International Mathematical Forum, Vol. 21, 2026, no. 1, 31 - 36
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2026.914565

Infinite Continued Fractions for the Euler-Mascheroni Constant γ

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This article shows how to derive infinite continued fractions for the Euler-Mascheroni constant.

Euler defined, in the year 1734, the constant γ as

$$\gamma = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \log n = \lim_{n\to\infty} H_n - \log n \approx 0{,}5772156649\ldots$$

with $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ harmonic series.

The above relationship can be written as:

$$\lim_{n\to\infty} [\log(e^{H_n}) - \log n] = \gamma$$

$$\lim_{n\to\infty} \left[\log\left(\frac{e^{H_n}}{n}\right)\right] = \gamma$$

so we have:

$$e^{\gamma} = \lim_{n\to\infty} \frac{e^{H_n}}{n}$$

Now consider the infinite generalized continued fraction for the calculation of $\log(x+1)$:

$$\log(x+1)=\cfrac{x}{1+\cfrac{x}{2+\cfrac{x}{3+\cfrac{4x}{4+\cfrac{4x}{5+\cfrac{9x}{6+\cfrac{9x}{7+\cfrac{16x}{8+\cfrac{16x}{9+\cfrac{25x}{10+\cfrac{25x}{11+\cdots..}}}}}}}}}}}$$

Place $(x+1)=\frac{e^{H_n}}{n}$, we have: $x=\frac{e^{H_n}}{n}-1$ and then you get

$$\gamma=\lim_{n\to\infty}\cfrac{\frac{e^{H_n}}{n}-1}{1+\cfrac{\frac{e^{H_n}}{n}-1}{2+\cfrac{\frac{e^{H_n}}{n}-1}{3+\cfrac{2^2\left(\frac{e^{H_n}}{n}-1\right)}{4+\cfrac{2^2\left(\frac{e^{H_n}}{n}-1\right)}{5+\cfrac{3^2\left(\frac{e^{H_n}}{n}-1\right)}{6+\cfrac{3^2\left(\frac{e^{H_n}}{n}-1\right)}{7+\cfrac{4^2\left(\frac{e^{H_n}}{n}-1\right)}{8+\cfrac{4^2\left(\frac{e^{H_n}}{n}-1\right)}{9+\cfrac{5^2\left(\frac{e^{H_n}}{n}-1\right)}{10+\cfrac{5^2\left(\frac{e^{H_n}}{n}-1\right)}{11+\cdots}}}}}}}}}}}$$

Which leads to the following infinite continued fraction:

$$\gamma=\cfrac{e^{\gamma}-1}{1+\cfrac{e^{\gamma}-1}{2+\cfrac{e^{\gamma}-1}{3+\cfrac{2^2(e^{\gamma}-1)}{4+\cfrac{2^2(e^{\gamma}-1)}{5+\cfrac{3^2(e^{\gamma}-1)}{6+\cfrac{3^2(e^{\gamma}-1)}{7+\cfrac{4^2(e^{\gamma}-1)}{8+\cfrac{4^2(e^{\gamma}-1)}{9+\cfrac{5^2(e^{\gamma}-1)}{10+\cfrac{5^2(e^{\gamma}-1)}{11+\cdots}}}}}}}}}}}$$

being that $e^{\gamma}=\prod_{n=1}^{\infty}\frac{\sqrt[n]{e}}{\left(1+\frac{1}{n}\right)}$ (C.S. Siracusa year 2015)

we obtain the following infinite generalized continued fraction as a function of the constant e
of Napier alone.

$$x=e^{\gamma}-1=\prod_{n=1}^{\infty}\frac{\sqrt[n]{e}}{\left(1+\frac{1}{n}\right)}-1=0{,}781072418\ldots$$

$K=0{,}781072418\ldots$

with these positions we have:

$$\gamma = \cfrac{\prod_{n=1}^{\infty}\frac{\sqrt[n]{e}}{\left(1+\frac{1}{n}\right)} - 1}{1 + \cfrac{\prod_{n=1}^{\infty}\frac{\sqrt[n]{e}}{\left(1+\frac{1}{n}\right)} - 1}{2 + \cfrac{\prod_{n=1}^{\infty}\frac{\sqrt[n]{e}}{\left(1+\frac{1}{n}\right)} - 1}{3 + \cfrac{2^2\left(\prod_{n=1}^{\infty}\frac{\sqrt[n]{e}}{\left(1+\frac{1}{n}\right)} - 1\right)}{4 + \cfrac{2^2\left(\prod_{n=1}^{\infty}\frac{\sqrt[n]{e}}{\left(1+\frac{1}{n}\right)} - 1\right)}{5 + \cfrac{3^2\left(\prod_{n=1}^{\infty}\frac{\sqrt[n]{e}}{\left(1+\frac{1}{n}\right)} - 1\right)}{6 + \cfrac{3^2\left(\prod_{n=1}^{\infty}\frac{\sqrt[n]{e}}{\left(1+\frac{1}{n}\right)} - 1\right)}{7 + \cfrac{4^2\left(\prod_{n=1}^{\infty}\frac{\sqrt[n]{e}}{\left(1+\frac{1}{n}\right)} - 1\right)}{8 + \cfrac{4^2\left(\prod_{n=1}^{\infty}\frac{\sqrt[n]{e}}{\left(1+\frac{1}{n}\right)} - 1\right)}{9 + \cfrac{5^2\left(\prod_{n=1}^{\infty}\frac{\sqrt[n]{e}}{\left(1+\frac{1}{n}\right)} - 1\right)}{10 + \cfrac{5^2\left(\prod_{n=1}^{\infty}\frac{\sqrt[n]{e}}{\left(1+\frac{1}{n}\right)} - 1\right)}{11 + \cdots}}}}}}}}}}}$$

Given the generic continued fraction

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \cdots}}}}$$

an equivalent transformation can be performed to make it arithmetic (with all $b_n = 1$).

$$a_0 + \cfrac{b_1 c_1}{a_1 c_1 + \cfrac{b_2 c_1 c_2}{a_2 c_2 + \cfrac{b_3 c_2 c_3}{a_3 c_3 + \cfrac{b_4 c_3 c_4}{a_4 c_4 + \cdots}}}}$$

For the transformation we have the following coefficients:

$$c_0 = 1; \quad c_n = \frac{1}{b_n c_{n-1}}$$

$$c_1 = \frac{1}{b_1 c_0} = \frac{1}{\prod_{k=1}^{\infty}\frac{\sqrt[k]{e}}{\left(1+\frac{1}{k}\right)} - 1}; \quad c_2 = \frac{1}{b_2 c_1} = \frac{1}{1^2};$$

$$c_3 = \frac{1}{b_3 c_2} = \frac{1}{\prod_{k=1}^{\infty}\frac{\sqrt[k]{e}}{\left(1+\frac{1}{k}\right)}-1}; \quad c_4 = \frac{1}{b_4 c_3} = \frac{1}{2^2};$$

$$c_5 = \frac{1}{b_5 c_4} = \frac{1}{\prod_{k=1}^{\infty}\frac{\sqrt[k]{e}}{\left(1+\frac{1}{k}\right)}-1}; \quad c_6 = \frac{1}{b_6 c_5} = \frac{1}{3^2};$$

$$c_7 = \frac{1}{b_7 c_6} = \frac{1}{\prod_{k=1}^{\infty}\frac{\sqrt[k]{e}}{\left(1+\frac{1}{k}\right)}-1}; \quad c_8 = \frac{1}{b_8 c_7} = \frac{1}{4^2};$$

$$c_9 = \frac{1}{b_9 c_8} = \frac{1}{\prod_{k=1}^{\infty}\frac{\sqrt[k]{e}}{\left(1+\frac{1}{k}\right)}-1}; \quad c_{10} = \frac{1}{b_{10} c_9} = \frac{1}{5^2};$$

$$c_{11} = \frac{1}{b_{11} c_{10}} = \frac{1}{\prod_{k=1}^{\infty}\frac{\sqrt[k]{e}}{\left(1+\frac{1}{k}\right)}-1}; \quad c_{12} = \frac{1}{b_{12} c_{11}} = \frac{1}{6^2};$$

$$c_{13} = \cdots; \quad \ldots\ldots.$$

$$c_0 = 1; \quad c_{2n-1} = \frac{1}{\prod_{k=1}^{\infty}\frac{\sqrt[k]{e}}{\left(1+\frac{1}{k}\right)}-1}; \quad c_{2n} = \frac{1}{n^2};$$

$$n \in N$$

With this transformation we obtain the following non-simple arithmetic continued fraction (the a_n are not integers):

$$\gamma = \cfrac{1}{\frac{1}{K} + \cfrac{1}{\frac{2}{1^2} + \cfrac{1}{\frac{3}{K} + \cfrac{1}{\frac{4}{2^2} + \cfrac{1}{\frac{5}{K} + \cfrac{1}{\frac{6}{3^2} + \cfrac{1}{\frac{7}{K} + \cfrac{1}{\frac{8}{4^2} + \cfrac{1}{\frac{9}{K} + \cfrac{1}{\frac{10}{5^2} + \cdots}}}}}}}}}}$$

Another infinite generalized continued fraction for γ is the following:

$$x = e^{\gamma} - 1 = \prod_{n=1}^{\infty} \frac{\sqrt[n]{e}}{\left(1 + \frac{1}{n}\right)} - 1 = 0{,}781072418\ldots$$

$$K = 0{,}781072418\ldots$$

$$\gamma = \cfrac{x}{1 + \cfrac{x}{2 + \cfrac{x}{3 + \cfrac{2x}{2 + \cfrac{2x}{5 + \cfrac{3x}{2 + \cfrac{3x}{7 + \cfrac{4x}{2 + \cfrac{4x}{9 + \cfrac{5x}{2 + \cfrac{5x}{11 + \cdots}}}}}}}}}}}$$

To transform it into a non-simple infinite arithmetic continued fraction the transformation coefficients are given by:

$$c_0 = 1; \quad c_n = \frac{1}{b_n c_{n-1}};$$

$$c_1 = \frac{1}{b_1 c_0} = \frac{1}{\prod_{k=1}^{\infty} \frac{\sqrt[k]{e}}{\left(1+\frac{1}{k}\right)} - 1}; \quad c_2 = \frac{1}{b_2 c_1} = 1;$$

$$c_3 = \frac{1}{b_3 c_2} = \frac{1}{\prod_{k=1}^{\infty} \frac{\sqrt[k]{e}}{\left(1+\frac{1}{k}\right)} - 1}; \quad c_4 = \frac{1}{b_4 c_3} = \frac{1}{2};$$

$$c_5 = \frac{1}{b_5 c_4} = \frac{1}{\prod_{k=1}^{\infty} \frac{\sqrt[k]{e}}{\left(1+\frac{1}{k}\right)} - 1}; \quad c_6 = \frac{1}{b_6 c_5} = \frac{1}{3};$$

$$\ldots\ldots\ldots$$

$$c_0 = 1; \quad c_{2n-1} = \frac{1}{\prod_{k=1}^{\infty} \frac{\sqrt[k]{e}}{\left(1+\frac{1}{k}\right)} - 1}; \quad c_{2n} = \frac{1}{n}; \quad n \in N$$

$$\gamma = \cfrac{1}{\frac{1}{K} + \cfrac{1}{\frac{2}{1} + \cfrac{1}{\frac{3}{K} + \cfrac{1}{\frac{2}{2} + \cfrac{1}{\frac{5}{K} + \cfrac{1}{\frac{2}{3} + \cfrac{1}{\frac{7}{K} + \cfrac{1}{\frac{2}{4} + \cfrac{1}{\frac{9}{K} + \cfrac{1}{\frac{2}{5} + \cfrac{1}{\frac{11}{K} + \cdots}}}}}}}}}}}$$

which is the same as the previous one.
Since the non-simple arithmetic continued fraction is infinite, it follows that the Euler-Mascheroni constant γ is irrational.

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Received: June 1, 2026; Published: June 14, 2026