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Teaching about Schimidt's Orthogonalization

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Abstract

Schimidt's orthogonalization is very important in linear algebra and has many applications. In this paper, we introduce Schimidt's orthogonalization from three aspects: background, step, application. And we describe the process of Schimidt's orthogonalization by elementary transformation of matrices.

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1 Introduction

In analytic geometry, we often choose Cartesian rectangular coordinate system, since the coordinates of a vector are exactly the projection on axis of coordinates and the length of a vector and expressions of some curves are particularly simple under it. But in an skew coordinate system, everything above becomes complicated, we are not familiar with the coordinates of a vector. So Cartesian rectangular coordinate system is very important in the study of metric space. As an extension, firstly, we introduce the concept of orthonormal basis.

Definition 1[1−2] Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be a basis of the $n-$dimensional vector space V, which is called a orthonormal basis if $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are orthogonal and unit vectors.

Example 1

$$\varepsilon_1 = (1,0,0)^T, \varepsilon_2 = (0,1,0)^T, \varepsilon_3 = (0,0,1)^T;$$

$$\varepsilon_1 = \frac{1}{\sqrt{3}}(1,1,1)^T, \varepsilon_2 = \frac{1}{\sqrt{2}}(1,0,-1)^T, \varepsilon_3 = \frac{1}{\sqrt{6}}(1,-2,1)^T.$$

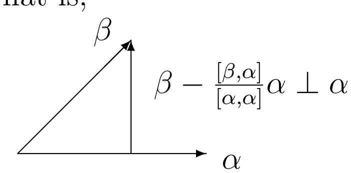
Orthonormal basis has outstanding advantages in dealing with problems. The expressions of coordinates and inner product for vectors are very simple under a orthonormal basis,

$$\alpha = \sum_{i=1}^{n}[\alpha, \varepsilon_i]\varepsilon_i, \quad [\alpha, \beta] = \sum_{i=1}^{n}[\alpha, \varepsilon_i][\beta, \varepsilon_i].$$

Lemma 1[1−2] Orthogonal vectors must be linearly independent.

However, linear independent vectors are not necessarily orthogonal. How to transform them into orthogonal vectors? In the following, the concept of projective vector is given.

Definition 2 Let nonzero vectors α, β be in vector space V, $\frac{[\beta,\alpha]}{\alpha,\alpha}\alpha$ is called a projective vector of β on α. That is,



For three vectors α, β, γ, set

$$\eta_1 = \alpha, \quad \eta_2 = \beta - \frac{[\beta_1, \eta_1]}{\eta_1, \eta_1}\eta_1, \quad \eta_3 = \gamma - \frac{[\gamma, \eta_1]}{[\eta_1, \eta_1]}\eta_1 - \frac{[\gamma, \eta_2]}{[\eta_2, \eta_2]}\eta_2,$$

then η_1, η_2, η_3 is a orthogonal vector group. Set

$$\varepsilon_1 = \frac{\eta_1}{\|\eta_1\|}, \varepsilon_2 = \frac{\eta_2}{\|\eta_2\|}, \varepsilon_3 = \frac{\eta_3}{\|\eta_3\|},$$

there by $\varepsilon_1, \varepsilon_2, \varepsilon_3$ is a orthonormal vector group. Repeat this process, we get Schimidt's orthogonalization.

2 Schimidt's Orthogonalization

Theorem 1 [1−2] *Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be a linear independent vector group, set*

$$\begin{aligned}
\eta_1 &= \alpha_1, \\
\eta_2 &= \alpha_2 - \frac{[\alpha_2, \eta_1]}{[\eta_1, \eta_1]}\eta_1, \\
&\vdots \\
\eta_k &= \alpha_k - \frac{[\alpha_k, \eta_1]}{[\eta_1, \eta_1]}\eta_1 - \cdots - \frac{[\alpha_k, \eta_{k-1}]}{[\eta_{k-1}, \eta_{k-1}]}\eta_{k-1},
\end{aligned}$$

then $\eta_1, \eta_2, \cdots, \eta_k$ is a orthogonal vector group. Set

$$\varepsilon_1 = \frac{\eta_1}{\|\eta_1\|}, \varepsilon_2 = \frac{\eta_2}{\|\eta_2\|}, \cdots, \varepsilon_k = \frac{\eta_k}{\|\eta_k\|},$$

then $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k$ is a orthonormal vector group. That is,

$$\alpha_1, \alpha_2, \ldots, \alpha_k \xrightarrow{\text{step1}} \eta_1, \eta_2, \cdots, \eta_k \xrightarrow{\text{step2}} \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k.$$

Note 1 (1)Step 1 is orthogonalization, step 2 is unitization, they are not interchangeable;

(2)$\alpha_1, \alpha_2, \cdots, \alpha_k$ is equivalent to $\eta_1, \eta_2, \cdots, \eta_k$ and $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k$.

Schmidt's orthogonalization not only illustrates the existence of orthonormal basis in any vector space, but also gives a concrete algorithm for constructing orthonormal basis.

Example 2 Transforming $\alpha_1 = (1, -1, 0, 0)^T, \alpha_2 = (1, 0, -1, 0)^T, \alpha_3 = (1, 0, 0, -1)^T$ into a orthonormal vector group.

Solution: By Theorem 1, we have

$$\begin{aligned}
\eta_1 &= \alpha_1, \\
\eta_2 &= \alpha_2 - \frac{[\alpha_2, \eta_1]}{[\eta_1, \eta_1]}\eta_1 = \frac{1}{2}(1, 1, -2, 0)^T, \\
\eta_3 &= \alpha_3 - \frac{[\alpha_3, \eta_1]}{[\eta_1, \eta_1]}\eta_1 - \cdots - \frac{[\alpha_3, \eta_2]}{[\eta_2, \eta_2]}\eta_2 = \frac{1}{3}(1, 1, 1, -3)^T,
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon_1 &= \frac{\eta_1}{\|\eta_1\|} = \frac{1}{\sqrt{2}}(1, -1, 0, 0)^T, \\
\varepsilon_2 &= \frac{\eta_2}{\|\eta_2\|} = \frac{1}{\sqrt{6}}(1, 1, -2, 0)^T, \\
\varepsilon_3 &= \frac{\eta_3}{\|\eta_3\|} = \frac{1}{\sqrt{12}}(1, 1, 1, -3)^T.
\end{aligned}$$

3 Matrix Form of Schimidt's Orthogonalization

In this section, using elementary transformation of matrices, we describe Schimidt's orthogonalization.

Let $(\eta_1,\cdots,\eta_k)=(\alpha_1,\cdots,\alpha_k)C$, where $C=\begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$, then

$$\Lambda=\begin{pmatrix} \eta_1' \\ \vdots \\ \eta_k' \end{pmatrix}(\eta_1,\cdots,\eta_k)=C'\begin{pmatrix} \alpha_1' \\ \vdots \\ \alpha_k' \end{pmatrix}(\alpha_1,\cdots,\alpha_k)C. \tag{1}$$

Denote $(\alpha_1,\cdots,\alpha_k)$ by A, (1) is equivalent to $C'(A'A)C=\Lambda$. Since A has full column rank, hence $A'A$ is a positive definite matrix and we have Theorem 2 as follows.

Theorem 2 *Let $\alpha_1,\alpha_2,\ldots,\alpha_k$ be a linear independent vector group, by contract transformation, there exists a matrix C as above such that $(\eta_1,\cdots,\eta_k)=(\alpha_1,\cdots,\alpha_k)C$ and $\eta_1,\cdots,\eta_k$ is a orthogonal vector group. Furthermore, there exists a matrix C such that $(\varepsilon_1,\cdots,\varepsilon_k)=(\alpha_1,\cdots,\alpha_k)C$ and $\varepsilon_1,\cdots,\varepsilon_k$ is a orthonormal vector group.*

Example 3 Transforming $\alpha_1=(1,-1,0,0)^T,\alpha_2=(1,0,-1,0)^T$, $\alpha_3=(1,0,0,-1)^T$ into a orthonormal vector group by Theorem 2.

Solution: $\begin{pmatrix} A'A \\ E \end{pmatrix}=\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\longrightarrow\begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \\ 1 & -\frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}\longrightarrow\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{12}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{12}} \\ 0 & 0 & \frac{3}{\sqrt{12}} \end{pmatrix}.$

Set $C_1=\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$, then

$$(\eta_1,\eta_2,\eta_3)=(\alpha_1,\alpha_2,\alpha_3)C_1=\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ -1 & \frac{1}{2} & \frac{1}{3} \\ 0 & -1 & \frac{1}{3} \\ 0 & 0 & -1 \end{pmatrix}.$$

Furthermore, set $C_2=\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$, then

$$(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(\alpha_1,\alpha_2,\alpha_3)C_2=\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{12}} \\ 0 & 0 & -\frac{3}{\sqrt{12}} \end{pmatrix}.$$

Note 2 *If A is a invertible, we can get $(\alpha_1, \cdots, \alpha_k) = (\varepsilon_1, \cdots, \varepsilon_k)C^{-1}$, let $C^{-1} = R$, $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k) = Q$, then A has $QR-$decomposition, where Q is a orthogonal matrix and R is a upper triangular matrix.*

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