Nonlinear Analysis and Differential Equations, Vol. 7, 2019, no. 1, 69 - 79
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/nade.2019.9710

New Modified Caccioppoili-Tilli Inequalities

Tiziano Granucci

ISIS Leonardo da Vinci, Via del Terzolle 77, Firenze, Italy

This article is distributed under the Creative Commons by-nc-nd Attribution License. Copyright © 2019 Hikari Ltd.

Abstract

In this paper we introduce and demonstrate new modified Caccioppoli-Tilli variational inequalities. Caccioppoli's inequalities are a fundamental element of the theorems on the regularity of weak solutions of elliptic partial differential equations. Knowing new modified Caccioppoili-Tilli inequalities we could get new regularity results.

Mathematics Subject Classification: 49N60, 35J60

Keywords: Local-minimizer, Caccioppoli Inequalities

1 Introduction

In this paper we introduce and prove new modified Caccioppoli-Tilli variational inequalities. Caccioppoli's inequalities are a fundamental element for theorems on the regularity of weak solutions of elliptic partial differential equations, refer to [1, 2, 4, 5, 6, 7].

In his article [12] P. Tilli, using a modified Caccioppoli inequality, proves, in an alternative way, the famous theorem of De Giorgi - Nash - Moser, refer to [2, 10, 11].

Furthermore, the use of Caccioppoli-Tilli inequalities has allowed an alternative proof of the regularity of scalar p-arminic functions, refer to [8, 9]. In this paper we will demonstrate some modified Caccioppoli-Tilli inequalities concerning the modulus of the gradient and Hessian of weak solutions of systems of elliptic partial differential equations. Let $u \in W^{1,p}(\Omega, \mathbb{R}^m) \cap W^{2,2}_{loc}(\Omega, \mathbb{R}^m)$ we define

$$T = \sqrt{1 + |\nabla u|^2} \tag{1}$$

and

$$V_p = \max\{T^p - a^p t, 0\} = (T^p - a^p t)_+ \tag{2}$$

with $a > 1$ and $t > 0$, where $|\nabla u|^2 = \sum_{k=1}^{m} \sum_{i=1}^{n} \left(\partial_i u^k\right)^2$; moreover, with Hu we denote the Hessian matrix of the function u and we define $|Hu|^2 = \sum_{k=1}^{m} \sum_{i,j=1}^{n} \left(\partial_{i,j} u^k\right)^2$. We will study the variational inequalities, which we will call modified Caccioppoli-Tilli inequality and which hold for the solutions of the following PDEs system

$$0 = \sum_{i=1}^{n} \sum_{k=1}^{m} \int_{\Omega} \partial_s \left(T^{p-2} \partial_i u^k\right) \partial_i \psi^k \, dx \qquad \forall \psi \in C_c(\Omega, \mathbb{R}^m) \tag{3}$$

for every $s = 1, ..., n$.

Our main result is the following:

Theorem 1 *Let $u \in W^{1,p}(\Omega, \mathbb{R}^m) \cap W^{2,2}_{loc}(\Omega, \mathbb{R}^m)$, with $n \geq 2$, $m \geq 1$ and $p > 2$, be a weak solution of the PDEs system*

$$0 = \sum_{i=1}^{n} \sum_{k=1}^{m} \int_{\Omega} \partial_s \left(T^{p-2} \partial_i u^k\right) \partial_i \psi^k \, dx \qquad \forall \psi \in C_c(\Omega, \mathbb{R}^m)$$

for every $s = 1, ..., n$, where $T = \sqrt{1 + |\nabla u|^2}$. If $|\nabla u| \in L^{\infty}_{loc}(\Omega)$ then tre positive constant depending only on p and n denoted by C_1, C_2 and C_3 exist, such that for every $x_0 \in \Omega$ the following modified Caccioppoili-Tilli inequailities hold

$$\int_{B_\varrho(x_0)} T^{p-2} V_p^{\gamma} |Hu|^2 \, dx \leq C_1 \int_{\partial B_\varrho(x_0)} T^{p-1} |\nabla T| \, V_p^{\gamma} \, d\mathcal{H}^{n-1} \tag{4}$$

$$\int_{B_\varrho(x_0)} T^{p-2} V_p^{\gamma} |\nabla T|^2 \, dx \leq C_2 \int_{\partial B_\varrho(x_0)} T^{p-1} |\nabla T| \, V_p^{\gamma} \, d\mathcal{H}^{n-1} \tag{5}$$

$$\int_{B_{\varrho,at}} V_p^{\gamma-1} T^{2p-2} |\nabla T|^2 \, dx \leq C_3 \int_{\partial B_\varrho(x_0)} T^{p-1} |\nabla T| \, V_p^{\gamma} \, d\mathcal{H}^{n-1} \tag{6}$$

for every $\gamma \geq 1$ and $0 < \varrho < \frac{1}{2} dist(x_0, \partial\Omega)$.

To prove the previous Theorem 1 we will use the following Proposition

Proposition 2 *Let $g : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{L}^n$-summable. Then*

$$\int_{\mathbb{R}^n} g \, dx = \int_0^{+\infty} \left(\int_{\partial B_r(0)} g \, d\mathcal{H}^{n-1} \right) dr \tag{7}$$

In particular, we see

$$\frac{d}{dr}\left(\int_{B_r(0)} g\,dx\right) = \int_{\partial B_r(0)} g\,d\mathcal{H}^{n-1} \tag{8}$$

for $\mathcal{L}^1$ *a.e.* $r > 0$.

Proof. See Proposition 1 pag 118 of [3]. ■

Remark 3 *The minima of the vectorial function*

$$J(u,\Omega) = \int_\Omega T^p\,dx \tag{9}$$

are solutions of (3); therefore the Theorem 1 could be the first step for an alternative proof of the continuity of the gradient of the p-harmonic functions in the case $p > 2$.

2 Proof of Theorem 1

Since

$$\partial_s\left(T^{p-2}\partial_i u^k\right) = \partial_s\left(T^{p-2}\right)\partial_i u^k + T^{p-2}\,\partial_{is}u^k \tag{10}$$

we get

$$0 = \sum_{i=1}^{n}\sum_{k=1}^{m}\int_\Omega \left(\partial_s\left(T^{p-2}\right)\partial_i u^k + T^{p-2}\,\partial_{is}u^k\right)\partial_i\psi^k\,dx \tag{11}$$

for every $\psi \in C_c^\infty(\Omega,\mathbb{R}^m)$ and for every $s = 1,...,n$. We now that

$$\partial_s\left(T^{p-2}\right) = (p-2)\,T^{p-3}\partial_s T = (p-2)\,T^{p-4}\sum_{j=1}^{n}\sum_{f=1}^{m}\partial_j u^f\,\partial_{js}u^f \tag{12}$$

then, since $u \in W^{1,p}(\Omega,\mathbb{R}^m) \cap W^{2,2}_{loc}(\Omega,\mathbb{R}^m)$ and $|\nabla u| \in L^\infty_{loc}(\Omega)$, we deduce that $\partial_s\left(T^{p-2}\right)\partial_i u^k \in L^2_{loc}(\Omega)$ and $T^{p-2}\,\partial_{is}u^k \in L^2_{loc}(\Omega)$ for every $i,s = 1,...n$ and $k = 1,...,m$. Since $W_0^{1,2}(\Omega,\mathbb{R}^m)$ is dense in $C_c^\infty(\Omega,\mathbb{R}^m)$ it follows that

$$0 = \sum_{i=1}^{n}\sum_{k=1}^{m}\int_\Omega \left(\partial_s\left(T^{p-2}\right)\partial_i u^k + T^{p-2}\,\partial_{is}u^k\right)\partial_i\psi^k\,dx \qquad \forall\psi \in W_0^{1,2}(\Omega,\mathbb{R}^m) \tag{13}$$

for every $s = 1, ..., n$. Let us fix $x_0 \in \Omega$ and $0 < \tau < \varrho < R_0 = \min\left\{1, \frac{dist(x_0,\partial\Omega)}{2\sqrt{n}}\right\}$, then, choose $\psi^k = \eta^2 V_p^\gamma \partial_s u^k$, with $\gamma > 1$ and $\eta \in C_c^\infty(B_\varrho(x_0))$, $\eta = 1$ on $B_\tau(x_0)$, $0 \leq \eta \leq 1$ on $B_\varrho(x_0)$ and $|\nabla\eta| \leq \frac{\widetilde{C}}{\varrho-\tau}$ with $\widetilde{C} > 0$, since

$$\eta^2 V_p^\gamma \partial_s u^k \in W_0^{1,2}(\Omega, \mathbb{R}^m)$$

and

$$\partial_i \psi^k = 2\eta\, \partial_i \eta\, V_p^\gamma\, \partial_s u^k + \eta^2 \left[\gamma V_p^{\gamma-1} \partial_i (T^p)\, 1_{\{T^p > at\}}\, \partial_s u^k + (T^p - at)_+\, \partial_{is} u^k\right]$$

we get

$$2\Theta_{1,s}(B_\varrho(x_0)) + 2\Theta_{2,s}(B_\varrho(x_0)) + \Theta_{3,s}(B_\varrho(x_0)) + \Theta_{4,s}(B_\varrho(x_0)) + \Theta_{5,s}(B_\varrho(x_0)) + \Theta_{6,s}(B_\varrho(x_0)) = 0$$

for every $s = 1, ..., n$ where

$$\begin{aligned}
\Theta_{1,s}(B_\varrho(x_0)) &= \sum_{i=1}^{n} \sum_{k=1}^{m} \int_{B_\varrho(x_0)} \partial_s (T^{p-2})\, \partial_i u^k\, \eta\, \partial_i \eta\, V_p^\gamma\, \partial_s u^k\, dx \\
\Theta_{2,s}(B_\varrho(x_0)) &= \sum_{i=1}^{n} \sum_{k=1}^{m} \int_{B_\varrho(x_0)} T^{p-2}\, \partial_{is} u^k\, \eta\, \partial_i \eta\, V_p^\gamma\, \partial_s u^k\, dx \\
\Theta_{3,s}(B_\varrho(x_0)) &= \sum_{i=1}^{n} \sum_{k=1}^{m} \int_{B_\varrho(x_0)} \partial_s (T^{p-2})\, \partial_i u^k\, \eta^2\, \gamma V_p^{\gamma-1}\, \partial_i (T^p)\, 1_{\{T^p > at\}}\, \partial_s u^k\, dx \\
\Theta_{4,s}(B_\varrho(x_0)) &= \sum_{i=1}^{n} \sum_{k=1}^{m} \int_{B_\varrho(x_0)} \partial_s (T^{p-2})\, \partial_i u^k\, \eta^2\, V_p^\gamma\, \partial_{is} u^k\, dx \\
\Theta_{5,s}(B_\varrho(x_0)) &= \sum_{i=1}^{n} \sum_{k=1}^{m} \int_{B_\varrho(x_0)} T^{p-2}\, \partial_{is} u^k\, \eta^2\, \gamma V_p^{\gamma-1}\, \partial_i (T^p)\, 1_{\{T^p > at\}}\, \partial_s u^k\, dx \\
\Theta_{6,s}(B_\varrho(x_0)) &= \sum_{i=1}^{n} \sum_{k=1}^{m} \int_{B_\varrho(x_0)} T^{p-2}\, \partial_{is} u^k\, \eta^2\, V_p^\gamma\, \partial_{is} u^k\, dx
\end{aligned} \tag{14}$$

Summing on $s = 1, ..., n$ it follows

$$\Theta_1(B_\varrho(x_0)) + \Theta_2(B_\varrho(x_0)) + \Theta_3(B_\varrho(x_0)) + \Theta_4(B_\varrho(x_0)) + \Theta_5(B_\varrho(x_0)) + \Theta_6(B_\varrho(x_0)) = 0 \tag{15}$$

where

$$\begin{aligned}
\Theta_1\left(B_\varrho\left(x_0\right)\right) &= \sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\int_{B_\varrho(x_0)} \partial_s\left(T^{p-2}\right)\partial_i u^k\,\eta\,\partial_i\eta\,V_p^\gamma\,\partial_s u^k\,dx\\
\Theta_2\left(B_\varrho\left(x_0\right)\right) &= \sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\int_{B_\varrho(x_0)} T^{p-2}\,\partial_{is} u^k\,\eta\,\partial_i\eta\,V_p^\gamma\,\partial_s u^k\,dx\\
\Theta_3\left(B_\varrho\left(x_0\right)\right) &= \sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\int_{B_\varrho(x_0)} \partial_s\left(T^{p-2}\right)\partial_i u^k\,\eta^2\,\gamma V_p^{\gamma-1}\,\partial_i\left(T^p\right)\,1_{\{T^p>at\}}\,\partial_s u^k\,dx\\
\Theta_4\left(B_\varrho\left(x_0\right)\right) &= \sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\int_{B_\varrho(x_0)} \partial_s\left(T^{p-2}\right)\partial_i u^k\,\eta^2\,V_p^\gamma\,\partial_{is} u^k\,dx\\
\Theta_5\left(B_\varrho\left(x_0\right)\right) &= \sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\int_{B_\varrho(x_0)} T^{p-2}\,\partial_{is} u^k\,\eta^2\,\gamma V_p^{\gamma-1}\,\partial_i\left(T^p\right)\,1_{\{T^p>at\}}\,\partial_s u^k\,dx\\
\Theta_6\left(B_\varrho\left(x_0\right)\right) &= \sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\int_{B_\varrho(x_0)} T^{p-2}\,\partial_{is} u^k\,\eta^2\,V_p^\gamma\,\partial_{is} u^k\,dx
\end{aligned} \tag{16}$$

Now let us analyze some terms of the last equation. Let us start with the term $\Theta_6\left(B_\varrho\left(x_0\right)\right)$ that we can write

$$\begin{aligned}
\Theta_6\left(B_\varrho\left(x_0\right)\right) &= \sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\int_{B_\varrho(x_0)} \eta^2\,T^{p-2}\,V_p^\gamma\,\left(\partial_{is} u^k\right)^2\,dx\\
&= \int_{B_\varrho(x_0)} \eta^2\,T^{p-2}\,V_p^\gamma\,\sum_{k=1}^{m}\sum_{s=1}^{n}\sum_{i=1}^{n}\left(\partial_{is} u^k\right)^2\,dx\\
&= \int_{B_\varrho(x_0)} \eta^2\,T^{p-2}\,V_p^\gamma\,\left|Hu\right|^2\,dx
\end{aligned}$$

then

$$\Theta_6\left(B_\varrho\left(x_0\right)\right) \geq 0 \tag{17}$$

Now, let us consider the term $\Theta_4\left(B_\varrho\left(x_0\right)\right)$, that we can write

$$\Theta_4\left(B_\varrho\left(x_0\right)\right) = \sum_{s=1}^{n}\int_{B_\varrho(x_0)} \eta^2\,V_p^\gamma\,\partial_s\left(T^{p-2}\right)\,\sum_{i=1}^{n}\sum_{k=1}^{m}\partial_i u^k\,\partial_{is} u^k\,dx$$

since we know that the following relationships hold

$$\sum_{i=1}^{n}\sum_{k=1}^{m}\partial_i u^k\,\partial_{is} u^k = T\,\partial_s T$$

and

$$\partial_s\left(T^{p-2}\right) = (p-2)\,T^{p-3}\,\partial_s T$$

then we get

$$\begin{aligned}\Theta_4\left(B_\varrho\left(x_0\right)\right) &= (p-2)\sum_{s=1}^{n}\int\limits_{B_\varrho(x_0)} \eta^2\, V_p^\gamma\, T^{p-2}\, \left(\partial_s T\right)^2\, dx \\ &= (p-2)\int\limits_{B_\varrho(x_0)} \eta^2\, V_p^\gamma\, T^{p-2}\, \sum_{s=1}^{n}\left(\partial_s T\right)^2\, dx \\ &= (p-2)\int\limits_{B_\varrho(x_0)} \eta^2\, V_p^\gamma\, T^{p-2}\, \left|\nabla T\right|^2\, dx\end{aligned}$$

It follows that

$$\Theta_4\left(B_\varrho\left(x_0\right)\right) \geq 0 \tag{18}$$

Let us consider the term

$$\Theta_7\left(B_\varrho\left(x_0\right)\right) = \Theta_5\left(B_\varrho\left(x_0\right)\right) + \Theta_3\left(B_\varrho\left(x_0\right)\right)$$

since

$$\begin{aligned}\Theta_5\left(B_\varrho\left(x_0\right)\right) &= \sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\int\limits_{B_\varrho(x_0)} \eta^2\, T^{p-2}\, \gamma V_p^{\gamma-1}\, \partial_i\left(T^p\right)\, 1_{\{T^{p-2}>at\}}\, \partial_{is}u^k\, \partial_s u^k\, dx \\ &= \Theta_5\left(B_{\varrho,at}\right) \\ &= \sum_{i=1}^{n}\int\limits_{B_{\varrho,at}} \eta^2\, T^{p-2}\, \gamma V_p^{\gamma-1}\, \partial_i\left(T^p\right)\, \sum_{s=1}^{n}\sum_{k=1}^{m}\partial_{is}u^k\, \partial_s u^k\, dx \\ &= \sum_{i=1}^{n}\int\limits_{B_{\varrho,at}} \eta^2\, T^{p-2}\, \gamma V_p^{\gamma-1}\, \partial_i\left(T^p\right)\, T\, \partial_i T\, dx \\ &= p\sum_{i=1}^{n}\int\limits_{B_{\varrho,at}} \eta^2\, T^{p-1} T^{p-1}\, \gamma V_p^{\gamma-1}\, \partial_i T\, \partial_i T\, dx \\ &= p\sum_{i=1}^{n}\int\limits_{B_{\varrho,at}} \eta^2\, T^{2p-2}\, \gamma V_p^{\gamma-1}\, \left(\partial_i T\right)^2\, dx\end{aligned}$$

and

$$\begin{aligned}
\Theta_3\left(B_\varrho\left(x_0\right)\right) &= \sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\int\limits_{B_{\varrho,at}} \eta^2\,\partial_s\left(T^{p-2}\right)\,\partial_i u^k\,\gamma V_p^{\gamma-1}\,\partial_i\left(T^p\right)\,\partial_s u^k\,dx \\
&= \Theta_3\left(B_{\varrho,at}\right) \\
&= (p-2)\,p\sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\int\limits_{B_{\varrho,at}} \eta^2\,T^{p-3}\,\gamma V_p^{\gamma-1}\,\partial_s T\,\partial_i u^k\,T^{p-1}\,\partial_i T\,\partial_s u^k\,dx \\
&= (p-2)\,p\sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\int\limits_{B_{\varrho,at}} \eta^2\,T^{2p-4}\,\gamma V_p^{\gamma-1}\,\partial_s u^k\,\partial_i u^k\,\partial_i T\,\partial_s T\,dx \\
&= (p-2)\,p\sum_{k=1}^{m}\int\limits_{B_{\varrho,at}} \eta^2\,T^{2p-4}\,\gamma V_p^{\gamma-1}\,\sum_{s=1}^{n}\sum_{i=1}^{n}\partial_s u^k\,\partial_i u^k\,\partial_i T\,\partial_s T\,dx \\
&\geq (p-2)\,p\sum_{k=1}^{m}\int\limits_{B_{\varrho,at}} \eta^2\,T^{2p-4}\,\gamma V_p^{\gamma-1}\left(\sum_{i=1}^{n}\partial_i u^k\,\partial_i T\right)^2 dx \geq 0
\end{aligned}$$

hold, where $B_{\varrho,at} = B_\varrho\left(x_0\right)\cap\{T^p > at\}$, then we get

$$\begin{aligned}
\Theta_7\left(B_\varrho\left(x_0\right)\right) &= \Theta_7\left(B_{\varrho,at}\right) \\
&\geq p\sum_{i=1}^{n}\int\limits_{B_{\varrho,at}} \eta^2\,\gamma V_p^{\gamma-1}\,T^{2p-2}\,\left(\partial_i T\right)^2\,dx \\
&= p\int\limits_{B_{\varrho,at}} \eta^2\,\gamma V_p^{\gamma-1}\,T^{2p-2}\,\left|\nabla T\right|^2\,dx
\end{aligned}$$

and

$$\Theta_7\left(B_\varrho\left(x_0\right)\right) \geq p\int\limits_{B_{\varrho,at}} \eta^2\,\gamma V_p^{\gamma-1}\,T^{2p-2}\,\left|\nabla T\right|^2\,dx \geq 0 \tag{19}$$

Considering the relations (15), (17), (18) and (19) we have

$$\begin{aligned}
&0 \leq \Theta_7\left(B_\varrho\left(x_0\right)\right)+\Theta_4\left(B_\varrho\left(x_0\right)\right)+\Theta_6\left(B_\varrho\left(x_0\right)\right) = -2\Theta_1\left(B_\varrho\left(x_0\right)\right)-2\Theta_2\left(B_\varrho\left(x_0\right)\right) \\
&\leq 2\left|\Theta_1\left(B_\varrho\left(x_0\right)\right)\right|+2\left|\Theta_2\left(B_\varrho\left(x_0\right)\right)\right|
\end{aligned} \tag{20}$$

Now we have to estimate $\left|\Theta_1\left(B_\varrho\left(x_0\right)\right)\right|$ e $\left|\Theta_2\left(B_\varrho\left(x_0\right)\right)\right|$.

Let us start with the term $|\Theta_1(B_\varrho(x_0))|$:

$$\begin{aligned}
|\Theta_1(B_\varrho(x_0))| &= \left|\sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\int_{B_\varrho(x_0)} \partial_s\left(T^{p-2}\right)\,\partial_i u^k\,\eta\,\partial_i\eta\,V_p^\gamma\,\partial_s u^k\,dx\right| \\
&= (p-2)\left|\sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\int_{B_\varrho(x_0)} T^{p-3}\,\partial_s T\,\partial_i u^k\,\eta\,\partial_i\eta\,V_p^\gamma\,\partial_s u^k\,dx\right| \\
&\leq (p-2)\sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\int_{B_\varrho(x_0)} T^{p-3}\,|\partial_s T|\,\left|\partial_i u^k\right|\,\eta\,|\partial_i\eta|\,V_p^\gamma\,\left|\partial_s u^k\right|\,dx \\
&\leq (p-2)\sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\int_{B_\varrho(x_0)} T^{p-3}\,|\nabla T|\,\left|\partial_i u^k\right|\,\eta\,|\nabla\eta|\,V_p^\gamma\,\left|\partial_s u^k\right|\,dx \\
&\leq (p-2)\sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\int_{B_\varrho(x_0)} T^{p-3}\,|\nabla T|\,\eta\,|\nabla\eta|\,V_p^\gamma\,\frac{1}{2}\left(\left|\partial_s u^k\right|^2+\left|\partial_i u^k\right|^2\right)\,dx
\end{aligned}$$

and using Young Inequality it follows

$$\begin{aligned}
|\Theta_1(B_\varrho(x_0))| &\leq \frac{(p-2)}{2}\left[\sum_{i=1}^{n}\int_{B_\varrho(x_0)} T^{p-3}\,|\nabla T|\,\eta\,|\nabla\eta|\,V_p^\gamma\,\sum_{s=1}^{n}\sum_{k=1}^{m}\left|\partial_s u^k\right|^2\,dx\right. \\
&\qquad \left.+\sum_{s=1}^{n}\int_{B_\varrho(x_0)} T^{p-3}\,|\nabla T|\,\eta\,|\nabla\eta|\,V_p^\gamma\,\sum_{i=1}^{n}\sum_{k=1}^{m}\left|\partial_i u^k\right|^2\,dx\right] \\
&\leq n(p-2)\int_{B_\varrho(x_0)} T^{p-3}\,|\nabla T|\,\eta\,|\nabla\eta|\,V_p^\gamma\,|\nabla u|^2\,dx \\
&\leq n(p-2)\int_{B_\varrho(x_0)} T^{p-1}\,|\nabla T|\,\eta\,|\nabla\eta|\,V_p^\gamma\,dx
\end{aligned}$$

then

$$|\Theta_1(B_\varrho(x_0))| \leq n(p-2)\int_{B_\varrho(x_0)} T^{p-1}\,|\nabla T|\,\eta\,|\nabla\eta|\,V_p^\gamma\,dx \tag{21}$$

Now let us consider $|\Theta_2(B_\varrho(x_0))|$;

$$\begin{aligned}
|\Theta_2(B_\varrho(x_0))| &\leq \left| \sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m} \int_{B_\varrho(x_0)} T^{p-2}\,\partial_{is}u^k\,\eta\,\partial_i\eta\,V_p^\gamma\,\partial_s u^k\,dx \right| \\
&= \left| \sum_{i=1}^{n} \int_{B_\varrho(x_0)} T^{p-2} \left(\sum_{s=1}^{n}\sum_{k=1}^{m} \partial_{is}u^k\,\partial_s u^k \right) \eta\,\partial_i\eta\,V_p^\gamma\,dx \right| \\
&= \left| \sum_{i=1}^{n} \int_{B_\varrho(x_0)} T^{p-1} \left(\frac{\sum_{s=1}^{n}\sum_{k=1}^{m} \partial_{is}u^k\,\partial_s u^k}{T} \right) \eta\,\partial_i\eta\,V_p^\gamma\,dx \right| \\
&= \left| \sum_{i=1}^{n} \int_{B_\varrho(x_0)} T^{p-1}\,(\partial_i T)\,\eta\,\partial_i\eta\,V_p^\gamma\,dx \right| \\
&\leq \sum_{i=1}^{n} \int_{B_\varrho(x_0)} T^{p-1}\,|\partial_i T|\,\eta\,|\partial_i\eta|\,V_p^\gamma\,dx \\
&\leq n \int_{B_\varrho(x_0)} T^{p-1}\,|\nabla T|\,\eta\,|\nabla\eta|\,V_p^\gamma\,dx
\end{aligned}$$

then it follows

$$|\Theta_2(B_\varrho(x_0))| \leq n \int_{B_\varrho(x_0)} T^{p-1}\,|\nabla T|\,\eta\,|\nabla\eta|\,V_p^\gamma\,dx \tag{22}$$

Using the relations (17), (20), (21) and (22) we get

$$\int_{B_\varrho(x_0)} \eta^2\,T^{p-2}\,V_p^\gamma\,|Hu|^2\,dx \leq n(p-1) \int_{B_\varrho(x_0)} T^{p-1}\,|\nabla T|\,\eta\,|\nabla\eta|\,V_p^\gamma\,dx \tag{23}$$

From the properties of the function η we deduce that

$$\int_{B_\tau(x_0)} T^{p-2}\,V_p^\gamma\,|Hu|^2\,dx \leq n(p-1)\widetilde{C}\frac{1}{\varrho-\tau} \int_{B_\varrho(x_0)\setminus B_\tau(x_0)} T^{p-1}\,|\nabla T|\,\eta\,|\nabla\eta|\,V_p^\gamma\,dx$$

Applying Proposition 2, for $\tau \to \varrho$ we have the Modified Caccioppoli-Tilli Inequality

$$\int_{B_\varrho(x_0)} T^{p-2}\,V_p^\gamma\,|Hu|^2\,dx \leq C_1 \int_{\partial B_\varrho(x_0)} T^{p-1}\,|\nabla T|\,V_p^\gamma\,d\mathcal{H}^{n-1}$$

where $C_1 = n(p-1)\widetilde{C}$.

Using the relations (18), (20), (21) and (22) we get

$$\int_{B_\varrho(x_0)} \eta^2 \, T^{p-2} \, V_p^\gamma \, |\nabla T|^2 \, dx \leq \frac{2n(p-1)}{(p-2)} \int_{B_\varrho(x_0)} T^{p-1} \, |\nabla T| \, \eta \, |\nabla \eta| \, V_p^\gamma \, dx \quad (24)$$

From the properties of the function η we deduce that

$$\int_{B_\tau(x_0)} T^{p-2} \, V_p^\gamma \, |\nabla T|^2 \, dx \leq \frac{2n(p-1)}{(p-2)} \frac{\tilde{c}}{\varrho - \tau} \int_{B_\varrho(x_0) \setminus B_\tau(x_0)} T^{p-1} \, |\nabla T| \, V_p^\gamma \, dx$$

Applying Proposition 2, for $\tau \to \varrho$ we have the Modified Caccioppoli-Tilli Inequality

$$\int_{B_\varrho(x_0)} T^{p-2} \, V_p^\gamma \, |\nabla T|^2 \, dx \leq C_2 \int_{\partial B_\varrho(x_0)} T^{p-1} \, |\nabla T| \, V_p^\gamma \, d\mathcal{H}^{n-1}$$

where $C_2 = \frac{2n(p-1)}{(p-2)} \tilde{c}$.

Using the relations (19), (20), (21) and (22) we get

$$\int_{B_{\varrho,at}} \eta^2 \, \gamma V_p^{\gamma-1} \, T^{2p-2} \, |\nabla T|^2 \, dx \leq \frac{2n(p-1)}{p} \int_{B_\varrho(x_0)} T^{p-1} \, |\nabla T| \, \eta \, |\nabla \eta| \, V_p^\gamma \, dx \quad (25)$$

From the properties of the function η we deduce that

$$\int_{B_{\tau,at}} V_p^{\gamma-1} \, T^{2p-2} \, |\nabla T|^2 \, dx \leq \frac{2n(p-1)}{p\gamma} \frac{\tilde{c}}{\varrho - \tau} \int_{B_\varrho(x_0) \setminus B_\tau(x_0)} T^{p-1} \, |\nabla T| \, V_p^\gamma \, dx$$

Applying Proposition 2, for $\tau \to \varrho$ we have the Modified Caccioppoli-Tilli Inequality

$$\int_{B_{\varrho,at}} V_p^{\gamma-1} \, T^{2p-2} \, |\nabla T|^2 \, dx \leq C_3 \int_{\partial B_\varrho(x_0)} T^{p-1} \, |\nabla T| \, V_p^\gamma \, d\mathcal{H}^{n-1}$$

where $C_3 = \frac{2n(p-1)}{p\gamma} \tilde{c}$.

References

[1] L. Ambrosio, *Lecture Notes on Partial Differential Equations*, http://vcgmt.sns.it/ papers/ ambcarmas10

[2] E. De Giorgi, Sulla differenziabilità e l'analicità delle estremali degli integrali multipli, *Mem.Accad. Sci Torino*, cl. Sci. Fis. Mat. Nat., **3** (1957), 25-43.

[3] L. C. Evans, R. F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, London New York Washington D. C., 1992.

[4] M. Giaquinta, E. Giusti, On the regularity of minima of variational integrals, *Acta Math.*, **148** (1982), 31-46. https://doi.org/10.1007/bf02392725

[5] M. Giaquinta, E. Giusti, Quasi-minima, *Ann. Inst. H. Poincarè (Analyse non lineaire)*, **1** (1984), 79-107. https://doi.org/10.1016/s0294-1449(16)30429-2

[6] M. Giaquinta, L. Martinazzi, *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, S.N.S. Press, Pisa, 2005. https://doi.org/10.1007/978-88-7642-443-4

[7] E. Giusti, *Metodi diretti nel Calcolo delle Variazioni*, U. M. I., Bologna, 1994.

[8] T. Granucci, On the regularity of the scalar p-Harmonic functions in n dimentions with $1 < p < 2$, *Nonlinear Analysis and Differential Equations*, **3**, no. 1 (2015), 45-51. https://doi.org/10.12988/nade.2015.41022

[9] T. Granucci, On the regularity of the scalar p-Harmonic functions in n dimentions with $2 < p < n$, *Annali Università di Ferrara*, **56** (2010), 283-293. https://doi.org/10.1007/s11565-010-0100-6

[10] J. Moser, A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations, *Comm. Pure Appl. Math.*, **13** (1960), 457-468. https://doi.org/10.1002/cpa.3160130308

[11] J. Nash, Continuity of solutions of parabolic and elliptic differential equations, *Amer. Journal of Math.,* **80** (1958), 931-953. https://doi.org/10.2307/2372841

[12] P. Tilli, Remarks on the Hölder continuity of solutions to elliptic equations in divergence form, *Cal. Var.*, **25** (3) (2006), 395-401. https://doi.org/10.1007/s00526-005-0348-3

Received: August 1, 2019; Published: September 3, 2019