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On the Recurrence of Tensor Products of Matrices

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Abstract

This article considers the recurrence of tensor products of matrices acting on finite-dimensional spaces over different numbers of fields, and in various forms.

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1 Introduction

Hypercyclicity is the most commonly studied notion in linear dynamics. Let X be an infinite-dimensional, separable Fréchet space, and let $T \in \mathcal{B}(X)$ be a linearly bounded operator on X. Then, the pair (X, T) is said to be a linear dynamical system. T is called a hypercyclic operator if there is a vector $x \in X$ such that $\overline{Orb(x, T)} := \{T^n x : n \geq 0\}$ is dense in X. In this case, $x \in X$ is said to be a hypercyclic vector for T. T is said to be topologically transitive if, for every pair (U, V) of non-empty open subsets of X, there exists an integer n such that $T^n(U) \cap V$ is non-empty. Because the space X is assumed to be separable, a simple Baire category argument shows that T is topologically transitive if and only if T is hypercyclic.

Research on linear systems dates back to the first half of the 20th century, when Birkhoff [1] showed that the translation operator $T_a : H(\mathbb{C}) \to H(\mathbb{C})$, where $a \neq 0$, defined by $T_a f(z) = f(z + a)$, is hypercyclic. This result was generalized by Godefroy and Shapiro [2], who proved that every operator on

$H(\mathbb{C}^N)$ that commutes with all translations, and is not a scalar multiple of identity, is chaotic. Following Birkhoff's 1929 result, Maclane [3] proved in 1952 that the differentiation operator acting on the space of entire functions is also hypercyclic. More precisely, he showed that the operator $D : H(\mathbb{C}) \to H(\mathbb{C})$, defined by $D(f) = f'$, is hypercyclic. Hypercyclicity on Banach spaces was researched by Rolewics [4] in 1969, and yielded the finding that λB is hypercyclic whenever $|\lambda| > 1$, where B is the unilateral backward shift on $l^p(1 \leq p < \infty)$ or c_0, and is defined by $B(x_0, x_1, x_2, \cdots) = (x_1, x_2, \cdots)$. A systematic investigation of hypercyclicity has been underway since the mid-1980s, and has attracted considerable attention from mathematicians around the world. For more results, see [5] and [6].

On the contrary, recurrence is a central notion in topological dynamics. This notion dates back to Poincare and Birkhoff, and refers to the existence of points in space in which parts of their orbits under a continuous map return to themselves. The notion of recurrence and variations of it were studied in the context of linear dynamics by [7]. Topological recurrence is a cornerstone of research on dynamical systems, and encapsulates the phenomenon of orbits returning to the neighborhoods of their initial points under continuous transformations. As a central theme in topological dynamics, it bridges the gap between the topological properties of spaces and the qualitative behavior of dynamical systems over time.

The dynamics of linear operators on a finite-dimensional space can be easily described by the Jordan decomposition theorem. Notably, there is no hypercyclic operator in finite-dimensional spaces (see [5, Theorem 2.58]). However, linear recurrence can occur on a finite-dimensional space (see [7, 8, 9, 10], etc.).

This article contributes to the ongoing discourse by investigating the recurrence of tensor products of matrices. Some results and definitions are summarized in Section 2, the recurrence of the tensor products of matrices in Jordan's form is considered in Section 3, while the recurrence of the tensor products of matrices in rational canonical form is considered in Section 4.

2 Preliminary Notes

Definition 2.1 ([7, Definition 1.1]). *Let X be a complex Banach space and $T : X \to X$ be a bounded linear operator acting on X. An operator T acting on X is then called recurrent if, for every open set $U \subset X$, there exists some $k \in \mathbb{N}$ such that*

$$U \cap T^{-k}U \neq \varnothing.$$

A vector $x \in X$ is called recurrent for T if there exists a strictly increasing sequence of positive integers $(k_n)_n$ such that

$$T^{k_n}x \to x,$$

as $n \to +\infty$. We denote by $Rec(T)$ the set of recurrent vectors for T.

Some spectral properties of recurrent operators are now summarized. We use $\sigma(T)$ to represent the spectrum of T, $\sigma_p(T)$ to denote the point spectrum of T, and $r(T)$ to represent its spectral radius.

Proposition 2.2 ([7, Proposition 2.9]). *Let $T : X \to X$ be an operator acting on X. If $r(T) < 1$, then T is not recurrent.*

Proposition 2.3 ([7, Proposition 2.11]). *If $T : X \to X$ is a recurrent operator, then every component of the spectrum of T, $\sigma(T)$, intersects with the unit circle $\mathbb{T}$.*

Theorem 2.4 ([7, Theorem 4.1]). *A matrix $T : \mathbb{C}^d \to \mathbb{C}^d$ is recurrent if and only if it is similar to a diagonal matrix with unimodular entries.*

Theorem 2.5 ([7, Theorem 4.3]). *A matrix $T : \mathbb{R}^d \to \mathbb{R}^d$ is recurrent if and only if it is similar to a block diagonal matrix of the form*

$$A = \begin{pmatrix} J_1 & 0 & \cdots & 0 & 0 \\ 0 & J_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{m-1} & 0 \\ 0 & 0 & \cdots & 0 & J_m \end{pmatrix}$$

where each J_j, $1 \leq j \leq m$ is either a 2×2 rotation matrix, or a 1×1 matrix with entries that are all either 1 or -1.

Definition 2.6. *Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$. The tensor product of A and B is then defined as the following block matrix:*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix}.$$

Proposition 2.7. $(A \otimes B)^T = A^T \otimes B^T; (A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Proposition 2.8. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

Theorem 2.9. *Assume that $A = (A_{ij})_{n\times m}$ is a block matrix. Then, for any matrix B, we have*

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1m}B \\ A_{21}B & A_{22}B & \cdots & A_{2m}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nm}B \end{pmatrix}.$$

Theorem 2.10. *Let $A = (A_{ij})_{n\times m}$ and $B = (B_{pq})_{s\times t}$ be two block matrices, where $A_{ij} \in M_{n_i,m_i}$, $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, m$; $B_{pq} \in M_{s_p,t_q}$, $p = 1, 2, \cdots, s$; $q = 1, 2, \cdots, t$. Then, there exist permutation matrices P and Q such that*

$$P(A\otimes B)Q = \begin{pmatrix} A_{11}\otimes B_{11} & \cdots & A_{11}\otimes B_{1t} & \cdots & A_{1m}\otimes B_{11} & \cdots & A_{1m}\otimes B_{1t} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{11}\otimes B_{s1} & \cdots & A_{11}\otimes B_{st} & \cdots & A_{1m}\otimes B_{s1} & \cdots & A_{1m}\otimes B_{st} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{n1}\otimes B_{11} & \cdots & A_{n1}\otimes B_{1t} & \cdots & A_{nm}\otimes B_{1t} & \cdots & A_{nm}\otimes B_{1t} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n1}\otimes B_{s1} & \cdots & A_{n1}\otimes B_{st} & \cdots & A_{nm}\otimes B_{st} & \cdots & A_{nm}\otimes B_{st} \end{pmatrix}.$$

3 Recurrence of Tensor Product of Matrices in Jordan's Form

Before characterizing the recurrence of the tensor products of matrices in different cases, we present a necessary condition for $T_1 \otimes T_2$ to be recurrent.

Theorem 3.1. *Let X and Y be F-spaces, and let $S : Y \to Y$ and $T : X \to X$ be operators. Suppose that there is an operator $\varphi : Y \to X$ of a dense range such that the following diagram commutes:*

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ \downarrow\psi & & \downarrow\psi \\ X & \xrightarrow{T} & X \end{array}$$

That is, if we have $T \circ \psi = \psi \circ S$, then the following hold:

1. *If S is hypercyclic, then so is T.*
2. *If S is weakly mixing, then so is T.*
3. *If S is chaotic, then so is T.*
4. *If S is mixing, then so is T.*
5. *If S is recurrent, then so is T.*

Proof. Items (1)–(4) can be found in Theorem [5, Definition 1.5]. We thus prove item (5) below. Let $y \in Rec(S)$ be a recurrent vector for S. Then, there exists a strictly increasing sequence of positive integers $(k_n)_n$ such that

$$S^{k_n} y \to y,$$

as $n \to +\infty$. According to $T \circ \psi = \psi \circ S$, we obtain

$$T^{k_n}[\psi(y)] = \psi(S^{n_k y}) \to \psi(y), \text{ as } n \to +\infty.$$

This shows that $\psi(y)$ is a recurrent vector for T. $\square$

Following [11], it is known that if $T \otimes R$ is hypercyclic, then neither T nor R is hypercyclic. This may happen in a recurrent case in which $T \otimes R$ is recurrent, but neither T nor R is recurrent. If we take $T := 2I$ and the backward shift $R := B$ as operators defined on l_2, then $T \otimes R = I \otimes 2B$ is hypercyclic and, thus, recurrent. However, $T := 2I$ is not recurrent because $T^{k_n} x = 2^{k_n} x \nrightarrow x$.

3.1 The complex field case

By building upon Theorem 2.4, we proceed to analyze the recurrence of the tensor products of two matrices.

Theorem 3.2. *Let T_1 and T_2 be two matrices in the complex number field. Assume that T_1 and T_2 are recurrent. Then, the tensor product $T_1 \otimes T_2$ is recurrent as well.*

Proof. According to Theorem 2.4, there exists an invertible matrix P_1 such that

$$P_1^{-1} T_1 P_1 = \begin{pmatrix} e^{i\theta_1^1} & 0 & \cdots & 0 & 0 \\ 0 & e^{i\theta_2^1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{i\theta_{m-1}^1} & 0 \\ 0 & 0 & \cdots & 0 & e^{i\theta_m^1} \end{pmatrix}.$$

Similarly, there exists an invertible matrix P_2 such that

$$P_2^{-1} T_2 P_2 = \begin{pmatrix} e^{i\theta_1^2} & 0 & \cdots & 0 & 0 \\ 0 & e^{i\theta_2^2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{i\theta_{m-1}^2} & 0 \\ 0 & 0 & \cdots & 0 & e^{i\theta_m^2} \end{pmatrix}.$$

According to Proposition 2.7 and Proposition 2.8, we have

$$
\begin{aligned}
&(P_1 \otimes P_1)^{-1}(T_1 \otimes T_2)(P_1 \otimes P_2) \\
&= \left(P_1^{-1}T_1P_1\right) \otimes \left(P_2^{-1}T_2P_2\right) \\
&= \begin{pmatrix} e^{i\theta_1^1} & 0 & \cdots & 0 & 0 \\ 0 & e^{i\theta_2^1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{i\theta_{m-1}^1} & 0 \\ 0 & 0 & \cdots & 0 & e^{i\theta_m^1} \end{pmatrix} \otimes \begin{pmatrix} e^{i\theta_1^2} & 0 & \cdots & 0 & 0 \\ 0 & e^{i\theta_2^2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{i\theta_{m-1}^2} & 0 \\ 0 & 0 & \cdots & 0 & e^{i\theta_m^2} \end{pmatrix} \\
&= \begin{pmatrix} e^{i(\theta_1^1+\theta_1^2)} & 0 & \cdots & 0 & 0 \\ 0 & e^{i(\theta_1^1+\theta_2^2)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{i(\theta_m^1+\theta_{m-1}^2)} & 0 \\ 0 & 0 & \cdots & 0 & e^{i(\theta_m^1+\theta_m^2)} \end{pmatrix}.
\end{aligned}
$$

We conclude that $T_1 \otimes T_2$ is similar to a diagonal matrix with unimodular entries. Applying Theorem 2.4 once again shows that $T_1 \otimes T_2$ is recurrent. $\square$

3.2 The real case

By building upon Theorem 2.5, we proceed to analyze the recurrence of the tensor products of two matrices.

Theorem 3.3. *Let T_1 and T_2 be two matrices in the real number field. Assume that T_1 and T_2 are recurrent in case the similarity matrix of one of T_1 and T_2 is diagonal matrix with all entries being either 1 or -1. Then, the tensor product $T_1 \otimes T_2$ is recurrent.*

Proof. For the sake of argument, we assume that T_1 is a diagonal matrix in which all entries are either 1 or -1. By Theorem 2.5, there then exists an invertible matrix P_1 such that

$$
P_1^{-1}T_1P_1 = \begin{pmatrix} c_1 & 0 & \cdots & 0 & 0 \\ 0 & c_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{m-1} & 0 \\ 0 & 0 & \cdots & 0 & c_m \end{pmatrix},
$$

where c_j is equal to -1 or 1 for $j = 1, 2, \cdots, m$. Assume also that there exists an invertible matrix P_2 such that

$$
P_2^{-1}T_2P_2 = \begin{pmatrix} J_1 & 0 & \cdots & 0 & 0 \\ 0 & J_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{m-1} & 0 \\ 0 & 0 & \cdots & 0 & J_m \end{pmatrix}
$$

where at least one of J_j, $1 \le j \le m$, is a 2×2 rotation matrix. According to Proposition 2.7 and Proposition 2.8, we then have

$$\begin{aligned}
&(P_1 \otimes P_1)^{-1} (T_1 \otimes T_2) (P_1 \otimes P_2) \\
&\quad = \left(P_1^{-1} T_1 P_1\right) \otimes \left(P_2^{-1} T_2 P_2\right) \\
&\quad = \begin{pmatrix} c_1 & 0 & \cdots & 0 & 0 \\ 0 & c_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{m-1} & 0 \\ 0 & 0 & \cdots & 0 & c_m \end{pmatrix} \otimes \begin{pmatrix} J_1 & 0 & \cdots & 0 & 0 \\ 0 & J_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{m-1} & 0 \\ 0 & 0 & \cdots & 0 & J_m \end{pmatrix} \\
&\quad = \begin{pmatrix} c_1 J_1 & 0 & \cdots & 0 & 0 \\ 0 & c_1 J_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c_m J_{m-1} & 0 \\ 0 & 0 & \cdots & 0 & c_m J_n \end{pmatrix}.
\end{aligned}$$

It is easy to see that for the block diagonal matrix of the last matrix, each $c_i J_j$, $1 \le i \le n$, $1 \le j \le m$, is either a 2×2 rotation matrix or a 1×1 matrix in which all entries are either 1 or -1. According to Theorem 2.5, we can then conclude that $T_1 \otimes T_2$ is recurrent. □

4 Recurrence of Companion Matrices and their Tensor Product

A square matrix P is a permutation matrix if exactly one entry in each of its rows and columns is 1, and all other entries are 0. For example,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let $f(\lambda) = \lambda^n - a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - a_{n-3}\lambda^{n-3} - \cdots - a_1\lambda - a_0$. Then, its companion matrix can be defined as

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & a_{n-1} \end{pmatrix}$$

Theorem 4.1. *Let A be a companion matrix and permutation matrix. That is,*

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

Then, A is recurrent.

Proof. The proof is straightforward because $A^n = E_n$ is an $n \times n$ identity matrix. $\square$

Remark 4.2. *A permutation matrix constitutes a sufficient condition for recurrence but the converse does not hold. Let A be*

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix},$$

It is then easy to see that A is the companion matrix of the polynomial $f(\lambda) = \lambda^3 - \lambda^2 + \lambda - 1$, and is recurrent because

$$A^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

However, A is not a permutation matrix.

Theorem 4.3. *An $n \times n$ matrix A over a number field is similar to a unique rational canonical form of A. That is,*

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_s \end{pmatrix},$$

where A_i $(i = 1, 2, \cdots, s)$ is companion matrix of some polynomial $f(\lambda)$. If A_i $(i = 1, 2, \cdots, s)$ is permutation matrix, then A is recurrent.

Proof. According to Theorem 4.1 above, we know that A_i $(i = 1, 2, \cdots, s)$ is recurrent. Assume that $A_i^{n_i} = E$, where $i = 1, 2, \cdots, s$. And let n be the least common multiple of $\{n_1, n_2, \cdots, n_s\}$. Then, we get $A^n = E$; therefore, A is recurrent. $\square$

Theorem 4.4. *Let A, B be the rational canonical form, i.e.,*

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_s \end{pmatrix}, B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_t \end{pmatrix},$$

where A_i $(i = 1, 2, \cdots, s)$ and B_j $(j = 1, 2, \cdots, t)$ are companion matrices of some polynomials $f(\lambda)$ and $g(\lambda)$, respectively. If A_i and B_j are permutation matrices, then $A \otimes B$ is recurrent.

Proof.

$$A \otimes B = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_s \end{pmatrix} \otimes \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_t \end{pmatrix}$$

$$= \begin{pmatrix} A_1B_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & A_1B_t & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & A_2B_1 & 0 & 0 & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & \ddots & 0 & 0 & \cdots \\ 0 & \cdots & \cdots & 0 & 0 & A_2B_t & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & A_sB_t \end{pmatrix}$$

As A_i and B_j are permutation matrices for $(i = 1, 2, \cdots, s)$ and $(j = 1, 2, \cdots, t)$, respectively, and the product of two permutation matrices is also a permutation matrix, A_iB_j is therefore a permutation matrix. Let n be the least common multiple of $\{n_{11}, n_{1t}, \cdots, n_{st}\}$, where $(A_iB_j)^{n_{ij}} = E$. We then have $(A \otimes B)^n = E$. Therefore, $A \otimes B$ is recurrent. □

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