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On the Parallel Curve of the Adjoint Curve in W_3

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Abstract

In this paper, we have defined the parallel curve D of the adjoint curve $\overline{C}$ of a curve C in three-dimensional Weyl space W_3. Also, we have obtained the conditions to be involute-evolute, Bertrand and Mannheim curve pair of the curve C and the parallel curve D of the adjoint curve $\overline{C}$ of the curve C.

Mathematics Subject Classification: 53A25, 53B25

Keywords: Weyl space, prolonged covariant derivative, adjoint curve, parallel curve

1 Introduction

A manifold with a conformal metric g_{ij} and a symmetric connection ∇_k satisfying the compatibility condition

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \tag{1}$$

is called a Weyl space which will be denoted by $W(g_{ij}, T_k)$. The vector field T_k is named the complementary vector field. Under a renormalization of the metric tensor g_{ij} in the form

$$\tilde{g}_{ij} = \lambda^2 g_{ij} \tag{2}$$

the complementary vector field T_k is transformed by the law

$$\tilde{T}_k = T_k + \partial_k ln\lambda \tag{3}$$

where λ is a scalar function [8].

If under the transformation (2), the quantity A is charged according to the rule

$$\tilde{A} = \lambda^P A \tag{4}$$

then, A is called a satellite of g_{ij} with weight $\{p\}$.

The prolonged derivative and prolonged covariant derivative of A are respectively defined by [3,9]

$$\dot{\partial}_k = \partial_k A - pT_k A \tag{5}$$

and

$$\dot{\nabla}_k A = \nabla_k A - pT_k A. \tag{6}$$

Let $\underset{r}{v}^i$ $(i, r = 1, 2, 3)$ be the contravariant components of the vector field $\underset{r}{v}$ in $W_3(g_{ij}, T_k)$. Suppose that the vector fields $\underset{r}{v}^i$ $(i, r = 1, 2, 3)$ are normalized by the conditions $g_{ij}\underset{r}{v}^i\underset{r}{v}^j = 1$ $(j = 1, 2, 3)$

The prolonged covariant derivative of the vector field $\underset{r}{v}$ is given by [10]

$$\dot{\nabla}_k \underset{r}{v}^i = \overset{s}{\underset{r}{T_k}} \underset{s}{v}^i. \tag{7}$$

The quantities

$$\overset{q}{\underset{rs}{T}} = \overset{s}{\underset{r}{T_k}} \underset{s}{v}^k \ (k, q, r, s = 1, 2, 3; r \neq s) \tag{8}$$

and

$$\overset{r}{\underset{s}{\mathfrak{z}}} = \overset{r}{\underset{s}{T_k}} \underset{s}{v}^k \tag{9}$$

are called the Chebyshev curvature of the first kind and the geodesic curvature of the lines of the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ [10], respectively.

The vector fields

$$\underset{rs}{a}^i = \overset{q}{\underset{rs}{T}} \underset{q}{v}^i \ , \ \underset{s}{c}^i = \overset{r}{\underset{s}{\mathfrak{z}}} \underset{r}{v}^i \ (i, q, r, s = 1, 2, 3) \tag{10}$$

are called the Chebyshev vector fields of the first kind and geodesic vector fields of the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$, respectively.

Since the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ is an orthogonal net, we have [10]

$$\overset{r}{\underset{r}{T_k}} = 0\ ,\ \overset{p}{\underset{r}{T_k}} + \overset{r}{\underset{p}{T_k}} = 0\ (r \neq p). \tag{11}$$

Matsuda and Yorozu showed that every circular helix in E^3 is a typical example of Bertrand curve. The circular helix is one in a family of special Frenet curves. They proved that no special Frenet curve in $E^n (n \geq 4)$ is a Bertrand curve. Besides, they gave generalization of Bertrand curve in 2003 [7].

Chrastinova presented parallel curves in three-dimensional space in 2007 [2].

Liu and Wang studied Mannheim partner curves in three-dimensional space. They obtained the necessary and sufficient conditions for the Mannheim partner curves in Euclidean E^3 in 2008 [4].

Tunçer and Ünal studied the properties of the spherical indicatrix of a Bertrand curve and its mate curve and presented some characteristic properties in the cases that Bertrand curve and its mate curve are slant helices, spherical indicatrices are slant helices and they also discussed that the spherical indicatrices made new curve pairs in the means of Mannheim, involute-evolute and Bertrand pairs. Additionally, they investigated the relations between the spherical images and introduced new representation of spherical indicatrices in 2012 [11].

Yüksel et al. studied parallel curves in Minkowski 3-space in 2014 [12].

Keskin et al. obtained characterization of the parallel curve of the adjoint curve of a given curve in E^3 and discussed the conditions to be involute-evolute, Bertrand and Mannheim curve pair of the given curve and parallel curve of the adjoint curve of the given curve in 2016 [5].

2 Preliminaries

Let $C\ :\ x^i = x^i(s)$ be a curve in three-dimensional Weyl space W_3 $(i = 1, 2, 3)$. s is the arc length parameter of C. Let us denote the Frenet apparatus of C by $\{\underset{1}{v}^i, \underset{2}{v}^i, \underset{3}{v}^i, \kappa, \tau\}$. Let $\overline{C}$:$y^i = y^i(s)$ be adjoint curve of C. Let us denote the Frenet apparatus of $\overline{C}$ by $\{\underset{1}{\overline{v}}^i, \underset{2}{\overline{v}}^i, \underset{3}{\overline{v}}^i, \overline{\kappa}, \overline{\tau}\}$. The adjoint curve $\overline{C}$ is defined as

$$\underset{1}{v}^k\ \dot{\nabla}_k\, y^i =\ \underset{3}{v}^i \tag{12}$$

[6]. Relations between Frenet apparatuses of C and $\overline{C}$ are in the following form

$$
\begin{aligned}
\underset{1}{\overline{v}}^i(s) &= \underset{3}{v}^i(s),\\
\underset{2}{\overline{v}}^i(s) &= -\underset{2}{v}^i(s),\\
\underset{3}{\overline{v}}^i(s) &= \underset{1}{v}^i(s),
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
\overline{\kappa}(s) &= \tau(s),\\
\overline{\tau} &= \kappa(s),
\end{aligned}
\tag{14}
$$

[6].

Let D:$z^i = z^i(s^*)$ be a curve ($i = 1, 2, 3; s^*$:arc length parameter of D). Let D be parallel curve at distance r from the curve $\overline{C}$. Then, the following equations are valid:

$$g_{ij}(y^i - z^i)(y^j - z^j) = r^2, \tag{15}$$

$$g_{ij}\underset{1}{\overline{v}}^i(y^j - z^j) = 0, \tag{16}$$

$$g_{ij}(\underset{1}{v}^k \,\dot{\nabla}_k\, \underset{1}{\overline{v}}^i)(y^j - z^j) + g_{ij}\underset{1}{\overline{v}}^i\underset{1}{\overline{v}}^j = 0. \tag{17}$$

Definition 2.1 *Let C : $x^i = x^i(s)$ and $\overline{C}$:$y^i = y^i(s)$ be two curves in W_3. $\{\underset{1}{v}^i, \underset{2}{v}^i, \underset{3}{v}^i\}$ and $\{\underset{1}{\overline{v}}^i, \underset{2}{\overline{v}}^i, \underset{3}{\overline{v}}^i\}$ are the Frenet frames of C and $\overline{C}$, respectively. $\{\underset{2}{v}^i, \underset{2}{\overline{v}}^i\}$ is linear dependent, then $(C, \overline{C})$ curve pair is called a Bertrand curve pair.*

Theorem 2.2 *Let $(C, \overline{C})$ be a Bertrand curve pair. $d(x^i(s), y^i(s))$ is constant.*

Definition 2.3 *Let C : $x^i = x^i(s)$ and $\overline{C}$:$y^i = y^i(s)$ be two curves in W_3. Let $\{\underset{1}{v}^i, \underset{2}{v}^i, \underset{3}{v}^i\}$ and $\{\underset{1}{\overline{v}}^i, \underset{2}{\overline{v}}^i, \underset{3}{\overline{v}}^i\}$ be the Frenet frames of C and $\overline{C}$, respectively. In tangent vector field $\underset{1}{v}^i$ of C coincides with principal normal vector field $\underset{2}{\overline{v}}^i$ of $\overline{C}$, in other words, if $g_{ij}\underset{1}{v}^i\underset{1}{\overline{v}}^j = 0$, $\overline{C}$ is involute of C and C is evolute of $\overline{C}$.*

Definition 2.4 *Let C : $x^i = x^i(s)$ and $\overline{C}$:$y^i = y^i(s)$ be two curves in W_3. The principal normal lines of C coincides with the binormal lines of $\overline{C}$, then C is called a Mannheim curve and $\overline{C}$ is a Mannheim partner curve of C. The pair $(C, \overline{C})$ is called a Mannheim pair.*

3 Parallel Curve of the Adjoint Curve in W_3

Using (16), $y^j - z^j$ can be written as the following form:

$$y^i - z^j = a_2 \underset{2}{\overline{v}}^i + a_3 \underset{3}{\overline{v}}^i \tag{18}$$

for appropriate coefficients a_2, a_3.

Theorem 3.1 *Let $\overline{C} : y^i = y^i(s)$ be a curve in W_3. Then, we have the following parallel curve D of the curve $\overline{C}$:*

$$z^j = \int_{s_0}^{s} \underset{1}{\overline{v}}^i \, ds \mp \sqrt{r^2 - \frac{1}{\overline{\kappa}^2}} \, \underset{3}{\overline{v}}^i + \frac{1}{\overline{\kappa}} \underset{2}{\overline{v}}^i. \tag{19}$$

Proof. Substituting (18) in (17), we have

$$g_{ij} \overline{\kappa} \underset{2}{\overline{v}}^i (a_2 \underset{2}{\overline{v}}^j + a_3 \underset{3}{\overline{v}}^j) + g_{ij} \underset{1}{\overline{v}}^i \underset{1}{\overline{v}}^j = 0 \tag{20}$$

or

$$\overline{\kappa} a_2 + 1 = 0 \tag{21}$$

where $\underset{1}{v}^k \dot{\nabla}_k \underset{1}{\overline{v}}^i = \overline{\kappa} \underset{2}{\overline{v}}^i$, $g_{ij} \underset{2}{\overline{v}}^i \underset{2}{\overline{v}}^j = 1$, $g_{ij} \underset{2}{\overline{v}}^i \underset{3}{\overline{v}}^j = 0$ and $g_{ij} \underset{1}{\overline{v}}^i \underset{1}{\overline{v}}^j = 1$.
From (21), it is obtained as $a_2 = \frac{-1}{\overline{\kappa}}$.
From (15), we have $g_{ij}(y^i - z^i)(y^j - z^j) = r^2 = a_2^2 + a_3^2$. Therefore, we obtain $a_3^2 = r^2 - a_2^2 = r^2 - \frac{1}{\overline{\kappa}^2}$ and $a_3 = \mp\sqrt{r^2 - \frac{1}{\overline{\kappa}^2}}$.

Corollary 3.2 *Let $C : x^i = x^i(s)$ be a curve in W_3. Let $\overline{C}$:$y^i = y^i(s)$ be adjoint curve of C and D:$z^i = z^i(s^*)$ be parallel curve of $\overline{C}$. From (13), (14), and (19), we have*

$$z^j = \int_{s_0}^{s} \underset{3}{v}^i \, ds \mp \sqrt{r^2 - \frac{1}{\tau^2}} \, \underset{1}{v}^i - \frac{1}{\tau^2} \underset{2}{v}^i. \tag{22}$$

Theorem 3.3 *Let $\overline{C} : y^i = y^i(s)$ be adjoint curve of C and D:$z^i = z^i(s^*)$ be the parallel curve of $\overline{C}$ in W_3. The arc-length s^* is given by differential equation:*

$$f(s) = \sqrt{(a_3 \overline{\tau} - \underset{1}{v}^k \dot{\nabla}_k a_2)^2 + (a_2 \overline{\tau} + \underset{1}{v}^k \dot{\nabla}_k a_3)^2} \tag{23}$$

where $a_2 = a_2(s)$ and $a_3 = a_3(s)$.
Proof. $\overline{C}$ and D satisfy (15), (16) and (17). From (19), we have

$$
\begin{aligned}
\underset{1}{v}^{k}\, \dot{\nabla}_k z^i &= (\overset{\star}{\underset{1}{v}}{}^{k} \overset{\overset{\bullet}{\star}}{\nabla}_k z^i) f(s) = \overset{\star}{\underset{1}{v}}{}^{i} f(s) = \underset{1}{\overline{v}}^{i} - (\underset{1}{v}^{k} \dot{\nabla}_k a_3) \underset{3}{\overline{v}}^{i} - a_3 \underset{1}{v}^{k} \dot{\nabla}_k \underset{3}{\overline{v}}^{i} \\
&\quad - (\underset{1}{v}^{k} \dot{\nabla}_k a_2) \underset{2}{\overline{v}}^{i} - a_2 \underset{1}{v}^{k} \dot{\nabla}_k \underset{2}{\overline{v}}^{i} \\
&= \underset{1}{\overline{v}}^{i} (1 + \overline{\kappa} a_2) + (a_3 \overline{\tau} - \underset{1}{v}^{k} \dot{\nabla}_k a_2) \underset{2}{\overline{v}}^{i} - \\
&\quad - (\underset{1}{v}^{k} \dot{\nabla}_k a_3 + a_2 \overline{\tau}) \underset{3}{\overline{v}}^{i}
\end{aligned} \tag{24}
$$

where $\overset{\star}{\underset{1}{v}}{}^{k} \overset{\overset{\bullet}{\star}}{\nabla}_k z^i = \overset{\star}{\underset{1}{v}}{}^{i}$. *Using* $a_2 = \frac{-1}{\overline{\kappa}}$ *in (24), we obtain*

$$
\underset{1}{v}^{k}\, \dot{\nabla}_k z^i = (\overset{\star}{\underset{1}{v}}{}^{k} \overset{\overset{\bullet}{\star}}{\nabla}_k z^i) f(s) = \overset{\star}{\underset{1}{v}}{}^{i} f(s) = (a_3 \overline{\tau} - \underset{1}{v}^{k} \dot{\nabla}_k a_2) \underset{2}{\overline{v}}^{i} - (\underset{1}{v}^{k} \dot{\nabla}_k a_3 + a_2 \overline{\tau}) \underset{3}{\overline{v}}^{i}. \tag{25}
$$

Taking norm of (25), we get

$$
f(s) = \sqrt{(a_3 \overline{\tau} - \underset{1}{v}^{k}\, \dot{\nabla}_k a_2)^2 + (\underset{1}{v}^{k}\, \dot{\nabla}_k a_3 + a_2 \overline{\tau})^2}. \tag{26}
$$

Corollary 3.4 *Let* $C : y^i = y^i(s)$ *be adjoint curve of* C *and* D:$z^i = z^i(s^*)$ *be the parallel curve of* $\overline{C}$. *From (22), we have*

$$
f(s) = \sqrt{(a_3 \kappa - \underset{1}{v}^{k}\, \dot{\nabla}_k a_2)^2 + (\underset{1}{v}^{k}\, \dot{\nabla}_k a_3 + a_2 \kappa)^2}. \tag{27}
$$

After this step, for convenience, let us take as a_2 *and* a_3 *are prolonged covariant constants. Hence,* $\underset{1}{v}^{k}\, \dot{\nabla}_k a_2 = \underset{1}{v}^{k}\, \dot{\nabla}_k a_3 = 0$. *Then, we have* $f(s)$ *as*

$$
f(s) = \sqrt{a_3^2 \kappa^2 + a_2^2 \kappa^2} = \sqrt{(a_3^2 + a_2^2) \kappa^2} = \sqrt{r^2 \kappa^2} = r\kappa. \tag{28}
$$

Theorem 3.5 *Let* $\overline{C} : y^i = y^i(s)$ *be adjoint curve of* C *and* D:$z^i = z^i(s^*)$ *be the parallel curve of* $\overline{C}$. D *and* C *are an involute-evolute curve pair if and only if* $\kappa = 0$ *and* $r = \mp \frac{1}{\tau^2}$ *where* $\tau \neq 0$.

Proof. Let us denote Frenet apparatuses of D *and* C *by* $\{\overset{\star}{\underset{1}{v}}{}^{i}, \overset{\star}{\underset{2}{v}}{}^{i}, \overset{\star}{\underset{3}{v}}{}^{i}, \overset{\star}{\kappa}, \overset{\star}{\tau}\}$ *and* $\{\underset{1}{v}^{i}, \underset{2}{v}^{i}, \underset{3}{v}^{i}, \kappa, \tau\}$, *respectively. If* D *and* C *are an involute-evolute curve pair,* $g_{ij} \overset{\star}{\underset{1}{v}}{}^{i} \underset{1}{v}^{j} = 0$.

We know that, $\underset{1}{v}^{k}\, \dot{\nabla}_k z^i = (\overset{\star}{\underset{1}{v}}{}^{k} \overset{\overset{\bullet}{\star}}{\nabla}_k z^i) f(s)$ *or* $\underset{1}{v}^{k}\, \dot{\nabla}_k z^i = \overset{\star}{\underset{1}{v}}{}^{i} f(s)$ *or* $\overset{\star}{\underset{1}{v}}{}^{i} = \frac{1}{f(s)} \underset{1}{v}^{k}\, \dot{\nabla}_k z^i$.

By means of (24), we get

$$\frac{1}{f(s)}\underset{1}{v}^k\,\dot{\nabla}_k z^i = \frac{1}{f(s)}[\underset{1}{\overline{v}}^i(1+\overline{\kappa}a_2) + a_3\overline{\tau}\,\underset{2}{\overline{v}}^i - a_2\overline{\tau}\,\underset{3}{\overline{v}}^i] = \underset{1}{\overset{\star}{v}}{}^i \tag{29}$$

Since $\overline{C}$ is the adjoint curve of C, the equations (13) and (14) are satisfied. Then,

$$\underset{1}{\overset{\star}{v}}{}^i = \frac{1}{f(s)}[\underset{3}{v}^i(1+\tau a_2) - a_3\kappa\underset{2}{v}^i - a_2\kappa\underset{1}{v}^i] \tag{30}$$

Since $g_{ij}\underset{1}{\overset{\star}{v}}{}^i\underset{1}{v}^j = 0$, from (30), we obtain $\kappa = 0$ or $a_2 = 0$. From here, $a_3 = 0$, i.e., $r = \mp\frac{1}{\tau}$ is obtained. The proof is completed.

Now, let us calculate κ and τ in terms of magnitudes in Weyl space:

$$\underset{1}{v}^k\,\dot{\nabla}_k z^i = \underset{3}{v}^i + a_2\underset{1}{v}^k\dot{\nabla}_k\underset{2}{v}^i - a_3\underset{1}{v}^k\dot{\nabla}_k\underset{1}{v}^i \tag{31}$$

$$\underset{1}{v}^k\,\dot{\nabla}_k z^i = \underset{3}{v}^i + a_2\underset{1}{v}^k\underset{21}{a}^i - a_3\underset{1}{c}^i \tag{32}$$

where

$$\underset{21}{a}^i = \underset{2}{\overset{1}{T}}_k\underset{1}{v}^k\underset{1}{v}^i + \underset{2}{\overset{3}{T}}_k\underset{1}{v}^k\underset{3}{v}^i \tag{33}$$

$$\underset{1}{c}^i = \underset{1}{\overset{2}{T}}_k\underset{1}{v}^k\underset{2}{v}^i + \underset{1}{\overset{3}{T}}_k\underset{1}{v}^k\underset{3}{v}^i. \tag{34}$$

$$\underset{1}{v}^k\dot{\nabla}_k\underset{2}{v}^i = \underset{21}{a}^i = \underset{2}{\overset{1}{T}}_k\underset{1}{v}^k\underset{1}{v}^i + \underset{2}{\overset{3}{T}}_k\underset{1}{v}^k\underset{3}{v}^i \tag{35}$$

$$(-\kappa\underset{1}{v}^i + \tau\underset{3}{v}^i)g_{ij}\underset{1}{v}^j = \underset{2}{\overset{1}{T}}_k\underset{1}{v}^k \tag{36}$$

$$-\kappa = g_{ij}\underset{21}{a}^i\underset{1}{v}^j = \underset{2}{\overset{1}{T}}_k\underset{1}{v}^k \tag{37}$$

$$\kappa = -\underset{2}{\overset{1}{T}}_k\underset{1}{v}^k \tag{38}$$

where $g_{ij}\underset{1}{v}^i\underset{1}{v}^j = 1$ and $g_{ij}\underset{3}{v}^i\underset{1}{v}^j = 0$.
Besides

$$\underset{1}{v}^k\dot{\nabla}_k\underset{3}{v}^i = -\tau\underset{2}{v}^i \tag{39}$$

$$\underset{3}{\overset{P}{T}}_k\underset{1}{v}^k\underset{P}{v}^i = \underset{31}{\overset{P}{\tau}}\underset{P}{v}^i = \underset{31}{a}^i \tag{40}$$

$$\underset{3}{\overset{1}{T}}_k \underset{1}{v}^k \underset{1}{v}^i + \underset{3}{\overset{2}{T}}_k \underset{1}{v}^k \underset{2}{v}^i = \underset{31}{a}^i = -\tau \underset{2}{v}^i \tag{41}$$

$$\underset{3}{\overset{1}{T}}_k \underset{1}{v}^k g_{ij} \underset{1}{v}^i \underset{2}{v}^j + \underset{3}{\overset{2}{T}}_k \underset{1}{v}^k g_{ij} \underset{2}{v}^i \underset{2}{v}^j = g_{ij} \underset{31}{a}^i \underset{2}{v}^j = -\tau g_{ij} \underset{2}{v}^i \underset{2}{v}^j \tag{42}$$

$$\underset{3}{\overset{2}{T}}_k \underset{1}{v}^k = g_{ij} \underset{31}{a}^i \underset{2}{v}^j = -\tau \tag{43}$$

$$\tau = -\underset{3}{\overset{2}{T}}_k \underset{1}{v}^k \tag{44}$$

where $g_{ij} \underset{1}{v}^i \underset{2}{v}^j = 0$ *and* $g_{ij} \underset{2}{v}^i \underset{2}{v}^j = 0$.

Theorem 3.6 *Let* $\overline{C}$ *be the adjoint curve of* C *and* D *be the parallel curve of* $\overline{C}$. D *and* C *are a Bertrand curve pair if and only if* $-a_2\kappa^2 - a_3(\underset{1}{v}^\ell \dot{\nabla}_\ell \kappa) + a_3 \overset{3}{\underset{1}{\mathfrak{Z}}} \tau = \lambda$,
where λ *is a constant number.*
Proof. *Let us denote Frenet apparatus of* D *and* C *by* $\{\overset{\star}{\underset{1}{v}}{}^i, \overset{\star}{\underset{2}{v}}{}^i, \overset{\star}{\underset{3}{v}}{}^i, \overset{\star}{\kappa}, \overset{\star}{\tau}\}$ *and* $\{\underset{1}{v}^i, \underset{2}{v}^i, \underset{3}{v}^i, \kappa, \tau\}$, *respectively. If* D *and* C *are a Bertrand curve pair, then* $g_{ij} \overset{\star}{\underset{2}{v}}{}^i \underset{2}{v}^j = \lambda$. *Let us calculate* $\overset{\star}{\underset{2}{v}}{}^i$ *and inner product of* $\overset{\star}{\underset{2}{v}}$ *and* $\underset{2}{v}$. *It is knowing* $z^i = \int_{s_0}^{s} \underset{3}{v}^i \, ds + a_2 \underset{2}{v}^i - a_3 \underset{1}{v}^i$ *and taking prolonged covariant derivative of* z^i *in the direction of* $\underset{1}{v}^k$ *we have*

$$\underset{1}{v}^k \dot{\nabla}_k z^i = \underset{3}{v}^i + a_2 \underset{1}{v}^k \dot{\nabla}_k \underset{2}{v}^i - a_3 \underset{1}{v}^k \dot{\nabla}_k \underset{1}{v}^i \tag{45}$$

$$= \underset{3}{v}^i + a_2 \underset{2}{\overset{P}{T}}_k \underset{1}{v}^k \underset{P}{v}^i - a_3 \underset{1}{\overset{P}{T}}_k \underset{1}{v}^k \underset{P}{v}^i \tag{46}$$

$$= \underset{3}{v}^i + a_2 \underset{2}{\overset{1}{T}}_k \underset{1}{v}^k \underset{1}{v}^i + a_2 \underset{2}{\overset{3}{T}}_k \underset{1}{v}^k \underset{3}{v}^i - a_3 \underset{1}{\overset{2}{T}}_k \underset{1}{v}^k \underset{2}{v}^i - a_3 \underset{1}{\overset{3}{T}}_k \underset{1}{v}^k \underset{3}{v}^i \tag{47}$$

$$= \underset{3}{v}^i - a_2 \kappa \underset{1}{v}^i + a_2 \tau \underset{3}{v}^i - a_3 \kappa \underset{2}{v}^i - a_3 \overset{3}{\underset{1}{\mathfrak{Z}}} \underset{3}{v}^i. \tag{48}$$

The second order prolonged covariant derivative of z^i *is*

$$\begin{aligned} \underset{1}{v}^\ell \dot{\nabla}_\ell (\underset{1}{v}^k \dot{\nabla}_k z^i) = & - a_2(\underset{1}{v}^\ell \dot{\nabla}_\ell \kappa) \underset{1}{v}^i - a_2 \kappa^2 \underset{2}{v}^i - a_2 \kappa \overset{3}{\underset{1}{\mathfrak{Z}}} \underset{3}{v}^i + \\ & + a_2 (\underset{1}{v}^\ell \dot{\nabla}_\ell \tau) \underset{3}{v}^i - a_3 (\underset{1}{v}^\ell \dot{\nabla}_\ell \kappa) \underset{2}{v}^i + \\ & + a_3 \kappa^2 \underset{1}{v}^i - a_3 \kappa \tau \underset{3}{v}^i - a_3 (\underset{1}{v}^\ell \dot{\nabla}_\ell \overset{3}{\underset{1}{\mathfrak{Z}}}) \underset{3}{v}^i + \\ & + a_3 (\overset{3}{\underset{1}{\mathfrak{Z}}})^2 \underset{1}{v}^i + a_3 \overset{3}{\underset{1}{\mathfrak{Z}}} \tau \underset{2}{v}^i. \end{aligned} \tag{49}$$

Since $\underset{1}{v}^{\ell}\dot{\nabla}_{\ell}(\underset{1}{v}^{k}\dot{\nabla}_{k}z^{i}) = \overset{\star}{\underset{2}{v}}{}^{i}$, *we get*

$$g_{ij}\overset{\star}{\underset{2}{v}}{}^{i}\underset{2}{v}^{j} = \lambda = -a_2\kappa^2 - a_3(\underset{1}{v}^{\ell}\dot{\nabla}_{\ell}\kappa) + a_3\overset{3}{\underset{1}{\mathfrak{Z}}}\tau \tag{50}$$

The proof is completed.

Theorem 3.7 *Let $\overline{C}$ be the adjoint curve of C and D be the parallel curve of $\overline{C}$. D and C are Mannheim curve pair if and only if*

$$\begin{aligned}&-a_3\kappa^2 - r^2\overset{3}{\underset{1}{\mathfrak{Z}}}\kappa^2 + a_2^2\kappa\underset{1}{v}^{\ell}\dot{\nabla}_{\ell}\tau - a_2a_3\kappa(\underset{1}{v}^{\ell}\dot{\nabla}_{\ell}\overset{3}{\underset{1}{\mathfrak{Z}}})+\\ &+a_2a_3\overset{3}{\underset{1}{\mathfrak{Z}}}(\underset{1}{v}^{\ell}\dot{\nabla}_{\ell}\kappa) - a_3^2(\overset{3}{\underset{1}{\mathfrak{Z}}})^2 = \lambda,\end{aligned} \tag{51}$$

where λ is a constant number.

Proof. *Let us denote Frenet apparatus of D and C by $\{\overset{\star}{\underset{1}{v}}{}^{i}, \overset{\star}{\underset{2}{v}}{}^{i}, \overset{\star}{\underset{3}{v}}{}^{i}, \overset{\star}{\kappa}, \overset{\star}{\tau}\}$ and $\{\underset{1}{v}^{i}, \underset{2}{v}^{i}, \underset{3}{v}^{i}, \kappa, \tau\}$, respectively.If D and C are a Mannheim curve pair, $g_{ij}\underset{2}{v}^{i}\overset{\star}{\underset{3}{v}}{}^{j} = \lambda$. We know that $\overset{\star}{\underset{3}{v}}{}^{i} = \varepsilon_{ijk}\overset{\star}{\underset{1}{v}}{}^{j}\overset{\star}{\underset{2}{v}}{}^{k}$ where $\underset{1}{v}^{k}\dot{\nabla}_{k}z^{i} = \overset{\star}{\underset{1}{v}}{}^{i}$ and $\underset{1}{v}^{\ell}\dot{\nabla}_{\ell}(\underset{1}{v}^{k}\dot{\nabla}_{k}z^{i}) = \overset{\star}{\underset{2}{v}}{}^{i}$.*

Now let us calculate inner product $\underset{2}{v}^{i}$ and $\overset{\star}{\underset{3}{v}}{}^{j}$. If the inner product is equal to λ, D and C are a Mannheim curve pair.

$$\begin{aligned}g_{ij}\underset{2}{v}^{i}\overset{\star}{\underset{3}{v}}{}^{j} = \lambda \Rightarrow &-a_3\kappa^2 - r^2\overset{3}{\underset{1}{\mathfrak{Z}}}\kappa^2 + a_2^2\kappa\underset{1}{v}^{\ell}\dot{\nabla}_{\ell}\tau - a_2a_3\kappa(\underset{1}{v}^{\ell}\dot{\nabla}_{\ell}\overset{3}{\underset{1}{\mathfrak{Z}}})+\\ &+a_2a_3\overset{3}{\underset{1}{\mathfrak{Z}}}(\underset{1}{v}^{\ell}\dot{\nabla}_{\ell}\kappa) - a_3^2(\overset{3}{\underset{1}{\mathfrak{Z}}})^2 = \lambda,\end{aligned} \tag{52}$$

where $a_2 = -\frac{1}{\kappa}$, $a_3 = \mp\sqrt{r^2 - \frac{1}{\kappa}^2}$.

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