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Composite Connections on Differentiable Manifolds

Gian Mario Gianella

Università di Torino, via Carlo Alberto n. 10, Italy

Abstract

In this article at the n.1 the notion of a 2nd order composite connection c.c. was introduced, associating with an affine connection a.c. of $\mathcal{D}$(M), a *derivation* of $\mathcal{D}$(M) of a new type, called of 2nd order. In the n.2 is dedicated to some theorems, with their consequences, concerning induced c.c.(on open submanifolds). In the n.3 were found tensor fields determinated by composite connections and in particular a tensor field that can be called: curvature tensor of the $\overset{(C)}{\nabla}_2$. In the n.4, tensor relations were found between the *covariant derivative* and the *composite derivative.*

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1. Composite connections of the 2nd order.

On the manifold M let it be assumed given, in addition to the ∇_X operator of an affine connection a.c. (cf. n. 2 of [1]), another $\mathcal{R}$-linear ∇_{XY} operator which, for each element (X, Y)$\in \mathcal{D}^1(M) \times \mathcal{D}^1$(M), associates to each tensor field T (on M) a new tensor field of the same type in such a way that the following conditions are satisfied:

$$(1{,}1)\begin{cases} (I_2)\begin{cases} \nabla_{(X+Y)Z} = \nabla_{XZ} + \nabla_{YZ}; \qquad \nabla_{X(Y+Z)} = \nabla_{XY} + \nabla_{XZ}; \\ \qquad\qquad X,\ Y,\ Z \in \mathcal{D}^1(M) \\ \nabla_{(fX)Y} = f\nabla_{XY}; \qquad \nabla_{X(fY)} = f\nabla_{XY} + (Xf)\nabla_Y; \quad f \in \mathcal{F} \end{cases} \\ (II_2)\nabla_{XY}f = X(Yf)); \\ (III_2)\ \ \nabla_{XY}(T \otimes S) = (\nabla_{XY}T) \otimes S + \nabla_X T \otimes \nabla_Y S + \\ \qquad\qquad \nabla_Y T \otimes \nabla_X S + T \otimes \nabla_{XY} S; \qquad T,\ S \in \mathcal{D}(M), \\ (IV_2)\ \ \nabla_{XY}(CT) = C\nabla_{XY}T \qquad C : \text{contraction.} \end{cases}$$

The ∇_{XY} operator, together with the ∇_X operator, determines a geometric entity $\overset{(C)}{\nabla}_2$ which we will say:*2nd order composite connection c.c.*(or also: composite derivation of the 2nd order); a c.c. is therefore the geometric entity determined by the following applications ($\mathcal{R}$-linears):

$$X \to \nabla_X$$

$$(X, Y) \to \nabla_{XY}$$

together with: (2,1) of [1], (1,1); ∇_X, ∇_{XY} will be called 1st and 2nd order operators, respectively, of the c.c. $\overset{(C)}{\nabla}_2$.

By analogy with the case of a a.c., it is also advisable to highlight the form taken by (III_2), (IV_2) in certain particular cases; for (III_2) the case that interests to highlight is obtained with T = f $\in \mathcal{F}$, for (IV_2) with T = Z $\in \mathcal{D}^1(M)$, S = $\omega \in D_1(M)$; thus are obtained the:

(III'$_2$) $$\nabla_{XY}(fS) = (XYf)S + Xf\nabla_Y S + Yf\nabla_X S + f\nabla_{XY}S;$$

(IV'$_2$) $$X(Y\omega(Z)) = \nabla_{XY}\omega(Z) + \nabla_X\omega(\nabla_Y Z) + \nabla_Y\omega(\nabla_X Z) + \omega(\nabla_{XY}Z).$$

2. Induced composite connections (2nd order) and local coordinates

Some propositions valid for the ∇_X operator of an a.c., are transported to the 2nd order operator ∇_{XY}; e.g. we have the:

Prop. I. *If X or Y, or S (*$\in \mathcal{D}^r_s(M)$*, vanishes on U (open submanifold of M) the same happens for* $\nabla_{XY}S$.

Proof: Let's take care of the case S = 0; let's choose $f \in \mathcal{C}^\infty$(M) such that: f = 0 in U and f = 1 in M-U; under these conditions we have (in U) ([4]). pag.9):

$$Xf = Yf = X(Yf) = 0;$$

since it can be written:

$$\mathrm{S} = \mathrm{f\ S}$$

is enough to apply (III'$_2$) to deduce equality:

$$\nabla_{XY} S = f\ \nabla_{XY} S$$

from which ∇_{XY} S $= 0$ is deduced in the submanifold U.

The proof relative to the hypothesis X $= 0$ or Y $= 0$ is done immediately by applying the (I$_2$) with the aid of the same function f used earlier.

Consequence of the previous Prop. and Prop. I of [1], is that:

A c.c. $\overset{(C)}{\nabla}_2$ *given on M, induces a connection (of the same type) of operators* $\nabla_{(U)X}$, $\nabla_{(U)XY}$ *on every open submanifold U of M.*

Indeed, assigned the fields: X, Y, S in U, for each p $\in$ U, there exist fields $\bar{X}$, $\bar{Y}$, $\bar{S}$, on M, which coincide respectively with X, Y, S in a certain neighborhood ([5], pag. 5) V of p; if we set:$(\nabla_{(U)X}S)_q = (\nabla_X \bar{S})_q$, $(\nabla_{(U)XY}S)_q = (\nabla_{\bar{X}\bar{Y}}S)_q$, $q \in V$, we obtain the induced connection in U ([1]).

Let us suppose U' the domain of a local chart φ; using notations already introduced for local coordinates and natural references, we can set:

(2,1) $$\nabla_{e_h e_k} e_j = L^p_{hkj} e_{p.}$$

the functions L^q_{hkj} are the components of the 2nd order (in local coordinates), of the c.c. .

Let us suppose U' coincides with the domain of another local card φ' ($x^{j'}$ are the corresponding coordinates), we will have (in U'):

(2,1') $$\nabla_{e_{h'} e_{k'}} e_{j'} = L^{p'}_{h'k'j'} e_{p'}, \qquad (e_{i'} = \tfrac{\partial}{\partial x^{i'}});$$

we suppose U $\cap$ U'$\neq \emptyset$ and we assume: $\vartheta^i_{i'} = \frac{\partial x^i}{\partial x^{i'}}$ and therefore: $e_{i'} = \vartheta^i_{i'} e_i$ (in U $\cap$ U'); substituting in (2,1') and taking into account some axioms related to $\overset{(C)}{\nabla}_2$, with formal steps that we do not consider necessary to report, we obtain the:

(2,2) $$L^{p'}_{h'k'j'}\vartheta^p_{p'} = \vartheta^p_{h'k'j'} + L^p_{hk}(\vartheta^h_{h'j'}\vartheta^k_{k'} + \vartheta^h_{h'}\vartheta^k_{k'j'}) + \\ L^p_{hj}\vartheta^h_{h'k'}\vartheta^j_{j'} + L^p_{hkj}\vartheta^h_{h'}\vartheta^k_{k'}\vartheta^j_{j'}$$

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[1] Note that the connection induced on U, by virtue of Prop. I, does not depend on the fields $\bar{X}$, $\bar{Y}$, $\bar{S}$, which extend X, Y, S to M.

that express the way in which the 2nd order components of $\overset{(C)}{\nabla}_2$ vary as a result of a change in local coordinates.

Let's now introduce the functions $\tilde{L}^p_{hkj}$ through the:

$$\nabla_{e_h e_k}\omega^p = \tilde{L}^p_{hkj}\omega^j$$

and we apply (IV'$_2$) setting, in it, X = e$_h$, Y = e$_k$ and $\omega = \omega^p$, Z = e$_j$; we have:

$$(2{,}3) \qquad \tilde{L}^p_{hkj} = -L^p_{hkj} + L^p_{hq}L^q_{kj} + L^p_{kq}L^q_{qj}. \qquad \text{(cf. (3,4) of [1])}$$

More generally, by applying (III$_2$), we can obtain the components of $\nabla_{e_h e_k}$ T (T $= e_{i_1} \otimes e_{i_2} ... \otimes e_{i_r} \otimes \omega_{j_1} ... \otimes \omega_{j_s}$) expressed using the components of $\nabla_{e_h} e_k, \quad \nabla_{e_h e_k} e_j$:

A c.c. $\overset{(C)}{\nabla}_2$ *is thus identified with the only components:* L^i_{hk} (1*st order*), L^i_{hkj} (2*nd order*).

The observation made at the end of n. 3 of [1] regarding the possibility of assigning (on M) a a.c. in a constructive way, that is, knowing in every open of a covering the related components, immediately extends to the $\overset{(C)}{\nabla}_2$.

3. Tensor fields determined by composite connections.

Consider the following operator:

$$(3{,}1) \qquad K(X,\ Y) = \nabla_{XY} - \nabla_X \nabla_Y, \qquad X,\ Y \in \mathcal{D}^1(M);$$

it is easy to verify, according to the norms of axioms: (2,1) of [1] and (1,1), that it is:

$$K(fX,\ gY)(hT) = fgh\{K(X,\ Y)T\},$$

for any f, g, h $\in \mathcal{F}$, $\quad X,\ Y \in \mathcal{D}^1(M), \quad T \in \mathcal{D}^r_s(M)$; moreover K(X, Y)T is linear (cf. (I$_2$)) in each of its arguments.

The application:

$$K : \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to E$$

with E being the space of endomorphisms of $\mathcal{D}^r_s$(M) (considered as an $\mathcal{F}$-module), has a tensorial character.

The tensor field K(X, Y) (with values in E), allows us to construct the connection $\overset{(C)}{\nabla}_2$ starting from the a.c. ∇; K(X, Y) one can say, for this:*fundamental* *tensor of the* $\overset{(C)}{\nabla}_2$.

It is therefore valid, in general, for the 2nd order operator of a c.c., the following decomposition:

$$\nabla_{XY} = \nabla_X \nabla_Y + K(X,\ Y) \tag{3,1'}$$

which constitutes the analogue of the decomposition of a generic derivation (of the 1st order) of $\mathcal{D}$ (M). (2)

It is obvious that:

$$K(X,\ Y) = 0$$

if and only if:

$$\nabla_{XY} = \nabla_X \nabla_Y,$$

in that case, the $\overset{(C)}{\nabla}_2$ is said to be, for obvious reasons, *deduced* from the ∇.

Once is chosen $\omega \in \mathcal{D}^s_r$ (M), one can proceed to the following application

($\mathcal{D}^s_r$(M) dual of $\mathcal{D}^r_s$ (M)):

$$(\omega,\ X,\ Y,\ T) \to \omega\{(K(X,\ Y)T).\}$$

of $\mathcal{D}^s_r \times \mathcal{D}^1 \times \mathcal{D}^1 \times \mathcal{D}^r_s$ in $\mathcal{F}$; such an application, $\mathcal{F}$-multilinear,

is an element of $\mathcal{D}^{r+s}_{r+s+2}$ and it is:

the fundamental tensor field (associated with K and taking values in $\mathcal{F}$) relative to $\overset{(C)}{\nabla}_2$.

Let's now consider the application:

$$R : \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to E$$

with:

$$R(X,\ Y) = \nabla_{XY} - \nabla_{YX} - \nabla_{[X,\ Y]}; \tag{3,2}$$

R(X, Y) *it is a tensor field, with values in E, that can be called: curvature tensor of the* $\overset{(C)}{\nabla}_2$.(3)

Considering then the curvature tensor B of the a.c.:

$$B(X,\ Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}, \tag{3,3}$$

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[2] cf. [4] pag. 124.

[3] The same designation can be used for the fields R(X, Y)T and: $(\omega,\ X,\ Y,\ T) \to \omega\{R(X,\ Y)T\},\ \ \omega \in \mathcal{D}^s_r(M),\ \ X,\ Y \in \mathcal{D}^1(M),\ \ T \in \mathcal{D}^r_s(M)$.

it is easy to deduce the relationshipe:

$$R(X,\ Y) = B(X,\ Y) + 2L(X,\ Y) \tag{3,4}$$

with:

$$2L(X,\ Y) = K(X,\ Y) - K(Y,\ X). \tag{3,5}$$

The (3,4) gives the decomposition of the curvature tensor of $\overset{(C)}{\nabla}_2$ through the curvature tensor of the a.c. ∇ and the fundamental tensor K(X, Y) (hemisymmetrized).

In particular:

$$R(X,\ Y) = B(X,\ Y)$$

if and only if K(X, Y) is symmetric: the equality of R(X, Y) and B(X, Y) does not require that $\overset{(C)}{\nabla}_2$ be derived.

4. Tensorial relations between covariant derivative and "composite derivative".

By covariant derivation we will mean the operation constrained to the use of the ∇_X operator (possibly applied repeatedly) and by (covariant) composition, the operation that involves both ∇_X and ∇_{XY}.

Let's first consider the following case:

$$\nabla_{XY}\omega, \qquad \omega \in \mathcal{D}_1(M).$$

If we write (IV'$_1$) with Y and Z instead of X and Y, we have:

$$Y\omega(Z) = \nabla_Y\omega(Z) + \omega(\nabla_Y Z);$$

from here, operating with X on both members and invoking the same (IV'$_1$) (4) we obtain:

$$XY\omega(Z) = \nabla_X\nabla_Y\omega(Z) + \nabla_X\omega(\nabla_Y Z) + \nabla_Y\omega(\nabla_X Z) + \omega(\nabla_X\nabla_Y Z);$$

by subtraction from (IV'$_2$) we have:

$$(\nabla_{XY}\omega - \nabla_X\nabla_Y\omega)(Z) = -\omega(K(X,\ Y)Z) \quad Z \text{ (any)} \tag{4,1}$$

Let's now introduce the application:

$$\tilde{K} : \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to \tilde{E}$$

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[4] That is, keeping in mind that it is: $X\nabla_Y\omega(Z) = \nabla_X\nabla_Y\omega(Z) + \nabla_Y\omega(\nabla_X Z), ...$

(with $\tilde{E}$ the space of endomorphisms of $\mathcal{D}_1$(M)) thus defined: ([5])

$$< \omega, K(X,\ Y)Z >=< \tilde{K}(X,\ Y)\omega, Z >,$$

($\tilde{K}$ endomorphism transport of K) and let's go to substitute in (4,1); it follows the:

(4,1') $$\nabla_{XY}\omega = \nabla_X\nabla_Y\omega - \tilde{K}(X,\ Y)\omega.$$

We now take another fundamental step for the continuation of the calculations of the sought relationships; let's consider (III$_2$) under the hypothesis T = Z $\in \mathcal{D}^1(M)$, $\quad S = \omega \in \mathcal{D}_1(M)$ subtracting the:

$$\nabla_X\nabla_Y(Z\otimes\omega) = (\nabla_X\nabla_Y Z)\otimes\omega + \nabla_X Z\otimes\nabla_Y\omega) + \nabla_Y Z\otimes\nabla_X\omega + Z\otimes\nabla_X\nabla_Y\omega$$

(easily deducible from (III$_2$)), it is derived:

$$\nabla_{XY}(Z\otimes\omega) = (\nabla_{XY} Z)\otimes\omega + \nabla_X Z\otimes\nabla_Y\omega) + \nabla_Y Z\otimes\nabla_X\omega + Z\otimes\nabla_{XY}\omega$$

$$\nabla_X\nabla_Y(Z\otimes\omega) = (\nabla_X\nabla_Y Z)\otimes\omega + \nabla_X Z\otimes\nabla_Y\omega) + \nabla_Y Z\otimes\nabla_X\omega + Z\otimes\nabla_X\nabla_Y\omega$$

(4,2) $$(\nabla_{XY} - \nabla_X\nabla_Y)(Z\otimes\omega) = \{K(X,\ Y)Z\}\otimes\omega - Z\otimes\{\tilde{K}(X,\ Y)\omega\}$$

that is:

(4,2') $$\nabla_{XY}(Z\otimes\omega) = \nabla_X\nabla_Y(Z\otimes\omega) + \{K(X,\ Y)Z\}\otimes\omega - Z\otimes\{\tilde{K}(X,\ Y)\omega.$$

According to what has been stated, appears evident the it possibility of obtaining:

$$\nabla_{XY}(X_{i_1}\otimes X_{i_2}...\otimes X_{i_r}\otimes\omega^{j_1}\otimes...\otimes\omega^{j_s})$$

and more generally:

$$\nabla_{XY}T, \quad T\in\mathcal{D}^r_s(M)$$

expressed by means of: $\nabla_X\nabla_Y T, \quad K(X,\ Y), \quad \tilde{K}(X,\ Y)$ that is the relationships that we set out to determine.

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[5] As is customary: $< \omega, K(X,\ Y)Z >= \omega\{K(X,\ Y)Z\}$

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