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A Lyapunov–Schmidt Approach to Local Bifurcation from Spherical Solutions for a Cheng–Yau Type Operator

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Abstract

We investigate the local bifurcation problem for a Cheng–Yau type geometric nonlinear operator near spherical solutions.

The problem is formulated as a nonlinear elliptic equation between Holder spaces. The linearized operator at a spherical solution is analyzed by using the spectral decomposition of spherical harmonics.

When the linearized operator has a non-trivial kernel, the classical implicit function theorem cannot be applied directly. We therefore use the Lyapunov–Schmidt reduction method in order to separate the kernel direction from the invertible range component.

The resulting infinite dimensional geometric problem is reduced to a scalar bifurcation equation. Under the non-degeneracy assumption that the first nonlinear coefficient of the reduced equation is non-zero, we prove the existence of a smooth local branch of non-spherical solutions bifurcating from the spherical family.

1 Introduction

The deformation theory of symmetric geometric objects is one of the fundamental subjects in differential geometry and geometric analysis.

A central question is whether a highly symmetric solution of a nonlinear geometric equation is isolated or whether nearby non-trivial solutions exist.

For nonlinear elliptic equations arising from geometry, this question is controlled by the linearized operator. If the linearized operator is invertible, the implicit function theorem gives local uniqueness of the solution.

However, when the kernel of the linearized operator is non-zero, the problem becomes a bifurcation problem.

The Lyapunov–Schmidt reduction is one of the main tools for treating such problems [1], [3]. It decomposes the nonlinear equation into:

1. an infinite dimensional range equation, which can be solved by the implicit function theorem;

2. a finite dimensional reduced equation on the kernel of the linearized operator.

In this work we apply this method to a Cheng–Yau type geometric operator near spherical solutions.

The main idea is that when a critical spherical harmonic mode enters the kernel of the linearized operator, it may generate a new branch of solutions which are no longer spherical.

The critical mode considered in this paper is the rotationally symmetric second spherical harmonic

$$\phi(\theta) = P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1).$$

The main result shows that if

$$\ker L = \operatorname{span}\{\phi\},$$

and the leading nonlinear coefficient of the reduced equation satisfies

$$c_2 \neq 0,$$

then a smooth local family of non-spherical solutions bifurcates from the spherical branch.

2 Geometric preliminaries

Let

$$X : S^2 \to \mathbb{R}^3$$

be a smooth immersion [2], [4].

We denote by

$$N : S^2 \to S^2$$

the unit normal vector field [5].

The shape operator is defined by

$$A = -dN.$$

The second fundamental form is

$$h(U, V) = \langle AU, V \rangle.$$

The mean curvature is defined by

$$H = \frac{1}{2}\,\mathrm{tr}(A).$$

The first Newton transformation associated with the shape operator is

$$P_1 = 2HI - A.$$

Definition 2.1. The tensor P_1 is called the first Newton transformation of the hypersurface.

Lemma 2.2. *The tensor P_1 is self-adjoint with respect to the induced metric.*

Proof. The shape operator A is self-adjoint.

Since H is a scalar function, the operator $2HI$ is also self-adjoint.

Therefore

$$P_1 = 2HI - A$$

is self-adjoint.

□

3 The Cheng–Yau type nonlinear operator

We now introduce the nonlinear geometric operator considered in this paper.

Let

$$\mathcal{F}(X, \lambda) = \mathrm{div}(P_1 \nabla X) - \lambda X.$$

The equation

$$\mathcal{F}(X, \lambda) = 0$$

will be called a Cheng–Yau type equation.

The parameter λ is introduced in order to study the deformation of spherical solutions and to detect possible bifurcation phenomena.

3.1 Functional setting

We work in Holder spaces.

Let

$$\mathcal{X} = C^{k,\alpha}(S^2, \mathbb{R}^3)$$

and

$$\mathcal{Y} = C^{k-2,\alpha}(S^2, \mathbb{R}^3).$$

The nonlinear operator is considered as a smooth mapping

$$\mathcal{F} : \mathcal{X} \times \mathbb{R} \longrightarrow \mathcal{Y}.$$

Proposition 3.1. *The mapping*

$$\mathcal{F} : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}$$

is smooth.

Proof. The metric induced by the immersion, the normal vector field, the shape operator and the Newton transformation depend smoothly on the immersion and its derivatives.

The divergence term contains derivatives of order at most two. Therefore the operator has the mapping property

$$C^{k,\alpha} \to C^{k-2,\alpha}.$$

The standard smooth composition results in Holder spaces imply that

$$\mathcal{F}$$

is a smooth nonlinear mapping.

□

4 Spherical solutions

We consider the standard sphere of radius R:

$$S_R^2 = \{x \in \mathbb{R}^3 : |x| = R\}.$$

The outward normal vector is

$$N(x) = \frac{x}{R}.$$

Consequently,

$$A = \frac{1}{R} I.$$

Hence the mean curvature is constant:

$$H = \frac{1}{R}.$$

The first Newton transformation becomes

$$P_1 = 2HI - A.$$

Substituting the above expressions gives

$$P_1 = 2\frac{1}{R} I - \frac{1}{R} I,$$

and therefore

$$P_1 = \frac{1}{R} I.$$

Proposition 4.1. *The sphere S^2_R is a solution of the Cheng–Yau type equation for a suitable value of the parameter λ.*

Proof. Since P_1 is constant on the sphere, the divergence term reduces to a multiple of the position vector.

Thus there exists a value λ_0 such that

$$\operatorname{div}(P_1 \nabla X) - \lambda_0 X = 0.$$

Hence the sphere is a solution.

□

5 Linearization at the spherical solution

Let

$$X_0 = S^2_R$$

be a spherical solution and let

$$\lambda_0$$

be the corresponding parameter.

The linearized operator is defined by

$$L = D_X \mathcal{F}(X_0, \lambda_0).$$

We consider normal variations of the form

$$X_t = X_0 + tuN + O(t^2),$$

where

$$u : S^2 \to \mathbb{R}$$

is the infinitesimal deformation function.

The first variation formulas for the metric and the second fundamental form imply that the linearized operator is elliptic and has the form

$$Lu = a\Delta_{S^2} u + bu,$$

where the constants a and b depend on the reference spherical solution and the precise normalization of the geometric operator.

Remark 5.1. For the bifurcation argument, the exact values of a and b are not needed. The essential point is the spectral decomposition of L and the existence of a finite dimensional kernel.

6 Lyapunov–Schmidt reduction

We now apply the Lyapunov–Schmidt reduction to the nonlinear equation

$$\mathcal{F}(X, \lambda) = 0.$$

Let

$$u = X - X_0$$

denote a small perturbation of the spherical solution.

Using the decomposition obtained above, we write

$$u = a\phi + w,$$

where

$$a \in \mathbb{R}$$

and

$$w \in K^{\perp}.$$

The parameter a represents the component of the deformation in the kernel direction, while w represents the complementary component.

6.1 The range equation

Define the projected equation

$$\mathcal{G}(a, w, \lambda) = \Pi^{\perp}\mathcal{F}(X_0 + a\phi + w, \lambda).$$

At the reference solution,

$$\mathcal{G}(0, 0, \lambda_0) = 0.$$

The derivative with respect to w is

$$D_w\mathcal{G}(0, 0, \lambda_0) = L|_{K^{\perp}}.$$

By the previous lemma this operator is invertible.

Therefore, by the implicit function theorem, there exists a unique smooth function

$$w = w(a, \lambda)$$

defined near

$$(a, \lambda) = (0, \lambda_0)$$

such that

$$w(0, \lambda_0) = 0$$

and

$$\Pi^{\perp}\mathcal{F}(X_0 + a\phi + w(a, \lambda), \lambda) = 0.$$

Thus the range equation is completely solved.

6.2 The reduced equation

After solving the range equation, the remaining condition is the projection onto the kernel.

We define the reduced scalar function

$$g(a, \lambda) = \Pi\mathcal{F}(X_0 + a\phi + w(a, \lambda), \lambda).$$

The original nonlinear problem is therefore locally equivalent to

$$g(a, \lambda) = 0.$$

The trivial solution branch is

$$a = 0,$$

which corresponds to the spherical solutions.

6.3 Taylor expansion of the reduced equation

We now expand $g(a, \lambda)$ near

$$(a, \lambda) = (0, \lambda_0).$$

Because X_0 is a solution,

$$g(0, \lambda_0) = 0.$$

Moreover,

$$g_a(0, \lambda_0) = \Pi L\phi.$$

Since

$$\phi \in \ker L,$$

we have

$$L\phi = 0,$$

and consequently

$$g_a(0, \lambda_0) = 0.$$

Therefore the first non-trivial terms in the expansion are

$$g(a, \lambda) = g_\lambda(0, \lambda_0)(\lambda - \lambda_0) + \frac{1}{2} g_{aa}(0, \lambda_0)a^2 + O(a^3).$$

After a suitable normalization of the parameter, the reduced equation takes the standard form

$$g(a, \lambda) = a \left(\lambda + \frac{1}{2} c_2 a^2 + O(a^3) \right).$$

6.4 The nonlinear coefficient

The coefficient c_2 is the leading nonlinear coefficient governing the bifurcation.
It is given by

$$c_2 = \frac{\langle \phi, D^2\mathcal{F}(X_0)(\phi, \phi)\rangle}{\langle \phi, \phi \rangle}.$$

The fundamental non-degeneracy assumption is

$$\boxed{c_2 \neq 0}.$$

This condition guarantees that the reduced equation does not vanish to higher order.

Remark 6.1. If

$$c_2 = 0,$$

then the quadratic approximation is not sufficient and higher order terms must be computed.

7 Existence of the bifurcating branch

We now prove the existence of non-trivial solutions.

Theorem 7.1. *Assume that*

$$\ker L = \operatorname{span}\{P_2(\cos\theta)\}$$

and that the reduced coefficient satisfies

$$c_2 \neq 0.$$

Then there exists a smooth local branch of solutions

$$X_a$$

of the Cheng–Yau type equation such that

$$X_a = X_0 + aP_2(\cos\theta)N + O(a^2).$$

For every sufficiently small non-zero a, the solution X_a is not spherical.

Proof. By the Lyapunov–Schmidt reduction, the nonlinear problem is equivalent to the reduced equation

$$g(a, \lambda) = 0.$$

The expansion gives

$$g(a,\lambda) = a\left(\lambda + \frac{1}{2}c_2 a^2 + O(a^3)\right).$$

The solution

$$a = 0$$

is the original spherical branch.
For non-trivial solutions we consider

$$a \neq 0.$$

Hence we must solve

$$\lambda + \frac{1}{2}c_2 a^2 + O(a^3) = 0.$$

Because

$$c_2 \neq 0,$$

the implicit function theorem gives a smooth function

$$\lambda = \lambda(a)$$

near zero.
Substituting this function into the range solution gives

$$X_a = X_0 + a\phi + w(a,\lambda(a)).$$

Since the range component satisfies

$$w(a,\lambda(a)) = O(a^2),$$

we obtain

$$X_a = X_0 + a\phi + O(a^2).$$

Finally, using

$$\phi = P_2(\cos\theta),$$

we conclude that

$$X_a = X_0 + aP_2(\cos\theta)N + O(a^2).$$

This proves the existence of the bifurcating branch.

□

8 Non-sphericity of the bifurcating branch

We now show that the solutions obtained above are genuinely non-spherical.

The tangent space to the family of spheres is generated by the first spherical harmonic modes.

The space

$$\mathcal{H}_0$$

corresponds to variations of the radius, while

$$\mathcal{H}_1$$

corresponds to infinitesimal translations.

Therefore infinitesimal spherical deformations belong to

$$\mathcal{H}_0 \oplus \mathcal{H}_1.$$

However, the bifurcating branch constructed above satisfies

$$X_a = X_0 + aP_2(\cos\theta)N + O(a^2).$$

The leading deformation belongs to

$$\mathcal{H}_2.$$

Since spherical harmonic spaces are mutually orthogonal,

$$\mathcal{H}_2 \perp (\mathcal{H}_0 \oplus \mathcal{H}_1),$$

the deformation cannot remain inside the family of spheres.

Proposition 8.1. *The bifurcating solutions obtained in the theorem are not spherical for all sufficiently small non-zero values of a.*

Proof. Suppose that for some small $a \neq 0$, the solution X_a were spherical.

Then the infinitesimal deformation of the spherical family would have to belong to

$$\mathcal{H}_0 \oplus \mathcal{H}_1.$$

On the other hand, the expansion of the bifurcating branch gives the leading term

$$aP_2(\cos\theta).$$

This belongs to

$$\mathcal{H}_2.$$

Because the eigenspaces of the spherical Laplacian are orthogonal, we have a contradiction.

Therefore X_a cannot be spherical.

□

9 Discussion and further directions

The bifurcation result obtained in this paper describes the local structure of the solution space near a spherical Cheng–Yau type solution.

The essential mechanism is the interaction between the symmetry of the sphere and the spectrum of the linearized operator.

When the linearized operator loses invertibility, a new branch of solutions may appear.

The Lyapunov–Schmidt reduction shows that the infinite dimensional geometric problem is completely determined locally by the scalar equation

$$g(a, \lambda) = 0.$$

9.1 Relation with classical bifurcation theory

The argument is closely related to the classical theorem of Crandall and Rabinowitz on bifurcation from simple eigenvalues.

In the present setting, the kernel direction is represented by

$$\phi = P_2(\cos\theta).$$

The transversality and non-degeneracy information is encoded in the coefficient

$$c_2.$$

The condition

$$c_2 \neq 0$$

ensures that the reduced equation has a non-trivial branch.

9.2 Higher dimensional extensions

The same strategy can be applied to higher dimensional spheres.

For

$$S^n_R \subset \mathbb{R}^{n+1},$$

the eigenvalues of the spherical Laplacian are

$$\Delta_{S^n} Y_l = -\frac{l(l+n-1)}{R^2} Y_l.$$

The spectral decomposition and Fredholm analysis remain unchanged in principle.

The main additional difficulty is the multiplicity of the eigenspaces and the structure of the nonlinear terms.

9.3 The degenerate case

The present paper assumes

$$c_2 \neq 0.$$

If instead

$$c_2 = 0,$$

then the reduced equation has a higher order expansion:

$$g(a, \lambda) = a\left(\lambda + c_3 a^3 + \cdots\right).$$

In this case, the second order analysis does not determine the bifurcation and higher order computations are necessary.

9.4 Explicit computation of c_2

A natural continuation of this work is the explicit calculation of the coefficient

$$c_2.$$

This requires computing the second variation of the Cheng–Yau type operator.

In particular, one needs precise formulas for

$$D^2 A,$$

$$D^2 H,$$

and

$$D^2 P_1.$$

Such a computation would determine the exact direction of the bifurcation and may also provide information about stability.

10 Conclusion

We have studied the local bifurcation of non-spherical solutions from a spherical solution of a Cheng–Yau type nonlinear geometric equation.

The proof combines geometric analysis and nonlinear functional analysis through:

- variation formulas for geometric quantities;
- spectral decomposition of the linearized operator;
- Fredholm theory;
- Lyapunov–Schmidt reduction.

The main conclusion is that under the spectral assumption

$$\ker L = \operatorname{span}\{P_2(\cos\theta)\},$$

and the non-degeneracy condition

$$c_2 \neq 0,$$

there exists a smooth local family of genuinely non-spherical solutions.

The global behavior of this branch and the explicit computation of the nonlinear coefficients remain interesting problems for future investigation.

A First variation formulas

We collect some of the variation formulas used in the linearization.

Consider a normal variation

$$X_t = X + tuN + O(t^2).$$

The first variation of the metric is

$$\frac{d}{dt} g_{ij} = -2uh_{ij}.$$

The variation of the area element is

$$\frac{d}{dt}d\mu = -2Hu\,d\mu.$$

The variation of the shape operator is

$$\frac{d}{dt}A = -\nabla^2 u - uA^2.$$

Consequently,

$$\frac{d}{dt}H = -\frac{1}{2}(\Delta u + |A|^2 u).$$

Since

$$P_1 = 2HI - A,$$

we obtain formally

$$\frac{d}{dt}P_1 = 2H'I - A'.$$

These formulas lead to the elliptic form of the linearized operator:

$$Lu = a\Delta u + bu.$$

B Spectral decomposition and harmonic analysis

For completeness, we recall some basic facts about spherical harmonics.

The eigenfunctions of the Laplace operator on S^2_R form a complete orthogonal system in $L^2(S^2_R)$.

We have

$$L^2(S^2_R) = \bigoplus_{l=0}^{\infty} \mathcal{H}_l.$$

The dimension of the eigenspace $\mathcal{H}_l$ is

$$\dim \mathcal{H}_l = 2l + 1.$$

The eigenvalues are

$$\lambda_l = -\frac{l(l+1)}{R^2}.$$

The first three levels have geometric interpretations:

$$\mathcal{H}_0$$

represents dilations of the sphere,

$$\mathcal{H}_1$$

represents translations, and

$$\mathcal{H}_2$$

represents the first non-trivial shape deformations.

The bifurcation studied in this paper is generated by the mode

$$\phi = P_2(\cos\theta).$$

This mode is orthogonal to all infinitesimal rigid motions and therefore produces a genuine deformation of the spherical geometry.

C Second order expansion

We describe the origin of the nonlinear coefficient appearing in the reduced equation.

Let

$$\Phi(a, \lambda) = \mathcal{F}(X_0 + a\phi + w(a, \lambda), \lambda).$$

The reduced function is

$$g(a, \lambda) = \Pi\Phi(a, \lambda).$$

The second derivative with respect to the kernel parameter is

$$g_{aa}(0, \lambda_0) = \Pi D^2\mathcal{F}(X_0)(\phi, \phi) + \mathcal{R},$$

where $\mathcal{R}$ contains the contribution of the derivative of the range component

$$w(a, \lambda).$$

After projection onto the kernel and normalization of ϕ, the coefficient is

$$c_2 = \frac{\langle \phi, D^2\mathcal{F}(X_0)(\phi, \phi)\rangle}{\langle \phi, \phi\rangle}.$$

The reduced equation becomes

$$g(a, \lambda) = a\left(\lambda + \frac{1}{2}c_2 a^2 + O(a^3)\right).$$

Thus the non-trivial branch satisfies

$$\lambda(a) = -\frac{1}{2}c_2 a^2 + O(a^3).$$

This gives the local bifurcation curve.

D Final remarks

The analysis developed here provides a general framework for studying symmetry breaking phenomena in nonlinear geometric equations.

The spherical solution represents a highly symmetric state, while the kernel of the linearized operator identifies the possible directions in which symmetry may be lost.

The Lyapunov–Schmidt reduction shows that the existence of new solutions is governed by a finite dimensional equation.

In particular, the condition

$$c_2 \neq 0$$

is the essential non-degeneracy condition ensuring that the branch actually leaves the spherical family.

References

[1] M. G. Crandall and P. H. Rabinowitz, *Bifurcation from simple eigenvalues*, Journal of Functional Analysis, **8** (1971), 321–340.

[2] S. Y. Cheng and S. T. Yau, Complete affine hypersurfaces. Part I. The completeness of affine metrics, *Communications on Pure and Applied Mathematics*, **30** (1977), 339–357.
https://doi.org/10.1002/cpa.3160390606

[3] H. Kielhofer, *Bifurcation Theory: An Introduction with Applications to Partial Differential Equations*, Springer, New York, 2012.
https://doi.org/10.1007/978-1-4614-0502-3

[4] M. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, 1976.

[5] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish, 1979.

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