tau_{k+2}} \|x_{k+1}\|^2 \leq \mu_1 e^{-\lambda_1 \tau_{k+2}} C_2 \|x_k\|^2 \leq \mu_1 e^{-\lambda_1 \tau_{k+2}} C_2 C_3 \|x_{k-1}\|^2 \end{aligned}$$

Clearly, if  $\tau_{k+2} \geq \tau_0$  then

$$\tau_{k+2} \geq \frac{\ln(\mu_1 C_2 C_3)}{\lambda_1}$$

which implies  $\mu_1 e^{-\lambda_1 \tau_{k+2}} C_2 C_3 \leq 1$  and so  $\|x_{k+2}\| \leq \|x_{k-1}\|$ .

Now we will prove that  $\lim_{k \rightarrow \infty} \|x_k\| = 0$ . For MDT  $\tau_0$  let  $\{t_{k'}\}_{k'=0}^{+\infty}$  be the switching time sequence with its dwell time greater than or equals  $\tau_1$  where  $\tau_1 > \tau_0$ , be such that  $\|x_{k'}\| \leq lb$ , where  $k' = 3, 4, \dots$ ,  $b = \max\{\|x_0\|, \|x_1\|, \|x_2\|\}$  and  $0 < l < 1$ .

Therefore,  $\lim_{k' \rightarrow \infty} \|x_{k'}\|^2 \leq \lim_{k' \rightarrow \infty} l^{2(k'/3)} b^2 = 0$ . This proves that, the switched system (2) is asymptotically stable under arbitrary switching signals with its dwell time greater than the MDT  $\tau_0$ .

**Theorem 3.2.** *Consider the nonlinear switched system (2) with  $n = 3$ , and the three subsystems are active in turn. If the subsystem  $\dot{x} = f_1(x)$  is exponentially asymptotically stable, and (A1) is satisfied for  $i = 1$ , and the subsystems  $\dot{x} = f_2(x)$  and  $\dot{x} = f_3(x)$  are unstable, and (A3) is satisfied for  $i = 2, 3$ . then we have three cases.*

- (i) *If  $\lambda_2 + \lambda_3 < \lambda_1$ , then system (2) is asymptotically stable under arbitrary switching with  $\tau_k^{(1)} \geq \tau_0$ ,  $\tau_k^{(2)} \leq \tau_0$  and  $\tau_k^{(3)} \leq \tau_0$ , where  $\tau_k^{(i)}$  is the dwell time of the  $i^{\text{th}}$  subsystem  $\dot{x} = f_i(x)$ , and the MDT is given by*

$$\tau_0 = \frac{\ln(\mu_1 \mu_2 \mu_3)}{\lambda_1 - (\lambda_2 + \lambda_3)}, \quad \mu_i = \frac{b_i}{a_i}, \quad i \in I \quad (5)$$

- (ii) *If  $\lambda_1 = \lambda_2 = \lambda_3$ , then system (2) is asymptotically stable under arbitrary switching with  $\tau_k^{(1)} - (\tau_k^{(2)} + \tau_k^{(3)}) \geq \tau_0$ , and the MDT is given by*

$$\tau_0 = \frac{\ln(\mu_1 \mu_2 \mu_3)}{\lambda_1}, \quad \mu_i = \frac{b_i}{a_i}, \quad i \in I \quad (6)$$

- (iii) *If  $\exists \lambda_i > \lambda_1, i = 2$  or  $3$ , then system (2) is asymptotically stable under arbitrary switching with  $\tau_k^{(1)} \geq \frac{\lambda_2 + \lambda_3 + \varepsilon}{\lambda_1} \tau_0$ ,  $\tau_k^{(2)} \leq \tau_0$  and  $\tau_k^{(3)} \leq \tau_0$ , and the MDT is given by*

$$\tau_0 = \frac{\ln(\mu_1 \mu_2 \mu_3)}{\varepsilon}, \quad \mu_i = \frac{b_i}{a_i}, \quad i \in I, \quad (7)$$

where  $\varepsilon > 0$  is an arbitrary positive real number.

**proof.** Assume that the 1<sup>st</sup> subsystem  $\dot{x} = f_1(x)$  is active on  $[t_{k+1}, t_{k+2})$ , the 2<sup>nd</sup> subsystem  $\dot{x} = f_2(x)$  is active on  $[t_k, t_{k+1})$ , and the 3<sup>rd</sup> subsystem

$\dot{x} = f_3(x)$  is active on  $[t_{k-1}, t_k]$ . According to the inequality (A3), we obtain that

$$\frac{\dot{V}_i(x)}{V_i(x)} \leq \lambda_i \quad \text{solve by integration}$$

$$\int_{t_k}^{t_{k+1}} \frac{\dot{V}_i(x)}{V_i(x)} dx \leq \int_{t_k}^{t_{k+1}} \lambda_i dx$$

$$\ln \left( \frac{V_i(x_{k+1})}{V_i(x_k)} \right) \leq \lambda_i (t_{k+1} - t_k)$$

$$V_i(x_{k+1}) \leq e^{\lambda_i \tau_{k+1}} V_i(x_k) \quad (8)$$

From (A1), (A3) and (8) it is easy to show that

$$\begin{aligned} \|x_{k+2}\|^2 &\leq \mu_1 e^{-\lambda_1 \tau_{k+2}} \|x_{k+1}\|^2 \leq \mu_1 e^{-\lambda_1 \tau_{k+2}} \frac{1}{a_2} V_2(x_{k+1}) \\ &\leq \mu_1 e^{-\lambda_1 \tau_{k+2}} \frac{1}{a_2} e^{\lambda_2 \tau_{k+1}} V_2(x_k) \leq \mu_1 e^{-\lambda_1 \tau_{k+2}} \frac{b_2}{a_2} e^{\lambda_2 \tau_{k+1}} \|x_k\|^2 \\ &\leq \mu_1 \mu_2 e^{-\lambda_1 \tau_{k+2}} e^{\lambda_2 \tau_{k+1}} \frac{1}{a_3} V_3(x_k) \\ &\leq \mu_1 \mu_2 e^{-\lambda_1 \tau_{k+2}} e^{\lambda_2 \tau_{k+1}} \frac{1}{a_3} e^{\lambda_3 \tau_k} V_3(x_{k-1}) \\ &\leq \mu_1 \mu_2 e^{-\lambda_1 \tau_{k+2}} e^{\lambda_2 \tau_{k+1}} \frac{b_3}{a_3} e^{\lambda_3 \tau_k} \|x_{k-1}\|^2 \\ &= \mu_1 \mu_2 \mu_3 e^{-\lambda_1 \tau_{k+2} + \lambda_2 \tau_{k+1} + \lambda_3 \tau_k} \|x_{k-1}\|^2 \end{aligned}$$

- (i) If  $\lambda_2 + \lambda_3 < \lambda_1$ , assume that  $\tau_{k+2} \geq \tau_0$ ,  $\tau_{k+1} \leq \tau_0$ , and  $\tau_k \leq \tau_0$ , then

$$\lambda_3 \tau_k + \lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2} \leq (\lambda_3 + \lambda_2 - \lambda_3) \tau_0 = (\lambda_3 + \lambda_2 - \lambda_1) \frac{\ln(\mu_1 \mu_2 \mu_3)}{\lambda_1 - (\lambda_2 + \lambda_3)}$$

which implies that

$$e^{\lambda_3 \tau_k + \lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2}} \leq (\mu_1 \mu_2 \mu_3)^{-1}$$

i.e.  $\mu_1 \mu_2 \mu_3 e^{\lambda_3 \tau_k + \lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2}} \leq 1$ , and therefore  $\|x_{k+2}\| \leq \|x_{k-1}\|$ .

- (ii) If  $\lambda_1 = \lambda_2 = \lambda_3$ , assume that  $\tau_{k+2} - \tau_{k+1} - \tau_k \geq \tau_0$  then

$$\tau_k + \tau_{k+1} - \tau_{k+2} \leq -\tau_0$$

and so,

$$\lambda_1 (\tau_k + \tau_{k+1} - \tau_{k+2}) \leq -\lambda_1 \frac{\ln(\mu_1 \mu_2 \mu_3)}{\lambda_1}$$

which implies  $e^{\lambda_1(\tau_k + \tau_{k+1} - \tau_{k+2})} \leq (\mu_1 \mu_2 \mu_3)^{-1}$ , i.e.  $\mu_1 \mu_2 \mu_3 e^{\lambda_1(\tau_k + \tau_{k+1} - \tau_{k+2})} \leq 1$ , therefore

$$\|x_{k+2}\| \leq \|x_{k-1}\|$$

- (iii)  $\lambda_i > \lambda_1$ ,  $i = 2$  or  $3$ , assume that  $\tau_{k+2} \geq \frac{\lambda_2 + \lambda_3 + \varepsilon}{\lambda_1} \tau_0$ ,  $\tau_{k+1} \leq \tau_0$ , and  $\tau_k \leq \tau_0$ , then

$$\lambda_3 \tau_k + \lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2} \leq \lambda_3 \tau_0 + \lambda_2 \tau_0 - \lambda_1 \left( \frac{\lambda_2 + \lambda_3 + \varepsilon}{\lambda_1} \tau_0 \right) = (\lambda_3 + \lambda_2 - \lambda_3 - \lambda_2 - \varepsilon) \tau_0 = -\varepsilon \tau_0$$

So,

$$\lambda_3 \tau_k + \lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2} \leq -\varepsilon \frac{\ln(\mu_1 \mu_2 \mu_3)}{\varepsilon}$$

which implies that  $e^{\lambda_3 \tau_k + \lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2}} \leq (\mu_1 \mu_2 \mu_3)^{-1}$  i.e.

$$\mu_1 \mu_2 \mu_3 e^{\lambda_3 \tau_k + \lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2}} \leq 1$$

and therefore  $\|x_{k+2}\| \leq \|x_{k-1}\|$

For all 3 cases we proved that  $\|x_{k+2}\| \leq \|x_{k-1}\|$  which implies  $\lim_{k \rightarrow \infty} \|x_k\| = 0$ .

**Theorem 3.3.** Consider the nonlinear switched system (2) with  $n = 3$ , and the three subsystems are active in turn. If the subsystem  $\dot{x} = f_1(x)$  is exponentially asymptotically stable, and (A1) is satisfied for  $i = 1$ . If the subsystems  $\dot{x} = f_2(x)$  and  $\dot{x} = f_3(x)$  are unstable, but (A3) is satisfied for  $i = 2$ , and (A2) is satisfied for  $i = 3$ , then we have three cases.

- (i) If  $\lambda_2 < \lambda_1$ , then system (2) is asymptotically stable under arbitrary switching with  $\tau_k^{(1)} \geq \tau_0$  and  $\tau_k^{(2)} \leq \tau_0$ , where  $\tau_k^{(i)}$  is the dwell time of the  $i^{\text{th}}$  subsystem  $\dot{x} = f_i(x)$ , and the MDT is given by

$$\tau_0 = \frac{\ln(\mu_1 \mu_2 C_3)}{\lambda_1 - \lambda_2}, \quad \mu_i = \frac{b_i}{a_i}, \quad i = \{1, 2\}, \quad (9)$$

- (ii) If  $\lambda_2 = \lambda_1$ , then system (2) is asymptotically stable under arbitrary switching with  $\tau_k^{(1)} - \tau_k^{(2)} \geq \tau_0$ , and the MDT is given by

$$\tau_0 = \frac{\ln(\mu_1 \mu_2 C_3)}{\lambda_1}, \quad \mu_i = \frac{b_i}{a_i}, \quad i = \{1, 2\}, \quad (10)$$

- (iii) If  $\lambda_2 > \lambda_1$ , then system (2) is asymptotically stable under arbitrary switching with  $\tau_k^{(1)} \geq \frac{\lambda_2 + \varepsilon}{\lambda_1} \tau_0$ ,  $\tau_k^{(2)} \leq \tau_0$ , and the MDT is given by

$$\tau_0 = \frac{\ln(\mu_1 \mu_2 C_3)}{\varepsilon}, \quad \mu_i = \frac{b_i}{a_i}, \quad i = \{1, 2\}, \quad (11)$$

where  $\varepsilon > 0$  is an arbitrary positive real number.

**proof.** Assume that the 1<sup>st</sup> subsystem  $\dot{x} = f_1(x)$  is active on  $[t_{k+1}, t_{k+2})$ , the 2<sup>nd</sup> subsystem  $\dot{x} = f_2(x)$  is active on  $[t_k, t_{k+1})$ , and the 3<sup>rd</sup> subsystem  $\dot{x} = f_3(x)$  is active on  $[t_{k-1}, t_k)$ . Then

$$\|x_{k+2}\|^2 \leq \mu_1 \mu_2 e^{-\lambda_1 \tau_{k+2}} e^{\lambda_2 \tau_{k+1}} \|x_k\|^2 \leq \mu_1 \mu_2 C_3 e^{\lambda_2 \tau_{k+1} - \lambda_1 \tau_k} \|x_{k-1}\|^2$$

- (i) If  $\lambda_2 < \lambda_1$ , assume that  $\tau_{k+2} \geq \tau_0$  and  $\tau_{k+1} \leq \tau_0$ , then

$$\lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2} \leq (\lambda_2 - \lambda_1) \tau_0 = (\lambda_2 - \lambda_1) \frac{\ln(\mu_1 \mu_2 C_3)}{\lambda_1 - \lambda_2}$$

which implies that  $e^{\lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2}} \leq (\mu_1 \mu_2 C_3)^{-1}$  i.e.  $\mu_1 \mu_2 C_3 e^{\lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2}} \leq 1$ , and therefore  $\|x_{k+2}\| \leq \|x_{k-1}\|$ .

- (ii) If  $\lambda_1 = \lambda_2$ , assume that  $\tau_{k+2} - \tau_{k+1} \geq \tau_0$  then  $\tau_{k+1} - \tau_{k+2} \leq -\tau_0$  and so,

$$\lambda_1 (\tau_{k+1} - \tau_{k+2}) \leq -\lambda_1 \frac{\ln(\mu_1 \mu_2 C_3)}{\lambda_1}$$

which implies that  $e^{\lambda_1 (\tau_{k+1} - \tau_{k+2})} \leq (\mu_1 \mu_2 C_3)^{-1}$  i.e.  $\mu_1 \mu_2 C_3 e^{\lambda_1 (\tau_{k+1} - \tau_{k+2})} \leq 1$ , therefore  $\|x_{k+2}\| \leq \|x_{k-1}\|$

- (iii)  $\lambda_2 > \lambda_1$ , assume that  $\tau_{k+1} \leq \tau_0$  and  $\tau_{k+2} \geq \frac{\lambda_2 + \varepsilon}{\lambda_1} \tau_0$ , then

$$\lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2} \leq \lambda_2 \tau_0 - \lambda_1 \left( \frac{\lambda_2 + \varepsilon}{\lambda_1} \tau_0 \right) = -\varepsilon \tau_0 = -\varepsilon \frac{\ln(\mu_1 \mu_2 C_3)}{\varepsilon}$$

which implies that  $e^{\lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2}} \leq (\mu_1 \mu_2 C_3)^{-1}$  i.e.  $\mu_1 \mu_2 C_3 e^{\lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2}} \leq 1$ , and therefore  $\|x_{k+2}\| \leq \|x_{k-1}\|$ .

For all 3 cases we proved that  $\|x_{k+2}\| \leq \|x_{k-1}\|$  which implies  $\lim_{k \rightarrow \infty} \|x_k\| = 0$ .

**Theorem 3.4.** Consider the nonlinear switched system (2) with  $n = 3$ , and the three subsystems are active in turn. If the subsystems  $\dot{x} = f_1(x)$  and  $\dot{x} = f_2(x)$  are exponentially asymptotically stable, and (A1) is satisfied for  $i = 1, 2$  and the subsystem  $\dot{x} = f_3(x)$  are unstable, and (A2) is satisfied for  $i = 3$ , then system (2) is asymptotically stable under arbitrary switching signals with minimum dwell time MDT  $\tau_0$ , where

$$\tau_0 = \frac{\ln(\mu_1 \mu_2 C_3)}{\lambda_1 + \lambda_2}, \quad \mu_i = \frac{b_i}{a_i}, \quad i = \{1, 2\} \quad (12)$$

**proof.** Assume that the 1<sup>st</sup> subsystem  $\dot{x} = f_1(x)$  is active on  $[t_{k+1}, t_{k+2})$ , the 2<sup>nd</sup> subsystem  $\dot{x} = f_2(x)$  is active on  $[t_k, t_{k+1})$ , and the 3<sup>rd</sup> subsystem  $\dot{x} = f_3(x)$  is active on  $[t_{k-1}, t_k)$ . Then

$$\|x_{k+2}\|^2 \leq \mu_1 \mu_2 C_3 e^{-\lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2}} \|x_{k-1}\|^2$$

clearly, if  $\tau_{k+2} \geq \tau_0, \tau_{k+1} \geq \tau_0$  then

$$-\lambda_1 \tau_{k+2} - \lambda_2 \tau_{k+1} \leq -\lambda_1 \tau_0 - \lambda_2 \tau_0 = -(\lambda_1 + \lambda_2) \tau_0 = -(\lambda_1 + \lambda_2) \frac{\ln(\mu_1 \mu_2 C_3)}{(\lambda_1 + \lambda_2)}$$

which implies that

$$e^{-\lambda_1 \tau_{k+2} - \lambda_2 \tau_{k+1}} \leq (\mu_1 \mu_2 C_3)^{-1}$$

i.e.

$$\mu_1 \mu_2 C_3 e^{-\lambda_1 \tau_{k+2}} e^{-\lambda_2 \tau_{k+1}} \leq 1$$

and so  $\|x_{k+2}\| \leq \|x_{k-1}\|$  and  $\lim_{k \rightarrow \infty} \|x_k\| = 0$ .

**Theorem 3.5.** Consider the nonlinear switched system (2) with  $n = 3$ , and the three subsystems are active in turn. If the subsystems  $\dot{x} = f_1(x)$  and  $\dot{x} = f_2(x)$  are exponentially asymptotically stable, and (A1) is satisfied for  $i = 1, 2$  and the subsystem  $\dot{x} = f_3(x)$  are unstable, and (A3) is satisfied for  $i = 3$ , then we have three cases.

- (i) If  $\lambda_3 < \lambda_1 + \lambda_2$ , then system (2) is asymptotically stable under arbitrary switching with  $\tau_k^{(1)} \geq \tau_0, \tau_k^{(2)} \geq \tau_0, \tau_k^{(3)} \leq \tau_0$ , where  $\tau_k^{(i)}$  is the dwell time of the  $i^{\text{th}}$  subsystem  $\dot{x} = f_i(x), i \in I$ , and the MDT is given by

$$\tau_0 = \frac{\ln(\mu_1 \mu_2 \mu_3)}{\lambda_1 + \lambda_2 - \lambda_3}, \quad \mu_i = \frac{b_i}{a_i}, \quad i \in I \quad (13)$$

- (ii) If  $\lambda_1 = \lambda_2 = \lambda_3$ , then system (2) is asymptotically stable under arbitrary switching with  $\tau_k^{(1)} + \tau_k^{(2)} - \tau_k^{(3)} \geq \tau_0$ , and the MDT is given by

$$\tau_0 = \frac{\ln(\mu_1 \mu_2 \mu_3)}{\lambda_1}, \quad \mu_i = \frac{b_i}{a_i}, \quad i \in I \quad (14)$$

- (iii) If  $\lambda_3 > \lambda_1 + \lambda_2$ , then system (2) is asymptotically stable under arbitrary switching with  $\tau_k^{(1)} \geq \frac{\lambda_3 + \varepsilon}{\lambda_1 + \lambda_2} \tau_0, \tau_k^{(2)} \geq \frac{\lambda_3 + \varepsilon}{\lambda_1 + \lambda_2} \tau_0, \tau_k^{(3)} \leq \tau_0$ , and the MDT is given by

$$\tau_0 = \frac{\ln(\mu_1 \mu_2 \mu_3)}{\varepsilon}, \quad \mu_i = \frac{b_i}{a_i}, \quad i \in I \quad (15)$$

where  $\varepsilon > 0$  is an arbitrary positive real number.

**proof.** Assume that the 1<sup>st</sup> subsystem  $\dot{x} = f_1(x)$  is active on  $[t_{k+1}, t_{k+2})$ , the 2<sup>nd</sup> subsystem  $\dot{x} = f_2(x)$  is active on  $[t_k, t_{k+1})$ , and the 3<sup>rd</sup> subsystem  $\dot{x} = f_3(x)$  is active on  $[t_{k-1}, t_k)$ . Then

$$\|x_{k+2}\|^2 \leq \mu_1 \mu_2 \mu_3 e^{\lambda_3 \tau_k - \lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2}} \|x_{k-1}\|^2$$

- (i) If  $\lambda_3 < \lambda_1 + \lambda_2$ , assume that  $\tau_k \leq \tau_0, \tau_{k+1} \geq \tau_0$ , and  $\tau_{k+2} \geq \tau_0$ , then

$$\lambda_3 \tau_k - \lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2} \leq (\lambda_3 - \lambda_2 - \lambda_1) \tau_0 = (\lambda_3 - \lambda_2 - \lambda_1) \frac{\ln(\mu_1 \mu_2 \mu_3)}{\lambda_1 + \lambda_2 - \lambda_3}$$

which implies that

$$e^{\lambda_3 \tau_k - \lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2}} \leq (\mu_1 \mu_2 \mu_3)^{-1}$$

i.e.  $\mu_1 \mu_2 \mu_3 e^{\lambda_3 \tau_k - \lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2}} \leq 1$ , and therefore  $\|x_{k+2}\| \leq \|x_{k-1}\|$ .

- (ii) If  $\lambda_1 = \lambda_2 = \lambda_3$ , assume that  $\tau_{k+2} + \tau_{k+1} - \tau_k \geq \tau_0$  then

$\tau_k - \tau_{k+1} - \tau_{k+2} \leq -\tau_0$  and so,

$$\lambda_1 (\tau_k - \tau_{k+1} - \tau_{k+2}) \leq -\lambda_1 \frac{\ln(\mu_1 \mu_2 \mu_3)}{\lambda_1}$$

which implies that

$$e^{\lambda_1 (\tau_k - \tau_{k+1} - \tau_{k+2})} \leq (\mu_1 \mu_2 \mu_3)^{-1}$$

i.e.  $\mu_1 \mu_2 \mu_3 e^{\lambda_1 (\tau_k - \tau_{k+1} - \tau_{k+2})} \leq 1$ , therefore  $\|x_{k+2}\| \leq \|x_{k-1}\|$ .

- (iii) If  $\lambda_3 > \lambda_1 + \lambda_2$ , assume that  $\tau_k \leq \tau_0, \tau_{k+1} \geq \frac{\lambda_3 + \varepsilon}{\lambda_1 + \lambda_2} \tau_0$ , and  $\tau_{k+2} \geq \frac{\lambda_3 + \varepsilon}{\lambda_1 + \lambda_2} \tau_0$ , then

$$\begin{aligned} \lambda_3 \tau_k - \lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2} &\leq \lambda_3 \tau_0 - \lambda_2 \frac{\lambda_3 + \varepsilon}{\lambda_1 + \lambda_2} \tau_0 - \lambda_1 \left( \frac{\lambda_3 + \varepsilon}{\lambda_1 + \lambda_2} \tau_0 \right) = -\varepsilon \tau_0 \\ &= -\varepsilon \frac{\ln(\mu_1 \mu_2 \mu_3)}{\varepsilon} \end{aligned}$$

which implies that

$$e^{\lambda_3 \tau_k - \lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2}} \leq (\mu_1 \mu_2 \mu_3)^{-1}$$

i.e.  $\mu_1 \mu_2 \mu_3 e^{\lambda_3 \tau_k - \lambda_2 \tau_{k+1} - \lambda_1 \tau_{k+2}} \leq 1$ , and therefore  $\|x_{k+2}\| \leq \|x_{k-1}\|$ .

In all cases we proved that  $\|x_{k+2}\| \leq \|x_{k-1}\|$  which implies  $\lim_{k \rightarrow \infty} \|x_k\| = 0$ .

## 4 Numerical Example

In this section, numerical examples are provided to illustrate some main results. We will only apply theorems (3.2) and (3.5).

**Example 4.1.** Consider the following switched system with  $i \in I = \{1, 2, 3\}$ ,

$$\dot{x} = A_i x, \quad (16)$$

and

$$A_1 = \begin{pmatrix} -1 & 10 \\ -2 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} 0.0556 & 3 \\ -9 & 0.5 \end{pmatrix}, A_3 = \begin{pmatrix} 0.125 & -2 \\ 4 & 0.25 \end{pmatrix}$$

Since  $A_1$  is exponentially asymptotically stable,  $A_2, A_3$  are unstable, we use theorem (3.2) to find the MDT for the system (16). Let  $V_1 = x_1^2 + 5x_2^2$  is a Lyapunov function for the 1<sup>st</sup> subsystem,  $V_2 = 3x_1^2 + x_2^2$  is a Lyapunov function for the 2<sup>nd</sup> subsystem, and  $V_3 = 2x_1^2 + x_2^2$  is a Lyapunov function for the 3<sup>rd</sup> subsystem. Next;

$$\begin{aligned} \dot{V}_1|_{(1)} &= 2x_1(-x_1 + 10x_2) + 10x_2(-2x_1 - x_2) \\ &= -2x_1^2 + 20x_1x_2 - 20x_1x_2 - 10x_2^2 \\ &= -2x_1^2 - 10x_2^2 = -2(x_1^2 + 5x_2^2) \leq -2V_1 \\ \dot{V}_2|_{(2)} &= 6x_1(0.0556x_1 + 3x_2) + 2x_2(-9x_1 + 0.5x_2) \\ &= 0.3336x_1^2 + 18x_1x_2 - 18x_1x_2 + x_2^2 \\ &= 0.3336x_1^2 + x_2^2 = 0.3336x_1^2 + x_2^2 \leq V_2 \\ \dot{V}_3|_{(3)} &= 4x_1(0.125x_1 - 2x_2) + 2x_2(4x_1 + 0.25x_2) \\ &= 0.5x_1^2 - 8x_1x_2 + 8x_1x_2 + 0.5x_2^2 \\ &= 0.5x_1^2 + 0.5x_2^2 = 0.5(x_1^2 + x_2^2) \leq 0.5V_3 \end{aligned}$$

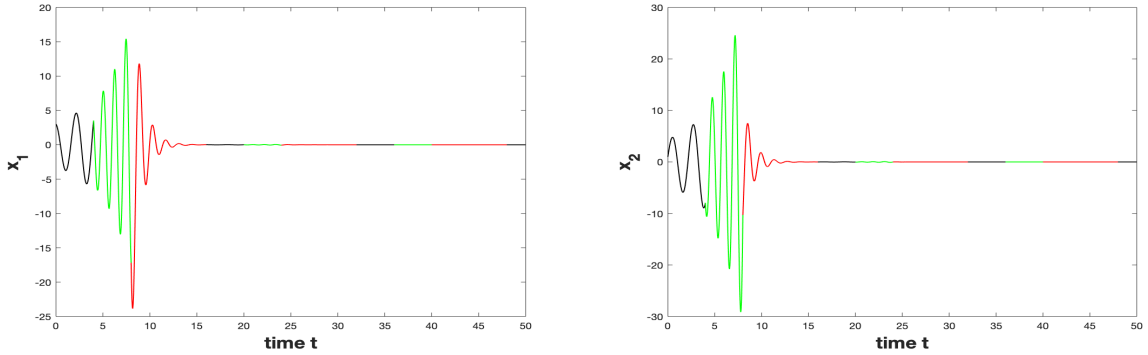
then,  $\lambda_1 = 2, \lambda_2 = 1$  and  $\lambda_3 = \frac{1}{2}$  and

$$\|x\|^2 \leq V_1(x) \leq 5\|x\|^2, \quad \|x\|^2 \leq V_2(x) \leq 3\|x\|^2, \quad \|x\|^2 \leq V_3(x) \leq 2\|x\|^2$$

where  $\|x\|^2 = x_1^2 + x_2^2$ , and then  $\mu_1 = 5, \mu_2 = 3, \mu_3 = 2$ . Since  $\lambda_2 + \lambda_3 < \lambda_1$  then

$$\tau_0 = \frac{\ln(5 \times 3 \times 2)}{2 - 1 - 0.5} = \frac{\ln(30)}{0.5} = 6.802394$$

The simulation is presented in Figure (1), where the initial condition is:  $[x_1(0), x_2(0)] = [3, 1]$ , and the switching signal is:

Figure 1: State response with MDT  $\tau_0 = 6.802394$ 

$$\sigma(t) = \begin{cases} 3, & t \in [t_{3m}, t_{3m+1}), t_{3m+1} - t_{3m} = 4 \\ 2, & t \in [t_{3m+1}, t_{3m+2}), t_{3m+2} - t_{3m+1} = 4 \\ 1, & t \in [t_{3m+2}, t_{3m+3}), t_{3m+3} - t_{3m+2} = 8 \quad \text{where } m = 0, 1, 2, \dots \end{cases}$$

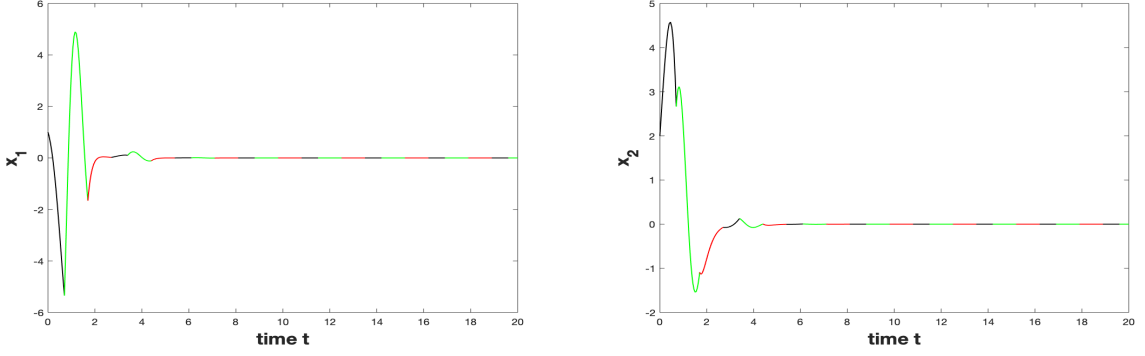
i.e., the 1<sup>st</sup> subsystem is activated for  $\tau_k = 8$  but the 2<sup>nd</sup> and the 3<sup>rd</sup> subsystems are activated for  $\tau_k = 4$ , then the switched system (16) is asymptotically stable for any switching signal with MDT  $\tau_0 = 6.802394$  as the stability degree of the stable subsystem is large enough to compensate the total instability effects of the other two subsystems.

**Example 4.2.** Consider the switched system (16) with  $i \in I = \{1, 2, 3\}$ , where

$$A_1 = \begin{pmatrix} -6 & -1 \\ 3 & -3 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 10 \\ -2 & -1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & -2 \\ 4 & 2 \end{pmatrix}$$

Since  $A_1, A_2$  are exponentially asymptotically stable,  $A_3$  is unstable, we use theorem (3.5) to find the MDT for system (16). Let  $V_1 = 3x_1^2 + x_2^2$  is a Lyapunov function for the 1<sup>st</sup>,  $V_2 = x_1^2 + 5x_2^2$  is a Lyapunov function for the 2<sup>nd</sup> subsystem, and  $V_3 = 2x_1^2 + x_2^2$  is a Lyapunov function for the 3<sup>rd</sup> subsystem. Next;

$$\begin{aligned} \dot{V}_1|_{(1)} &= 6x_1(-6x_1 - x_2) + 2x_2(3x_1 - 3x_2) \\ &= -36x_1^2 - 6x_1x_2 + 6x_1x_2 - 6x_2^2 \\ &= -36x_1^2 - 6x_2^2 = -6(6x_1^2 + x_2^2) \leq -6V_1 \end{aligned}$$

Figure 2: State response with MDT  $\tau_0 = 0.850299$ 

$$\begin{aligned}
 \dot{V}_2|_{(2)} &= 2x_1(-x_1 + 10x_2) + 10x_2(-2x_1 - x_2) \\
 &= -2x_1^2 + 20x_1x_2 - 20x_1x_2 - 10x_2^2 \\
 &= -2x_1^2 - 10x_2^2 = -2(x_1^2 + 5x_2^2) \leq -2V_2 \\
 \dot{V}_3|_{(3)} &= 4x_1(x_1 - 2x_2) + 2x_2(4x_1 + 2x_2) \\
 &= 4x_1^2 - 8x_1x_2 + 8x_1x_2 + 4x_2^2 \\
 &= 4x_1^2 + 4x_2^2 = 4(x_1^2 + x_2^2) \leq 4V_3
 \end{aligned}$$

then,  $\lambda_1 = 6$ ,  $\lambda_2 = 2$  and  $\lambda_3 = 4$  and

$$\|x\|^2 \leq V_1(x) \leq 3\|x\|^2, \quad \|x\|^2 \leq V_2(x) \leq 5\|x\|^2, \quad \|x\|^2 \leq V_3(x) \leq 2\|x\|^2$$

where  $\|x\|^2 = x_1^2 + x_2^2$ , and then  $\mu_1 = 3, \mu_2 = 5, \mu_3 = 2$ . Since  $\lambda_1 + \lambda_2 > \lambda_3$  then

$$\tau_0 = \frac{\ln(3 \times 5 \times 2)}{6 + 2 - 4} = \frac{\ln(30)}{4} = 0.850299$$

The simulation is presented in Figure (2), where the initial condition is:  $[x_1(0), x_2(0)] = [1, 2]$ , and the switching signal is:

$$\sigma(t) = \begin{cases} 3, & t \in [t_{3m}, t_{3m+1}), t_{3m+1} - t_{3m} = 0.70 \\ 2, & t \in [t_{3m+1}, t_{3m+2}), t_{3m+2} - t_{3m+1} = 1 \\ 1, & t \in [t_{3m+2}, t_{3m+3}), t_{3m+3} - t_{3m+2} = 1 \quad \text{where } m = 0, 1, 2, \dots \end{cases}$$

i.e., the 1<sup>st</sup> and the 2<sup>nd</sup> subsystems are activated for  $\tau_k = 1$  but the 3<sup>rd</sup> subsystem is activated for  $\tau_k = 0.70$  then the switched system (16) is asymptotically stable for any switching signal with MDT  $\tau_0 = 0.850299$

## 5 Conclusion

In this paper, we have investigated the stability of nonlinear switched systems for three subsystems, two of which are exponentially stable, and one is unstable or vice versa. we have used the multiple Lyapunov functions to obtain an estimate for the MDT, the stable subsystem should be activated longer than MDT,  $\tau_0$ , and the unstable mode should be activated less than this MDT. This condition which assures the exponential stability of switched nonlinear systems. Finally, we have provided an illustrative examples has been given.

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**Received: February 11, 2020; Published: February 23, 2020**