

The Inequalities for Volume Sums of Mixed Complex Intersection Bodies

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Abstract

Recently, Koldobsky, Paouris and Zymonopoulou introduced the notion of complex intersection bodies. From then, a number of important geometric inequalities for them are established. In the present paper we prove several new isoperimetric type inequalities for volume sums of mixed complex intersection bodies.

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1 Introduction

The concept of an intersection body was introduced by Lutwak [12] in 1988. Let K and L be origin symmetric star bodies in R^n . We say that K is the intersection body of L and write $K = I(L)$ if the radius of K in every direction is equal to the volume of the central hyperplane section of L perpendicular to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\|\xi\|_{I(L)}^{-1} = V_{n-1}(L \cap \xi^\perp).$$

Intersection bodies were used to solve the important Busemann-Petty problem (see [3], [4], [5], [8], [17]).

The study of complex convex bodies just appears recent years, and results appear only occasionally (see, e.g., [1, 7, 9, 10, 15, 16, 19]). To formulate the

complex version, we need several definitions. Origin-symmetric convex bodies in C^n are the unit balls of norms on C^n . We denote by $\|\cdot\|_K$ the norm corresponding to the body K :

$$K = \{z \in C^n : \|\cdot\|_K \leq 1\}.$$

We identify C^n with R^{2n} using the standard mapping

$$\xi = (\xi_1, \dots, \xi_n) = (\xi_{11} + i\xi_{12}, \dots, \xi_{n1} + i\xi_{n2}) \mapsto (\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n2}).$$

Since norms on C^n satisfy the equality

$$\|\lambda z\| = |\lambda| \|z\|, \quad \forall z \in C^n, \forall \lambda \in C,$$

origin symmetric complex convex bodies correspond to those origin symmetric convex bodies K in R^{2n} that are invariant with respect to any coordinate-wise two-dimensional rotation, namely for each $\theta \in [0, 2\pi]$ and each $(\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n2}) \in R^{2n}$

$$\|\xi\|_K = \|R_\theta(\xi_{11}, \xi_{12}), \dots, R_\theta(\xi_{n1}, \xi_{n2})\|_K, \quad (1.1)$$

where R_θ stands for the counterclockwise rotation of R^2 by the angle θ with respect to the origin. We shall say that K is a complex convex body in R^{2n} if K is a convex body and satisfies equations (1.1).

For $\xi \in C^n$, $|\xi| = 1$, denote by

$$H_\xi = \{z \in C^n : (z, \xi) = \sum_{k=1}^n z_k \bar{\xi}_k = 0\},$$

the complex hyperplane through the origin, perpendicular to ξ . Under the standard mapping from C^n to R^{2n} , the hyperplane H_ξ turns into a $(2n-2)$ -dimensional subspace of R^{2n} orthogonal to the vectors

$$\xi = (\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n2}) \quad \text{and} \quad \xi^\perp = (-\xi_{12}, \xi_{11}, \dots, -\xi_{n2}, \xi_{n1}).$$

The orthogonal two-dimensional subspace $H_\xi^\perp$ has orthonormal basis $\{\xi, \xi^\perp\}$. A star body K in R^{2n} is a complex star body if and only if, for every $\xi \in S^{2n-1}$, the section $K \cap H_\xi^\perp$ is a two-dimensional Euclidean circle with radius $\rho_K(\xi) = \|\xi\|_K^{-1}$.

Given a convex body $C \subset C$ and convex bodies $K_1, \dots, K_{2n} \subset C^n$, the mixed projection body $\Pi^C(K_1, \dots, K_{2n})$ in a complex vector space was defined by Abardia and Bernig in [1]. Moreover, they established the Minkowski, Brunn-Minkowski and Aleksandrov-Fenchel type inequality for mixed complex projection bodies.

Theorem A^[1]. *Let K and L be convex bodies in C^n and $C \subset C$ is a convex subset and is not a point. Then*

$$V(\Pi_1^C(K, L))^{2n-1} \geq V(\Pi^C K)^{2n-2} V(\Pi^C L), \quad (1.2)$$

with equality if and only if K and L are homothetic.

Theorem B^[1]. Let K and L be convex bodies in C^n and $C \subset C$ is a convex subset and is not a point. Then

$$V(\Pi^C(K + L))^{\frac{1}{2n(2n-1)}} \geq V(\Pi^C K)^{\frac{1}{2n(2n-1)}} + V(\Pi^C L)^{\frac{1}{2n(2n-1)}}, \quad (1.3)$$

equality if and only if K and L are homothetic.

Theorem C^[1]. Let K_i be convex bodies in C^n , ($i = 1, \dots, 2n-1$) and $C \subset C$ is a convex subset and is not a point. If $2 \leq r \leq 2n-2$, then

$$V(\Pi^C(K_1, \dots, K_{2n-1}))^r \geq \prod_{j=1}^r V(\Pi^C(\underbrace{K_j, \dots, K_j}_r, K_{r+1}, \dots, K_{2n-1})).$$

Koldobsky, Paouris and Zymonopoulou [10] firstly introduced the complex intersection body and considered the complex Busemann-Petty problem.

For mixed complex intersection operator, Wang [15] established the following dual Minkowski, Brunn-Minkowski type inequalities for complex intersection bodies:

Theorem D^[15]. Let K and L be star bodies in C^n . If $0 \leq i \leq 2n-2$, $1 \leq j \leq 2n-3$, then

$$\widetilde{W}_i(I_j^C(K, L))^{2n-2} \leq \widetilde{W}_i(I^C K)^{2n-j-2} \widetilde{W}_i(I^C L)^j, \quad (1.4)$$

with equality if and only if K and L are dilates.

Theorem E^[15]. Let K and L be star bodies in C^n . If $0 \leq i \leq 2n-2$, $0 \leq j \leq 2n-4$, then,

$$\widetilde{W}_i(I_j^C(K \widetilde{+}_2 L))^{\frac{4}{(2n-i)(2n-j-2)}} \leq \widetilde{W}_i(I_j^C K)^{\frac{4}{(2n-i)(2n-j-2)}} + \widetilde{W}_i(I_j^C L)^{\frac{4}{(2n-i)(2n-j-2)}}, \quad (1.5)$$

with equality if and only if K and L are dilates.

Wu [16] obtained the following Aleksandrov-Fenchel type inequalities for mixed complex intersection bodies in C^n :

Theorem F^[16]. Let $K_1, \dots, K_{2n-2}$ be star bodies in C^n . If $0 \leq i \leq 2n-1$, $2 \leq r \leq 2n-2$, then

$$\widetilde{W}_i(I^C(K_1, \dots, K_{2n-2}))^r \leq \prod_{j=1}^r \widetilde{W}_i(I^C(\underbrace{K_j, \dots, K_j}_r, K_{r+1}, \dots, K_{2n-2})), \quad (1.6)$$

with equality hold if and if K_i , ($i = 1, \dots, r$) are all dilations of each other.

In 2004, Leng [11] defined the volume difference function of convex bodies $D, L \in R^n$ where $D \subseteq L$, by

$$D_V(K, D) = V(K) - V(D)$$

and obtained Minkowski and Brunn-Minkowski type inequalities. Since this seminal paper, inequalities for differences of geometric functionals have become the focus of increased attention (see e.g., [13], [18], [19], [20], [21], [22]).

In 2005, the volume sum function was first introduced in [23] as follows: If K and D are star bodies, then the dual quermassintegral sum function of the star bodies K and D , $S_{\widetilde{W}_i}(K, D)$, is defined by

$$S_{\widetilde{W}_i}(K, D) = \widetilde{W}_i(K) + \widetilde{W}_i(D),$$

where $0 \leq i \leq n-1$. When $i=0$, we have

$$S_V(K, D) = V(K) + V(D),$$

which is called the volume sum function of the star bodies K and L .

The aim of this paper is to establish the following new Minkowski, Brunn-Minkowski and Aleksandrov-Fenchel type inequalities for volume sums of complex intersection bodies.

Theorem 1.1. *Let K , L and D be star bodies in C^n . Let D be a dilated copy of D' . If $0 \leq i \leq 2n-2$, $1 \leq j \leq 2n-3$, then*

$$\begin{aligned} & S_{\widetilde{W}_i}(I_j^C(K, L), I_j^C(D, D'))^{2n-2} \\ & \leq S_{\widetilde{W}_i}(I^C K, I^C D)^{2n-j-2} S_{\widetilde{W}_i}(I^C L, I^C D')^j, \end{aligned}$$

with equality if and only if K and L are dilates and $(\widetilde{W}_i(I^C K), \widetilde{W}_i(I^C L)) = \mu(\widetilde{W}_i(I^C D), \widetilde{W}_i(I^C D'))$, where μ is a constant.

Theorem 1.2. *Let K , L and D be star bodies in C^n . Let D be a dilated copy of D' . If $0 \leq i \leq 2n-2$, $0 \leq j \leq 2n-4$, then*

$$\begin{aligned} & S_{\widetilde{W}_i}(I_j^C(K \widetilde{+}_2 L), I_j^C(D \widetilde{+}_2 D'))^{\frac{4}{(2n-i)(2n-j-2)}} \\ & \leq S_{\widetilde{W}_i}(I_j^C K, I_j^C D)^{\frac{4}{(2n-i)(2n-j-2)}} + S_{\widetilde{W}_i}(I_j^C L, I_j^C D')^{\frac{4}{(2n-i)(2n-j-2)}}, \end{aligned}$$

with equality if and only if K and L are dilates and $(\widetilde{W}_i(I_j^C K), \widetilde{W}_i(I_j^C L)) = \lambda(\widetilde{W}_i(I_j^C D), \widetilde{W}_i(I_j^C D'))$, where λ is a constant.

Theorem 1.3. *Let $K_1, \dots, K_{2n-2}$ and $D_1, \dots, D_{2n-2}$ be star bodies in C^n . Let the bodies D_j , ($j = 1, \dots, r$), $2 \leq r \leq 2n-2$, be dilated copies of each other. If $0 \leq i \leq 2n-1$, then*

$$\begin{aligned} & \left[S_{\widetilde{W}_i}(I^C(K_1, \dots, K_{2n-2}), I^C(D_1, \dots, D_{2n-2})) \right]^r \\ & \leq \prod_{j=1}^r S_{\widetilde{W}_i}(I^C(\underbrace{K_j, \dots, K_j}_r, K_{r+1}, \dots, K_{2n-2}), I^C(\underbrace{D_j, \dots, D_j}_r, D_{r+1}, \dots, D_{2n-2})), \end{aligned}$$

with equality holds if and if $K_1, \dots, K_r$ are dilates and

$$\begin{aligned} & \left(\widetilde{W}_i(I^C(\underbrace{K_1, \dots, K_1}_r, K_{r+1}, \dots, K_{2n-2})), \dots, \widetilde{W}_i(I^C(\underbrace{K_r, \dots, K_r}_r, K_{r+1}, \dots, K_{2n-2})) \right) \\ &= \gamma \left(\widetilde{W}_i(I^C(\underbrace{D_1, \dots, D_1}_r, D_{r+1}, \dots, D_{2n-2})), \dots, \widetilde{W}_i(I^C(\underbrace{D_r, \dots, D_r}_r, D_{r+1}, \dots, D_{2n-2})) \right), \end{aligned}$$

where γ is a constant.

2 Preliminary Notes

A body is a compact set with nonempty interior. A set K is star-shaped with respect to the origin if every line through the origin which meets K does so in a closed line segment (see [14]). Let $K \subset R^n$ be a star shaped body (with respect to the origin), the radial function $\rho(K, \cdot) : R^n \setminus \{o\} \rightarrow R$ is defined by

$$\rho(K, x) = \rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}.$$

The Minkowski functional $\|\cdot\|_K : R^n \rightarrow R$ is defined by $\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}$. Obviously, $\rho_K(x) = \|x\|_K^{-1}$, for $x \in R^n \setminus \{o\}$. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If the Minkowski functional of a star body K in R^{2n} is R_θ -invariant (that is, satisfies equation (1.1)), then we say that K is a complex star body in C^n .

For K and L be star bodies in C^n and $\lambda_1, \lambda_2 \geq 0$ (not both zero), the radial sum $\lambda_1 \cdot K \tilde{+} \lambda_2 \cdot L$ is a star body whose radial function is given by (see [10], [15])

$$\rho(\lambda_1 \cdot K \tilde{+} \lambda_2 \cdot L, \cdot)^2 = \lambda_1 \rho(L_1, \cdot)^2 + \lambda_2 \rho(L_2, \cdot)^2. \quad (2.1)$$

Let $K_1, \dots, K_{2n}$ be star bodies in C^n , the dual mixed volume $\tilde{V}(K_1, \dots, K_{2n})$ has the following integral representation (see [15]):

$$\tilde{V}(K_1, \dots, K_{2n}) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_{2n}, u) du, \quad (2.2)$$

where du is the spherical Lebesgue measure on S^{2n-1} . We write $\tilde{V}_2(K_1, \dots, K_{2n-2}, L)$ for $\tilde{V}(K_1, \dots, K_{2n-2}, L, L)$, where K_i ($i = 1, 2, \dots, 2n-2$) appear once and L appear twice. For $i \geq 0, j \geq 0$ and $i+j \leq 2n$, we write $\tilde{W}_{i,j}(K, L)$ for the dual mixed volume $V(K, \dots, K, B, \dots, B, L, \dots, L)$, where K appears $2n-i-j$ times, B appears i times and L appears j times. The dual mixed volume $\tilde{W}_{i,j}(K, K)$ will be written as $\tilde{W}_i(K)$ and is called the dual i th quermassintegral of K .

Let $K_1, \dots, K_{2n-2}$ be star bodies in C^n . The mixed complex intersection body $I^C(K_1, \dots, K_{2n-2})$ is defined by

$$\rho(I^C(K_1, \dots, K_{2n-2}), \xi)^2 = \frac{1}{2(n-1)\pi} \int_{S^{2n-1} \cap H_\xi} \rho(K_1, u) \cdots \rho(K_{2n-2}, u) du,$$

where du is the spherical Lebesgue measure on S^{2n-1} and H_ξ is a $2n-2$ dimensional subspace of R^{2n} orthogonal to the vectors ξ and $\xi^\perp$.

Obviously, every mixed complex intersection body corresponds to an origins symmetric convex body R^{2n} which is invariant with respect to any coordinate-wise two dimensional rotation.

If $K_1 = \cdots = K_{2n-i-2} = K$ and $K_{2n-i-1} = \cdots = K_{2n-2} = L$, the mixed complex intersection body $I^C(K_1, \dots, K_{2n-2})$ is written as $I_i^C(K, L)$. If $L = B$, $I_i^C(K, L)$ is written as $I_i^C(K)$ and is called the i th complex intersection body of K . We simply write $I^C(K)$ rather than $I_0^C(K)$, which was first defined by Koldobsky, Paouris and Zymonopoulou [10].

3 Main Results

In this section, we complete the proofs of Theorems 1.1-1.3. For the proof of Theorem 1.1, we require the following lemma.

Lemma 3.1^[23]. *Let $a, b \geq 0$, $c, d > 0$, then for $0 < p < 1$, then*

$$a^p c^{1-p} + b^p d^{1-p} \leq (a+b)^p (c+d)^{1-p}, \quad (3.1)$$

with equality if and only if $a/b = c/d$.

Proof of Theorem 1.3. By inequality (1.6), we have for $2 \leq r \leq 2n-2$,

$$\widetilde{W}_i(I^C(K_1, \dots, K_{2n-2})) \leq \left[\prod_{j=1}^r \widetilde{W}_i(I^C(\underbrace{K_j, \dots, K_j}_r, K_{r+1}, \dots, K_{2n-2})) \right]^{\frac{1}{r}}. \quad (3.12)$$

Since D_j , ($j = 1, \dots, r$) are dilates of each other, it follows from inequality (1.6) that

$$\widetilde{W}_i(I^C(D_1, \dots, D_{2n-2})) = \left[\prod_{j=1}^r \widetilde{W}_i(I^C(\underbrace{D_j, \dots, D_j}_r, D_{r+1}, \dots, D_{2n-2})) \right]^{\frac{1}{r}}. \quad (3.13)$$

Combining (3.12) and (3.13), and using inequality (3.11), we obtain

$$\widetilde{W}_i(I^C(K_1, \dots, K_{2n-2})) + \widetilde{W}_i(I^C(D_1, \dots, D_{2n-2}))$$

$$\begin{aligned}
&\leq \left[\prod_{j=1}^r \widetilde{W}_i(I^C(\underbrace{K_j, \dots, K_j}_r, K_{r+1}, \dots, K_{2n-2})) \right]^{\frac{1}{r}} + \\
&\quad \left[\prod_{j=1}^r \widetilde{W}_i(I^C(\underbrace{D_j, \dots, D_j}_r, D_{r+1}, \dots, D_{2n-2})) \right]^{\frac{1}{r}} \\
&\leq \left[\prod_{j=1}^r \left[\widetilde{W}_i(I^C(\underbrace{K_j, \dots, K_j}_r, K_{r+1}, \dots, K_{2n-2})) + \widetilde{W}_i(I^C(\underbrace{D_j, \dots, D_j}_r, D_{r+1}, \dots, D_{2n-2})) \right] \right]^{\frac{1}{r}}.
\end{aligned}$$

Thus

$$\begin{aligned}
&S_{\widetilde{W}_i}(I^C(K_1, \dots, K_{2n-2}), I^C(D_1, \dots, D_{2n-2}))^r \\
&\leq \prod_{j=1}^r S_{\widetilde{W}_i}(I^C(\underbrace{K_j, \dots, K_j}_r, K_{r+1}, \dots, K_{2n-2}), I^C(\underbrace{D_j, \dots, D_j}_r, D_{r+1}, \dots, D_{2n-2})).
\end{aligned}$$

According to the equality conditions of inequalities (3.11) and (3.12), we see that equality holds if and only if $K_1, \dots, K_r$ are dilates and

$$\begin{aligned}
&\left(\widetilde{W}_i(I^C(\underbrace{K_1, \dots, K_1}_r, K_{r+1}, \dots, K_{2n-2})), \dots, \widetilde{W}_i(I^C(\underbrace{K_r, \dots, K_r}_r, K_{r+1}, \dots, K_{2n-2})) \right) \\
&= \gamma \left(\widetilde{W}_i(I^C(\underbrace{D_1, \dots, D_1}_r, D_{r+1}, \dots, D_{2n-2})), \dots, \widetilde{W}_i(I^C(\underbrace{D_r, \dots, D_r}_r, D_{r+1}, \dots, D_{2n-2})) \right),
\end{aligned}$$

where γ is a constant.

From Theorem 1.3 we directly obtain

Corollary 3.5. *Let K_i and D_i be star bodies in C^n , $1 \leq i \leq 2n-2$. If the bodies D_j , $(j = 1, \dots, r)$, $2 \leq r \leq 2n-2$, are dilates of each other, then*

$$\begin{aligned}
&\left[S_V(I^C(K_1, \dots, K_{2n-2}), I^C(D_1, \dots, D_{2n-2})) \right]^r \\
&\leq \prod_{j=1}^r S_V(I^C(\underbrace{K_j, \dots, K_j}_r, K_{r+1}, \dots, K_{2n-2}), I^C(\underbrace{D_j, \dots, D_j}_r, D_{r+1}, \dots, D_{2n-2})),
\end{aligned}$$

with equality holds if and if $K_1, \dots, K_r$ are dilates and

$$\begin{aligned}
&\left(V(I^C(\underbrace{K_1, \dots, K_1}_r, K_{r+1}, \dots, K_{2n-2})), \dots, V(I^C(\underbrace{K_r, \dots, K_r}_r, K_{r+1}, \dots, K_{2n-2})) \right) \\
&= \gamma \left(V(I^C(\underbrace{D_1, \dots, D_1}_r, D_{r+1}, \dots, D_{2n-2})), \dots, V(I^C(\underbrace{D_r, \dots, D_r}_r, D_{r+1}, \dots, D_{2n-2})) \right),
\end{aligned}$$

where γ is a constant.

4 Conclusion

In this paper, we prove the Minkowski, Brunn-Minkowski and Aleksandrov-Fenchel type inequalities for volume sums of mixed complex intersection bodies. We study a complex version of the inequalities for volume sums of mixed (real) intersection bodies. It seems that not much has been done in this direction.

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