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Sobolev-Il'in Inequality for a Class of Generalized Shift Subadditive Operators

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Abstract

We study a problem of establishment of Sobolev-Il'in inequalities type strong and weak inequalities for subadditive operators with majorizing operators from certain class of Riesz potential type integral convolutions with almost monotone kernels, generated by both ordinary and generalized shift operators, associated with Laplace-Bessel differential operator.

Keywords: Sobolev-Il'in inequalities, subadditive operators, majorizing operators, Riesz potential, monotone kernels, generalized shift operators, Laplace-Bessel differential operator.

1 Introduction

Partial equation containing the Laplace-Bessel differential operator $\Delta_{B_{m+k,k}}$, using the Fourier-Bessel's many dimensional transforms was first studied in the papers of I.A. Kipriyanov (see [1]). For further investigations they introduced the weight spaces $L_{p,\nu}$.

Construction of fundamental solutions of B- elliptic equations was given in the paper of I.A. Kipriyanov and L.A. Ivanov [2], where it is proved that the

solution of the equation $\Delta_{B_{m+k,k}} u(x) = f(x)$ is a cylindrical potential type integral operator

$$u(x) \equiv I_B^{\alpha}(f)(x) = \int_{R^+_{m+k,k}} |y|^{\alpha-n-|\gamma_{k.m+k}|} T^y f(x)\, y^{\gamma_{k,m+k}} dy,$$

$$0 < \alpha < m + k + |\gamma_{k,m+k}|$$

(when $\alpha = 2$) called the Riesz potential, that contains the transformation T^y, in one-dimensional case introduced by B.M. Levitan [3] and called the generalized or Bessel shift operator. In [4], I.A. Kipriyanov in fact reduced the problem of obtaining a priori estimates to estimates of Riesz's generalized potentials and their appropriate derivatives.

Estimates of Hardy-Littlewood-Sobolev and Sobolev-Il'in type inequalities (generalizing the one-dimensional Hardy-Littlewood inequalities) for potential type integrals is one of the main elements of integral representations method developed first by S.L. Sobolev (see [5]). Hardy-Littlewood-Sobolev- type inequalities for the Riesz B-potential I_B^{α} in the scale of $L_{p,\nu}$ spaces were obtained in the paper of A.D. Gadjiev and I.A. Aliyev [6]. A special place in establishing Hardy-Littlewood-Sobolev and Sobolev-Il'in type estimates for integral operators of B-harmonic analysis in different metrics is occupied by the works of V.S. Guliyev and his followers (see [7], [8]).

For the first time, in the papers of S.K. Abdullayev and Z.A. Damirova [9], S.K. Abdullayev and B.K. Agarzayev [10], these estimated were extended for the case of the Riesz potential, with nonpower kernels of the form

$$I_B^{\omega}(f)(x) = \int_{R^+_{m+k,k}} T^y(f(x))\, \omega(|y|)\, |y|^{-(m+k+|v|)}\, d\mu(y)$$

in the case of ordinary and generalized shift T^y, respectively.

In the paper, for subadditive operators majorized with operators of certain class of integral convolutions of Riesz potentials I_B^{α}, with almost monotone kernels, we prove the validity of Sobolev-Il'in type estimates. Note that for this class of subadditive operators, the Hardy-Hittlewood-Sobolev type estimates were established in [11].

As is known, the Riesz-Bessel generalized potentials (even ordinary Riesz potentials, see Nakai [2]) with nonpower kernels don't act, generally speaking in the scale of $L_{p,\nu}$ spaces.

2 Some designation and preliminary information

Let R^l be Euclidean space of dimension $lm, k \geq 0$ be integers,

$$n = m + k \geq 1, \;\; p \geq 1,$$

$$R^+_{m+k,k} = \left\{(x_1, ..., x_{m+k}) \in R^{m+k} : x_{m+i} > 0, i = 1, ..., k\right\},$$

$$R^+_{m+0,0} \equiv R^m$$

$$-T^y_{\gamma_{n,k}}(u(x)) =$$

$$= c_\nu \int_0^\pi ... \int_0^\pi u\left(x' - y', (x_{m+1}, y_{m+1})_{\alpha_1}, ..., (x_{m+k}, y_{m+k})_{\alpha_k}\right) \times$$

$$\times \sin^{\gamma_{m+1}-1}\alpha_1 ... \sin^{\gamma_{m+k}-1}\alpha_k d\alpha_1 ... d\alpha_k$$

be a generalized shift operator generated by the Laplace-Bessel operator

$$\Delta_{B_{m+k,k}}(x) = \sum_{j=1}^m \frac{\partial^2}{\partial x_j^2} + \sum_{j=m+1}^{m+k}\left(\frac{\partial^2}{\partial x_j^2} + \frac{\gamma_j}{x_j}\frac{\partial}{\partial x_j}\right),$$

$$x \in R^+_{m+k,k}, \gamma_{m+1} > 0, ..., \gamma_{m+k} > 0,$$

where

$$x', y' \in R_m, \;\; x = (x, x_{m+1}, ..., x_{m+k}), \;\; y = (y, y_{m+1}, ..., y_{m+k}),$$

$$(x_{m+i}, y_{m+i})_{\alpha_i} = \sqrt{x^2_{m+i} - 2x_{m+i}y_{m+i}\cos\alpha_i + y^2_{m+i}}$$

$i = 1, ..., k, C_\nu$ is a normalizing factor.

Further we assume

$$\gamma_{n,k} = (0, ..., 0, \gamma_{m+1}, ..., \gamma_{m+k}) \in R^+_{m+k,k},$$

$$|\gamma_{n,k}| = \sum_{i=1}^k \gamma_{m+i}, a = n + |\gamma_{n,k}|,$$

$$y^{\gamma_{n,k}} = \prod_{i=1}^{m+k} y_i^{\gamma_i} = y_{m+1}^{\gamma_{m+1}} ... y_{m+k}^{\gamma_{m+k}}, \quad if \;\; y \in R^+_{m+k,k}.$$

In designation $\gamma_{n,k}$ the index n indicates the dimension of this vector, the index k the amount of its positive coordinates; $\gamma n, k = (0, ..., 0) \in R^m$ if $k = 0$.

$L^\Phi_{\gamma_{n,k}}\left(R^+_{n,k}\right)$ is Orlicz space [12] determined by the N-function Φ:

$$L^\Phi_{\gamma_{n,k}}\left(R^+_{n,k}\right) = \left\{f - izm. : \int_{R^+_{n,k}} \Phi(\varepsilon |f(x)|) d\mu_{\gamma_{n,k}}(y) < \infty, \; \varepsilon > 0\right\} =$$

$$\|f\|_{L^\Phi_\gamma\left(R^+_{n,k}\right)} = \inf\left\{\lambda > 0 : \int_{R^+_{n,k}} \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu_{\gamma_{n,k}}(y) \le 1\right\},$$

$$d\mu_{n,k}(y) = y^{\gamma_{n,k}} dy \equiv y_{m+1}^{\gamma_{m+1}} ... y_{m+k}^{\gamma_{m+k}} dy_1 ... dy_{m+k}.$$

The function $\Phi : [0,\infty) \to (0,\infty]$ is said to be N-function if there exists a non-decreasing left-continuous function $q : [0,\infty) \to (0,\infty]$ such that $q(0) = 0,\ q(t) \to \infty\ (t \to \infty)$ and

$$\Phi(r) = \int_0^r q(t)\, dt.$$

In the case $\Phi(t) = |t|^p$, $t > 0$ and $1 \le p < +\infty$ we denote the space $L^{\Phi}_{\gamma_{n,k}}\left(R^+_{n,k}\right)$ by $L_{p,\gamma_{n,k}}\left(R^+_{n,k}\right)$ -space of functions integrable in p-th degree with the weight $y^{\nu_{m+1}}_{m+1} ... y^{\nu_{m+k}}_{m+k}$:

$$L_{p,\gamma_{n,k}}\left(R^+_{n,k}\right) =$$

$$= \left[f - izm. : \|f\|_{L_{p,\gamma_{n,k}}\left(R^+_{n,k}\right)} = \left(\int_{R^+_{n,k}} |f(y)|^p \, d\mu_{\gamma_{n,k}}(y) \right)^{1/p} < +\infty \right].$$

Definition 1. The positive function $g(t)$ almost decreases (almost increases) on the set $X \subset (-\infty, +\infty)$ if there exists a constant such that for any $c_d > 0\ (c_l > 0)$, and also the relation $f \sim g\ (x \in X)$ means that there exists a constant $C > 0$ such that

$$C^{-1} f(x) \le g(x) \le C f(x),\ x \in X.$$

Let $\Omega_{p,\alpha}\left(\tilde{\Omega}_{p,\alpha}\right)\ (p \ge 1, \alpha > 0)$ be a union of the functions $\omega : (0,+\infty) \to (0,+\infty)$ such that $\omega(t)$ increases (almost increases), $t^{\{-\alpha/p\}+\varepsilon}\omega(t)$ decreases (almost decreases) for small $\varepsilon > 0$ and the integral $\int_0 t^{-1}\omega(t)\, dt$ converges.

Obviously, $\Omega_{p,\alpha} \subset \Omega_{1,\alpha}$ and $\tilde{\Omega}_{p,\alpha} \subset \tilde{\Omega}_{1,\alpha}$ and also if $\omega \in \tilde{\Omega}_{p,\alpha}$ then $\omega(2t) \le C\omega(t)$, where C is independent of $t \in (0,+\infty)$.

Definition 2. Let $p \ge 1, \alpha > 0$. It is said that the subadditive operator A belongs to the class $K\gamma_{n,k}\left(p, \tilde{\Omega}_{p,\alpha}\right)$ if

1. $Af(x)$ exists for almost all $x \in R^+_{n,k}$, when $L_{p,\gamma_{n,k}}\left(R^+_{n.k}\right)$ and

2. there exist $\omega \in \tilde{\Omega}_{p,\alpha}$ and $C > 0$ that for almost all $x \in R^+_{n,k}$,

$$|Af(x)| \le C \int_{R^+_{n+k}} T^y_{\gamma_{n,k}}(f(x))\, \omega(|y|)\, |y|^{-\alpha}\, d\mu_{\gamma_{n,k}}(y).$$

Denote $\alpha_{n,k} = n + |\gamma_{n,k}|$.

In the case when $\omega \in \tilde{\Omega}_{p,\alpha}$, it directly follows from the definition (see [12]) that the generalized Riesz potential

$$I^{\omega}_B(f)(x) = \int_{R^+_{n,k}} T^y_{\gamma_{n,k}}(f(x))\, \omega(|y|)\, |y|^{-\alpha}\, d\mu_{\gamma_{n,k}}(y),$$

the Bessel potential

$$\left(J_B^{\omega} f\right)(x) = \int_{R_{m+k,k}^{+}} T_{\gamma_{n,k}}^{y}\left(f(x)\right) G_{\gamma}^{\omega}(y)\, d\mu_{\gamma_{n,k}}(y),$$

$$G_{\gamma}^{\omega}(x) = c_{\gamma}^{\omega} \int_0^{+\infty} \frac{\omega\left(\delta^{1/2}\right)}{\delta^{\left(n+|\gamma_{n,k}|\right)/2}} e^{-\frac{\delta}{4\pi} - \frac{|x|^2 \pi}{\delta}} \frac{d\delta}{\delta},$$

and the generalized B-fractional-maximum function

$$M_{\gamma}^{\omega} f(x) = \sup_{r>0} \frac{\omega\left(|B(0,r)|_{\gamma_{n,k}}^{1/\alpha}\right)}{|B(0,r)|_{\gamma_{n,k}}} \int_{B(0,r)} T_{\gamma_{n,k}}^{y} |f(x)|\, d\mu_{\gamma_{n,k}}(y),$$

$$|B(0,r)|_{\gamma_{n,k}} = \int_{B(0,r)} d\mu_{\gamma_{n,k}}(y),$$

belong to the class $K_{\gamma_{n,k}}\left(p, \tilde{\Omega}_{p,\alpha_{n,k}}\right)$.

Note that when $\omega(t) = t^s$, $0 < s < \alpha_{n,k}$, I_B^{ω} is the Riesz potential of order s, J_B^{ω} is the Bessel potential of order s, while $M_{\gamma}^{\omega} f(x)$ is the B fractional maximum function $M_{\gamma}^{s} f(x)$.

3 Hardy-Littlewood-Sobolev- inequalities

The following theorem, where Hardy-Littlewood-Sobolev estimates are established for the operators from the class $K_{\gamma_{n,k}}\left(p, \tilde{\Omega}_{p,\alpha_{n,k}}\right)$, was proved in the paper [11].

Theorem 1. Let $1 \le p < +\infty$ and $A \in K_{\gamma_{n,k}}\left(p, \tilde{\Omega}_{p,\alpha_{n,k}}\right)$,$k \ge 0$. Then there exists the N -function Φ such that

$$\Phi^{-1}\left(r^{-a}\right) \sim r^{-a/p} \int_0^r \omega(t)\, t^{-1} dt \ (r > 0),$$

where Φ^{-1} is the inverse of Φ, ω is a function from definition 2 corresponding to the operator A, and
a) if $p > 1$, then $\exists C > 0$, $\forall f \in L_{p,\gamma_{n,k}}\left(R_{n,k}^{+}\right)$

$$\|Af\|_{L_{\nu}^{\Phi}\left(R_{n,k}^{+}\right)} \le C \|f\|_{L_{p,\gamma_{n,k}}\left(R_{n,k}^{+}\right)}$$

b) if $p = 1$, then $\exists C > 0$, $\forall f \in L_{1,\gamma_{n,k}}\left(R_{n,k}^{+}\right)$, $\forall \beta > 0$

$$\int_{\{x:|Af(x)|>2\beta\}} d\mu(x) \le \left\{\Phi\left[\left(\frac{c}{\beta}\|f\|_{L_{1,\gamma_{n,k}}}\right)^{-1}\right]\right\}^{-1}.$$

Remark 1. This theorem is exact for the generalized Riesz potential I_B^{ω} when $\omega \in \tilde{\Omega}_{p,\alpha}$ see [12]. We also note that this theforem covers the case of ordinary shift in all variables, if we put $k = 0$ that is necessary in establishing the Sobolev-Il'in inequalities in general case.

4 Sobolev-Il'in-type theorems

Let $n = m + k \geq 2$ and $s \in \{1, ..., m + k - 1\}$. Then we partition the space $R^+_{m+k,k}$ of the points $x = (x_1, ..., x_{m+k})$ into the direct sum of the space R^+_{s,k_s} of the points ${}_sx = (x_{n_1}, ..., x_{n_s})$ with coordinates $x_{n_1}, ..., x_{n_s}$ (where $k_s = rang\,(\{n_1, ..., n_s\} \bigcap \{m + 1, ..., m + k\})$ and $1 \leq n_1 < ... < n_s \leq m + k$) and the spaces $R^+_{n-s,(k-k_s)}$ of the point ${}_sx'$ such that $x =\uparrow ({}_sx', {}_s\,x') \in R^+_{n,k}$ (for denotations see [5]).

Note that at certain choice of the vector $\gamma_{n,k} = (0, ..., 0, \gamma_{m+1}, ..., \gamma_{m+k})$ even at one and the same value of the parameters s, m_s, k_s, the expansion $x =\uparrow ({}_sx, {}_s\,x')$ is determined nonuniquely. This circumstance does not influence of final result, but the in specific cases the coordinates $x_{n_1}, ..., x_{n_s}$ are fixed.

Assume

$$m_s = rang\left(\{n_1, ..., n_s\} \bigcap \{1, ..., m\}\right).$$

Then m_s, k_s are integers such that $0 \leq m_s \leq m,\ 0 \leq k_s \leq k$ and $m_s + k_s = s$. If $m_s > 0\ \ (k_s > 0)$, we assume

$$\{n_1, ..., n_s\} \bigcap \{1, ..., m\} = \{j_1, ..., j_{m_s}\},\ \ j_1 < ... < j_{m_s},$$

$$\{n_1, ..., n_s\} \bigcap \{m + 1, ..., m + k\} = \{m + i_1, ..., m + i_{k_s}\},$$

$$i_1 < ... < i_{k_s},$$

then obviously,

$${}_sy = \left(y_{j_1}, ..., y_{j_{m_s}}, y_{m+i_1}, ..., y_{m+i_{k_s}}\right)$$

and

$$d_sy = dy_{j_1}...dy_{j_{m_s}} dy_{m+i_1}...dy_{m+i_{k_s}}.$$

And also, if $k_s > 0$, then

$$\gamma_{s,k_s} = \left(0, ..., 0, \gamma_{i_1}, ..., \gamma_{i_{k_s}}\right) \in R^+_{s,k_s},\ \ |\gamma_{s,k_s}| = \gamma_{i_1} + ... + \gamma_{i_{k_s}},$$

$${}_sy^{\gamma_{s,k_s}} = y^{\gamma_{i_1}}_{m+i_1}...y^{\gamma_{i_{k_s}}}_{m+i_{k_s}}, d\mu_{s,k_s}\,({}_sy) =_s y^{\gamma_{s,k_s}} d_sy.$$

In these denotations we assume also

$$m^{'}_s = m - m_s,\ \ k^{'}_s = k - k_s,\ \ R^+_{s,k_s} \equiv R^+_{m_s+k_s,k_s},$$

$$R^+_{n-s,k-k_s} \equiv R^+_{m'_s+k'_s,k'_s} \equiv R^+_{n-s,k'_s}.$$

Further, ${}_sy',\ y_{n-s,k'_s},\ ({}_sy')^{\gamma_{n-s,k_s}}$ and $d\mu_{n-s,k'_s}\,({}_sy')$ are determined from the equalities

$$y =\uparrow ({}_sy, {}_s\,y')\,\gamma_{n,k} =\uparrow (\gamma_{s,k_s}, \gamma_{n-s,k_s}),$$

$$y^{\gamma_{n,k}} \equiv y^{\gamma_{m+1}}_{m+1}...y^{\gamma_{m+k}}_{m+k} =_s y^{\gamma_{s,k_s}}\,({}_sy')^{\gamma_{n-s,k'_s}}$$

and

$$d\mu_{n,k}(y) = d\mu_{s,k_s}({}_s y)\, d\mu_{n-s,k'_s}({}_s y'),$$

respectively.

And also if $m_s = 0$, then $\{j_1, ..., j_{m_s}\} = \emptyset,\ k_s = s$ and

$${}_s y = \left(y_{m+i_1}, ..., y_{m+i_{k_s}}\right),$$

$$d\mu_{s,k_s}({}_s y) = y_{m+i_1}^{\gamma_{i_1}} ... y_{m+i_{k_s}}^{\gamma_{i_{k_s}}} dy_{m+i_1} ... dy_{m+i_{k_s}} =_s y^{\gamma_{s,k_s}} d_s y.$$

Note that at certain choice of the parameter $\gamma_{n,k} = (0, ..., 0, \gamma_{m+1}, ..., \gamma_{m+k})$ even at one and the same value of the parameters s, m_s, k_s the expansion $x = \uparrow ({}_s x, {}_s x')$ is determined non-uniquely. This circumstance doesn't influence on final result, but in specific cases, the coordinates $x_{n_1}, ..., x_{n_s}$ are fixed.

Introduce the denotation (see the denotation $\alpha_{m+k,k}$)

$$a_{s,k_s} = s + |\gamma_{s,k_s}|,\ a_{n-s,k'_s} = n - s + \left|\gamma_{n-s,k'_s}\right|, \left(a_{s,k_s} + a_{n-s,k'_s} = \alpha_{m+k,k}\right),$$

$$\omega_{p,a_{s,k_s}}(t) = \omega(t)\, t^{-a_{s,k_s}/p}, \quad \omega_{p,a_{n-s,k'_s}}(t) = \omega(t)\, t^{-a_{n-s,k_s}/p},$$

and note some properties of the operator T^y that we will repeatedly use in the sequel:

$T1)\ T^y 1 = 1;\quad T2)\, T^y(Cf) = CT^y(y),\ \ C \in R;\quad T3)\ \ if\ \ |f| \le g$, then

$$T^y(|f|) \le T^y(|g|);\quad T4)\, (|T(f)|)^p \le T(|f|)^p;$$

$$T5)\ \left(\int_{R^+_{m+k,k}} (T^y(|f(x)|))^p\, d\mu(y)\right)^{1/p} \le \|f\|_{p,\nu};$$

$$T6)\, T^{({}_s y, {}_s y')}_{\gamma_{n,k}} f({}_s x, {}_s x') = T^{{}_s y}_{\gamma_s,k_s}\left(T^{{}_s y'}_{\gamma_{n-s},k_s} f({}_s x, {}_s x')\right) =$$

$$= T^{{}_s y'}_{\gamma_{n-s},k'_s}\left(T^{{}_s y}_{\gamma_s,k_s} f({}_s x, {}_s x')\right).$$

5 Main results.

Theorem 4 (main). Let $1 \le p < +\infty$, $k \ge 0$, $m \ge 0$, $n = m + k \ge 2$, $A \in K_{\gamma_{n,k}}\left(p, \tilde{\Omega}_{p,\alpha_{n,k}}\right)$ and ω an be appropriate function. If $s \in \{1, ..., m + k - 1\}$, $0 \le m_s \le m$, $0 \le k_s \le k$ $(k_s + m_s = s)$ and $\omega_{p,a_{n-s,k_s}} \in \tilde{\Omega}_{p,a_{s,k_s}}$, then there exists the N-function $\Phi = \Phi_{p,s}$ such that

$$\Phi^{-1}_{p,s}\left(r^{-a_{s,k_s}}\right) \sim r^{a_{s,k_s}/p} \int_0^r \omega_{p,a_{n-s,k_s}}(t)\, t^{-1} dt\ r > 0$$

and

a) if $p > 1$, then there exists $C > 0$ such that for any function

$$f \in L_{p,\gamma_{n,k}}\left(R^{+}_{m+k,k}\right)$$

and

$$\left\|(Af)\left(\cdot,_s x'\right)\right\|_{L^{\Phi_{p,s}}_{\gamma_{s,k_s}}\left(R^{+}_{s,k_s}\right)} \leq C\left\|f\right\|_{L_{p,\gamma_{n,k}}\left(R^{+}_{m+k,k}\right)},$$

b) there exists $C > 0$ such that for any function $f \in L_{1,\gamma_{n,k}}\left(R^{+}_{n,k}\right)$ for any $\beta > 0$ and ${}_s x' \in R^{+}_{n-s,k'_s}$,

$$\int_{\{x:|(Af)(\cdot,_s x)|>2\beta\}} d\mu\left(x\right) \leq \left\{\Phi_{1,s}\left[\left(\frac{c}{\beta}\left\|f\right\|_{L_{1,\gamma_{n,k}}\left(R^{+}_{n,k}\right)}\right)^{-1}\right]\right\}^{-1}.$$

Let $\omega : (0,+\infty) \to (0,+\infty)$.

Let introduce the subadditive operator $I^{\omega}_{\gamma_{n,k}} : f \to I^{\omega}_{\gamma_{n,k}}(f)$, where

$$I^{\omega}_{\gamma_{n,k}}(f)(x) = \int_{R^{+}_{n,k}} T^{y}_{\gamma_{n,k}}\left(\omega\left(x\right)|x|^{-\alpha_{n,k}}\right)|f\left(y\right)|\, d\mu_{\gamma_{n,k}}\left(y\right).$$

The following lemma is a starting point for proving the analoges of Sobolev's second theorem on potentials.

Lemma C. Let $k \geq 0$, $S \in \{1,...,m+k-1\}$, $0 \leq k_s \leq k$, $0 \leq m_s \leq m$, $k_s + m_s = s$, $1 \leq p < \infty$ and $\omega \in \tilde{\Omega}_{p,n+\left|\gamma_{n,k}\right|}$. Then there $c_1 > 0$ exists such that for any function $f \in L^{loc}_{1,\gamma_{n,k}}\left(R^{+}_{n,k}\right)$ and any $x = \uparrow\left({}_s x, {}_s x'\right) \in R^{+}_{n,k}$ the following inequality is valid

$$I^{\omega}_{\gamma_{n,k}}(f)(x) \leq C_1 I^{\overset{\omega_{p,a}}{}}_{\gamma_{s,k_s}{}_{n-s,k'_s}}\left(f_{p,s}\right)\left({}_s x\right),$$

where

$$f_{p,s}\left(t\right) = \left\|f\left(t,\cdot\right)\right\|_{L_{p,\gamma_{n-s,k'_s}}\left(R^{+}_{n-s,k'_s}\right)}, \quad t \in R^{+}_{s,k_s}$$

$$\left(I^{\omega_{p,a}{}_{n-s,k_s}}_{\gamma ls,k_s}\left(f_{p,s}\right)\left({}_s x\right) =\right.$$

$$\left. = \int_{R^{+}_{s,k_s}} T^{{}_s y}\frac{\omega_{p,a}{}_{n-s,k'_s}\left(\left|{}_s x\right|\right)}{\left|{}_s x\right|^{s+\left|y_{s,k_s}\right|}} \cdot \left\|f\left({}_s y,\cdot\right)\right\|_{L_{p,\gamma_{n-s,k_s}}\left(R^{+}_{n-s,k_s}\right)} d\mu_{k_s,s}\left(y\right)\right).$$

Proof Let $B\left(t\right) = \omega\left(t\right) t^{-\alpha_{n,k}}$. Denote

$$F\left(x,y\right) = \int_{R^{+}_{n-s,k'_s}} T^{y}_{\gamma_{n,k}}\left(B\left(|x|\right)\right)|f\left({}_s y,{}_s y'\right)|\, d\mu_{n-s,k'_s}\left({}_s y'\right). \tag{1}$$

Then taking into account the equality $d\mu_{nk}(y) = d\mu_{s,k_s}({}_s y)\, d\mu_{n-s,k'_s}({}_s y')$, by using the Foubini theorem, we have

$$I^{\omega}_{\gamma_{n,k}}(f)(x) = \int_{R^+_{n,k}} T^y_{\gamma_{n,k}}(B(|x|))\,|f(y)|\, d\mu_{\gamma_{n,k}}(y) =$$

$$= \int_{R^+_{s,k_s}} \left(\int_{R^+_{n-s,k'_s}} T^y_{\gamma_{n,k}}(B(|x|))\,|f({}_s y,{}_s y')|\, d\mu_{n-s,k'_s}({}_s y') \right) d\mu_{s,k_s}({}_s y).$$

Thus,

$$I^{\omega}_{\gamma_{n,k}}(f)(x) = \int_{R^+_{n,k}} T^y(B(|x|)) \cdot |f(y)|\, d\mu_{n,k}(y) =$$

$$= \int_{R^+_{s,k_s}} (F(x,{}_s y))\, d\mu_{s,k_s}({}_s y). \tag{2}$$

Let $p = 1$.

Taking into account the properties $T1) - T6)$ of the generalized shift operator T^y and almost decrease of $B(t) = \omega(t)\, t^{-\left(n+|\gamma_{n,k}|\right)}$, we have

$$T^y_{\gamma_{n,k}}(B(|x|)) = T^{({}_s y,{}_s y')}_{\gamma_{n,k}}(B(|({}_s x,{}_s x')|)) =$$

$$= T^{{}_s y'}_{\gamma_{n-s,k'_s}} \left(T^{{}_s y}_{\gamma_{n-s,k'_s}} B(|({}_s x,{}_s x')|) \right) \le$$

$$\le T^{{}_s y'}_{\gamma_{n-s,k'_s}} \left(T^{{}_s y}_{\gamma_{s,k_s}} (CB(|{}_s x|)) \right) = C T^{{}_s y}_{\gamma_{s,k_s}} (B(|{}_s x|)).$$

Whence, allowing for

$$B(t) = \omega(t)\, t^{-\alpha_{m+k,k}} = \omega(t)\, t^{-\alpha_{n-s,k'_s}} t^{-a_{s,k_s}} = \omega_{p,a_{n-s,k'_s}}(t)\, t^{-a_{s,k_s}},$$

we get

$$I^{\omega}_{\gamma_{n,k}}(f)(x) = \int_{R^+_{n,k}} T^y_{\gamma_{n,k}}(B(|x|))\,|f(y)|\, y^{\gamma_{n,k}} dy \le$$

$$\le C \int_{R^+_{s,k_s}} T^{s,k}_{\gamma_{s,k_s}}(B(|{}_s x|)) \left(\int_{R^+_{n-s,k'_s}} |f({}_s y,{}_s y')|\, d\mu_{n-s,k'_s}({}_s y') \right) d\mu_{s,k_s}(y) =$$

$$= \int_{R^+_{s,k_s}} T_{\gamma_{s,k_s}}(B(|{}_s x|))\, \|f({}_s y,\cdot)\|_{L_{1\gamma_{n-s,k'_s}}}\, d\mu_{s,k_s}({}_s y) = I^{\omega_{p,\alpha_{s,k_s}}}_{\gamma ls}(f_{1,s})({}_s x).$$

By this, in the case $p = 1$ we proved lemma C.

Let $p > 1$.

Using (1), having applied the Holder inequality, we get

$$F(x,{}_s y) \le \left\| T^y_{\gamma_{n,k}}(B(|x|)) \right\|_{L_{p',\gamma_{n-s,k'_s}}(R^+_{n-s,k_s})} \|f({}_s y,\cdot)\|_{L_{p,\gamma_{s,k_s}}(R^+_{s,k_s})}. \tag{3}$$

Estimate from above $\left\|T^{y}_{\gamma_{n,k}}\left(B\left(|x|\right)\right)\right\|_{L_{p',\gamma_{n-s,k'_s}}\left(R^{+}_{n-s,k_s}\right)}$.

Allowing for the properties $T1)-T6)$ of the generalized shift operator T^{y}, we get

$$T^{y}_{\gamma_{n,k}}\left(B\left(|x|\right)\right)\equiv T^{({}_sy,{}_sy')}_{\gamma_{n,k}}\left(B\left(|({}_sx,{}_s\,x')|\right)\right)=$$

$$=T^{{}_sy}_{\gamma_{s,k_s}}\left(T^{{}_sy}_{\gamma_{n-s,k'_s}}\left(B\left(|({}_sx,{}_s\,x')|\right)\right)\right)=$$

$$=\int_0^{\pi}...\int_0^{\pi}T^{{}_sy'}_{\gamma_{n-s,k'_s}}\left(B\left(|({}_s\tilde{y},{}_s\,x')|\right)\right)\prod_{l=1}^{k_s}\sin g\gamma_{ij-1}\alpha_{i_l}d\alpha_{i_l},$$

where

$${}_s\bar{y}=\left(x_{j_1}-y_{j_1},...,x_{j_{m_s}}-y_{j_{m_s}},\left(x_{m+i_1},y_{m+i_1}\right)_{\alpha_{i_1}},...,\left(x_{m_i_{k_s}},y_{m_{+i_{k_s}}}\right)_{\alpha_{i_{k_s}}}\right),$$

$$\left(x_{m+i},s_{m+i}\right)_{\alpha_i}=\sqrt{x^2_{m+i}-2x_{m+i}s_{m+i}\cos\alpha_i+s^2_{m+i}},\left|({}_s\tilde{y},x')\right|=$$

$$=\left(\left|{}_s\tilde{y}\right|^2+\left|{}_sx'\right|^2\right)^{1/2},$$

$$\left|{}_s\tilde{y}\right|=\left(\sum_{i=1}^{m_s}\left|x_{j_i}-y_{j_i}\right|^2+\sum_{l=1}^{k_s}\left(x^2_{m+i_1}-2x_{m+i_l}y_{m+i_l}\cos\alpha_{i_l}+y^2_{m+i_l}\right)\right)^{1/2}.$$

Then

$$\left\|T^{y}_{\gamma_{n,k}}\left(B\left(|x|\right)\right)\right\|_{L_{p',\gamma_{n-s,k'_s}}}=$$

$$=\left(\int_{R^{+}_{n-s,k'_s}}\left(\int_0^{\pi}...\int_0^{\pi}T^{{}_sy'}_{\gamma_{n-s,k'_s}}\left(B\left(|({}_s\tilde{y},{}_s\,x')|\right)\right)\prod_{l=1}^{k_s}\sin^{\gamma_{ij-1}}\alpha_{i_l}d\alpha_{i_l}\right)^{p'}d\mu_{n-s,k'_s}\left({}_sy'\right)\right)^{1/p'}.$$

Having applied the Minkovsky inequality, from the last one we get

$$\left\|T^{y}_{\gamma_{n,k}}\left(B\left(|x|\right)\right)\right\|_{L_{p',\gamma_{n-s,k'_s}}\left(R^{+}_{n-s,k'_s}\right)}\leq C\int_0^{\pi}...\int_0^{\pi}J\prod_{l=1}^{k_s}\sin^{\gamma_{il-1}}\alpha_{il}d\alpha_{il},$$

$$J=\left[\int_{R^{+}_{n-s,k'_s}}\left(T^{{}_sy'}_{\gamma_{n-s,k'_s}}B\left(|({}_s\tilde{y},{}_s\,x')|\right)\right)^{p'}\left({}_sy'\right)^{\gamma_{n-s,k'_s}}d_sy'\right]^{1/p'}. \tag{4}$$

Taking into account the property $T5)$, estimate J

$$J=\left[\int_{R^{+}_{n-s,k'_s}}\left(T^{{}_sy'}_{\gamma_{n-s,k'_s}}\left(B\left(|({}_s\tilde{y},{}_s\,x')|\right)\right)\right)^{p'}\left({}_sy'\right)^{\gamma_{n-s,k'_s}}d_sy'\right]^{1/p'}\leq$$

$$\leq \left[\int_{R^+_{n-s,k'_s}} \left(B\left(\left|\left({}_s\tilde{y},{}_s y'\right)\right|\right)\right)^{p'} \left({}_s y'\right)^{\gamma_{n-s,k'_s}} d_s y'\right]^{1/p'} =$$

$$= \left(\int_{R^+_{n-s,k'_s}} \left[\frac{\omega\left(\left|\left({}_s\tilde{y},{}_s y'\right)\right|\right)}{\left(\left|\left({}_s\tilde{y},{}_s y'\right)\right|\right)^{\frac{n+\left|\gamma_{n,k}\right|}{p}-\varepsilon}} \cdot \frac{1}{\left(\left|\left({}_s\tilde{y},{}_s y'\right)\right|\right)^{\frac{n+y_{n,k}}{p'}+\varepsilon}}\right]^p \left({}_s y'\right)^{\gamma_{n-s,k'_s}} d_s y'\right)^{\frac{1}{p'}}.$$

Hence, allowing for the condition $\omega \in \tilde{\Omega}_{p,n+\left|\gamma_{k,n}\right|}$ and

$$\left|\left({}_s\tilde{y},{}_s y'\right)\right| = \left|\left({}_s\tilde{y},{}_s y'\right)\right| = \left(\left|{}_s\tilde{y}\right|^2 + \left|{}_s y'\right|^2\right)^{1/2} \geq \left|{}_s\tilde{y}\right|,$$

we get

$$J \leq C \frac{\omega\left(\left|{}_s\tilde{y}\right|\right)}{\left|{}_s\tilde{y}\right|^{\frac{n+\left|y_{k,n}\right|}{p}-\varepsilon}} D$$

$$D = \left(\int_{R^+_{n-s,k'_s}} \left(\left|\left({}_s\tilde{y},{}_s y'\right)\right|\right)^{-\left(\frac{n+\gamma_{k,n}}{p'}+\varepsilon\right)p'} \left({}_s y'\right)^{\gamma_{n-s,k'_s}} d_s y'\right)^{\frac{1}{p'}}. \tag{5}$$

Estimate D

$$D = \left(\int_{R^+_{n-s,k'}} \left(\left[\left|{}_s\tilde{y}\right| + \left|{}_s y'\right|^2\right]^{\frac{1}{2}}\right)^{-\left[+\left|\gamma_{n,k}\right|+\varepsilon p'\right]} \left({}_s y'\right)^{\gamma_{n-s,k'_s}} d_s y'\right)^{\frac{1}{p'}} \leq$$

$$\leq C \left[\int_{R^+_{n-s,k'_s}} \left(1 + \frac{\left|{}_s y'\right|}{\left|{}_s\tilde{y}\right|}\right)^{-\left(n+\left|\gamma_{n,k}\right|\right)} \left({}_s y'\right)^{\gamma_{n-s,k'_s}} d_s y'\right]^{\frac{1}{p'}} \times$$

$$\times \left|{}_s\tilde{y}\right|^{\frac{-\left[n+\left|\gamma_{n,k}\right|+\varepsilon p'\right]}{p'}} = C D_1 \left|{}_s y'\right|^{\frac{-\left[n+\left|\gamma_{n,k}\right|+\varepsilon p'\right]}{p'}},$$

$$D_1 = \left[\int_{R^+_{n-s,k'_s}} \left(1 + \frac{\left|{}_s y'\right|}{\left|{}_s\tilde{y}\right|}\right)^{-\left(n+\left|\gamma_{n,k}\right|\right)} \left({}_s y'\right)^{\gamma_{n-s,k'_s}} d_s y'\right]^{\frac{1}{p'}}. \tag{6}$$

Estimate D_1 from upper .
Making change of variables $\frac{{}_s y'}{\left|{}_s\tilde{y}\right|} =_s z' \left({}_s y' \to_s z'\right)$, we have

$$\left({}_s y'\right)^{\gamma_{k'_s,n-s}} d_s y' = \left|{}_s\tilde{y}\right|^{\left|\gamma_{n-s,k'_s}\right|} \left|{}_s\tilde{y}\right|^{n-s} \left({}_s z'\right)^{\gamma_{n-s,k'_s}} d_s z'.$$

Whence

$$D_1 = \left|{}_s\tilde{y}\right|^{\frac{n-s+\left|\gamma_{n-s,k'_s}\right|}{p'}} D_{11},$$

$$D_{11}=\left(\int\limits_{R^{+}_{n-s,k'_s}}(1+|{}_sz'|)^{-\left(n+|\gamma_{n,k}|\right)}\,({}_sz')^{\gamma_{n-s,k_s}}\,d_sz'\right)^{\frac{1}{p}}. \tag{7}$$

Passing to spherical coordinates ${}_sz'=t\theta$, we get

$$D_{11}=\left(\int_s ds\int_0^{\infty}(1+t)^{-\left(n+|\gamma_{n,k}|\right)}\,t^{\left|\gamma_{n-s,k'_s}\right|+[(n-s)-1]}dt\right)^{\frac{1}{p'}}$$

and this integral converges, as the function

$$(1+t)^{-\left(n+|\gamma_{n,k}|\right)}\,t^{\left|\gamma_{n-s,k'_s}\right|+[(n-s)-1]}$$

is of order $\left|\gamma_{n-s,k'_s}\right|+[(n-s)-1]>0$ as $t\to 0$ and $-\left(n+|\gamma_{n,k}|\right)+\left|\gamma_{n-s,k'_s}\right|+[(n-s)-1]\le -(s+1)\le -2$ as $t\to\infty$.
Thus, taking sequentially into account (7), (6) and (5), we get

$$D_1\le C\,|{}_s\tilde{y}|^{\frac{n-s+\left|\gamma_{n-s,k'_s}\right|}{p'}},$$

$$D\le CD_1\,|{}_s\tilde{y}|^{\frac{-\left[n-s+\left|\gamma_{n,k}\right|+\varepsilon p'\right]}{p'}}\le C\,|{}_s\tilde{y}|^{\frac{-n-s+\left|\gamma_{n-s,k'_s}\right|}{p'}}\,|{}_s\tilde{y}|^{\frac{-\left[n+\left|\gamma_{n,k}\right|+\varepsilon p'\right]}{p'}}=$$
$$=|{}_s\tilde{y}|^{\frac{-\left[s+\left|\gamma_{n,k_s}\right|+\varepsilon p'\right]}{p'}}=|{}_s\tilde{y}|^{\frac{-\left[s+\left|\gamma_{s,k_s}\right|\right]}{p'}-\varepsilon},$$

whence

$$J\le C\frac{\omega\left(|{}_s\tilde{y}|\right)}{|{}_s\tilde{y}|^{\frac{n+\left|\gamma_{k,n}\right|}{p}-\varepsilon}}D=C\frac{\omega\left(|{}_s\tilde{y}|\right)}{|{}_s\tilde{y}|^{\frac{n+\left|\gamma_{k,n}\right|}{p}-\varepsilon}}\,|{}_s\tilde{y}|^{\frac{-\left[s+\left|\gamma_{s,k_s}\right|\right]}{p'}-\varepsilon}=$$

$$=C\frac{\omega\left(|{}_s\tilde{y}|\right)}{|{}_s\tilde{y}|^{\frac{n+\left|\gamma_{k,n}\right|}{p}+\frac{s+\left|\gamma_{s,k_s}\right|}{p'}}}=$$

$$=C\frac{\omega\left(|{}_s\tilde{y}|\right)}{|{}_s\tilde{y}|^{\frac{h-s+\left|\gamma_{n-s,k'_s}\right|}{p}}}\frac{1}{|{}_s\tilde{y}|^{s+\left|\gamma_{s,k_s}\right|}}=C\frac{\omega_{p,a_{n-s,k'_s}}\left(|{}_s\tilde{y}|\right)}{|{}_s\tilde{y}|^{s+\left|\gamma_{s,k_s}\right|}}.$$

In the last passage we take into account

$$\frac{n+|\gamma_{n,k}|}{p}+\frac{s+|\gamma_{s,k_s}|}{p'}=$$

$$=\frac{n-s+\left|\gamma_{n-s,k'_s}\right|+s-\left|\gamma_{n-s,k'_s}\right|+|\gamma_{n,k}|}{p}+\frac{s+|\gamma_{s,k_s}|}{p'}=$$

$$= \frac{n-s+\left|\gamma_{n-s,k_s'}\right|}{p} + \frac{s+\left|\gamma_{s,k_s}\right|}{p'} + \frac{s+\left|\gamma_{s,k_s'}\right|}{p'} =$$

$$= \frac{n-s+\left|\gamma_{n-s,k_s'}\right|}{p} + \left(s+\left|\gamma_{s,k_s}\right|\right).$$

Substituting the obtained representation in (4), we have

$$\left\|T^{y}_{\gamma_{n,k}}\left(B\left(|x|\right)\right)\right\|_{L_{p',\gamma_{n-s,k_s'}}\left(R^{+}_{n-s,k_s'}\right)} \leq C\int_0^{\pi}...\int_0^{\pi} J\prod_{l=1}^{k_s}\sin^{\gamma_{il-1}}\alpha_{il}d\alpha_{il},$$

$$\leq C\int_0^{\pi}...\int_0^{\pi}\frac{\omega\left(|{}_s\tilde{y}|\right)}{|{}_s\tilde{y}|^{\beta}}\cdot\prod_{i=1}^{k_s}\left(\sin^{\gamma_{m_s+i-1}}\alpha_i d\alpha_i\right) = CT^{s^y}_{s^x}\left(\frac{\omega_s\left(|{}_sx|\right)}{|{}_sx|^{s+|\gamma_{s,k_s}|}}\right).$$

We take into account the last estimation in (3) and get:

$$F\left(x,{}_s\,y\right) \leq CT^{{}_sy}_{\gamma_{s,k_s}}\left(\frac{\omega_s\left(|{}_sx|\right)}{|{}_sx|^{s+|\gamma_{s,k_s}|}}\right)\|f\left({}_sy,\cdot\right)\|_{L_{p,\gamma_s k_s}}\left(R^{+}_{s,k_s}\right).$$

Taking this into account in (2), we easily get:

$$I^{\omega}_{\gamma_{n,k}}\left(f\right)\left(x\right) \leq$$

$$\leq C\int_{R^{+}_{s,k_s}} T^{{}_sy}_{\gamma_{s,k_s}}\left(\frac{\omega_s\left(|{}_sx|\right)}{|{}_sx|^{s+|\gamma_{s,k_s}|}}\right)\|f\left({}_sy,\cdot\right)\|_{L_{p,\gamma_{s,k_s}}\left(R^{+}_{s,k_s}\right)}d\mu_{s,k_s}\left({}_sy\right) =$$

$$= CI^{\omega_{p,a_{n-s},k_s}}_{\gamma_{s,k_s}}\left(f_{p,s}\right)\left({}_sx\right).$$

Thus, the inequality

$$I^{\omega}_{\gamma_{n,k}}\left(f\right)\left(x\right) \leq C_1 I^{\omega_{p,a_{n-s},k_s'}}_{\gamma_{s,k_s}}\left(f_{p,s}\right)\left({}_sx\right)$$

and Lemma C are proved in the case if $p>1$, as well.

Theorem 2 directly follows from Lemma C. By applying. Theorem 1 and that from the relation $\omega_{p,\alpha_{s'}} \in \tilde{\Omega}_{p,\alpha_s}$ it follows $\omega \in \tilde{\Omega}_{p,\alpha_{m+k,k}}$.

Note that if $\omega = t^{\alpha}$, then all the results of the paper [8] (Theorem 2 and Theorem 3) belonging to relating to Sobolev-Il'in estimation for the Riesz potentials

$$I^{\omega}_{B}\left(f\right)\left(x\right) = \int_{R^{+}_{m+k,k}}|y|^{\alpha-n-|\gamma_{k,m+k}|}\,T^{y}_{\gamma_{k,m+k}}f\left(x\right)y^{y_{k,m+k}}dy,$$

$$0<\alpha<m+k+\left|\gamma_{k,m+k}\right|,$$

where the case $0<m_s<m$, $0<k_s<k$ was not considered at all, follows from Theorem 2.

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