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Heron's Problem in an Inner Product Space

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Abstract

An optimization problem attributed to Heron entails minimizing the distance between two fixed points in the plane subject to the constraint that any path between them must include a stop at a given line. While Fermat's theorem from calculus may be applied, the solution "from The Book" is almost surely a coordinate-free approach from transformation geometry. In this article, we recast Heron's problem in the context of a finite dimensional inner product space. When the constraint is represented by a hyperplane, the problem responds to a natural analogue of the geometric approach in the classical case. We conclude by offering a modest extension of the method for more general constraining subspaces.

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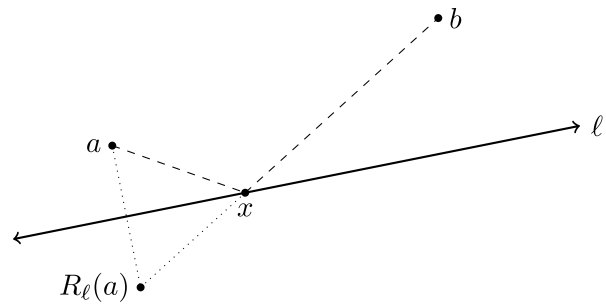
A classical optimization problem attributed to Heron of Alexandria (1075 AD) states: given a line ℓ in the plane and points a and b not on ℓ, determine the point(s) x on ℓ for which the sum of lengths $ax + xb$ is minimal.

In 1917, the problem appeared as *the milkmaid puzzle* in [3]: the milkmaid and her milking-stool are at point a and the dairy "where the extract has to be deposited" is at point b.

> *But it has been noticed that the young woman always goes down to the river with her pail before returning to the dairy. ... It is quite easy to indicate the exact spot on the bank of the river to which she should*

> *direct her steps if she wants as short a walk as possible. Can you find that spot?*

When a and b are on opposite sides of ℓ, the optimal point is, of course, $x = \overline{ab} \cap \ell$, where $\overline{ab}$ denotes the line segment joining points a and b, and when they are on the same side of ℓ, a straightforward approach from transformation geometry yields a viable candidate: reflect point a, say, over the line ℓ to the point $R_\ell(a)$ and let x be the point of intersection of ℓ and the line segment $\overline{R_\ell(a)b}$. The triangle inequality confirms that x provides the unique optimal solution. The article [11] offers interactive applets to explore and visualize the geometry of Heron's problem.



Heron's problem and its variations are somewhat ubiquitous: the problem appears as exercise 1.5.19 of [4], and, in the subsequent section, Eccles notes that "our method for solving this minimum distance problem applies equally well to the solution of several other seemingly unrelated problems ..." These involve a light beam reflected off a mirror, a ball caroming off the wall of a billiard table, and one calling attention to general situations in which the principle 'angle of incidence = angle of reflection' applies. Calculus students working problem 4.7.72 of [12] encounter the principle where the expectation is the solution of a canonical critical point equation and an application of Fermat's theorem (Theorem 4.1.4 of [12]). Further aspects of Heron's problem are explored in section 6.1 (Milkmaids and Elliptical Mirrors) of [7], and a more general version of the problem appears, for example, as exercise 4.172 in section 4.9 (Lagrange multipliers) of [5]. The fertility, longevity, and sustained interest in Heron's problem are evidenced by the generalizations in articles such as [2], [6], and [9], for example.

It turns out that the success of the canonical geometric approach in the classical setting of the plane is guaranteed by the fact that the constraint line ℓ is a one dimensional (affine) subspace of the two dimensional space $\mathbb{R}^2$.

In this article, we recast Heron's problem in the framework of a real finite dimensional inner product space with a linear subspace in the role of the constraint. The case of an affine subspace serving as the constraint can easily be reduced to our setting by a simple translation. When the constraining subspace is a *hyperplane* in the ambient space, a natural analogue of the geometric approach from the plane produces the unique solution to Heron's problem in this context. We conclude by offering a modest variation of the solution in the event that the constraint is represented by a subspace that is not a hyperplane.

1 Inner product space preliminaries.

1.1 Line segments

If V is a vector space and $a, b \in V$, then the *line segment* $\overline{ab}$ is the set

$$\overline{ab} = \{a + \lambda(b - a) : 0 \leq \lambda \leq 1\}.$$

Of some interest in its own right is the fact that if V is equipped with an inner product $\langle \cdot, \cdot \rangle$, and $\| \cdot \|$ denotes the norm that it induces, then the line segment $\overline{ab}$ is characterized by

$$\overline{ab} = \{v \in V : \|a - b\| = \|a - v\| + \|v - b\|\}. \tag{1}$$

Indeed, if $v = a + \lambda(b - a)$ for some $\lambda \in [0, 1]$, then

$$\|a-v\|+\|v-b\| = \|\lambda(a-b)\|+\|a+\lambda(b-a)-b\| = \lambda\|a-b\|+(1-\lambda)\|a-b\| = \|a-b\|.$$

Conversely, if $v \in V$ is a vector for which the equality $\|a - b\| = \|a - v\| + \|v - b\|$ holds, and neither $a - v$ nor $v - b$ is the zero vector, then one of $a - v$ or $v - b$ is a positive scalar multiple of the other (Lemma 3.2-1(b) of [8]). But if, say, $a - v = k(v - b)$ for some $k > 0$, then $v = a + \frac{k}{k+1}(b - a)$ which implies that $v \in \overline{ab}$.

If U is a subspace of V, and a and b are points in $V \setminus U$ for which the line segment $\overline{ab}$ intersects U, then, as one might expect, $\overline{ab} \cap U$ consists of exactly one point: if there are distinct numbers $\lambda_1, \lambda_2 \in (0, 1)$ for which $u_1 = a + \lambda_1(b - a)$ and $u_2 = a + \lambda_2(b - a)$ are both in U, then the linear combination $u_1 + \frac{\lambda_1}{\lambda_2 - \lambda_1}(u_1 - u_2) = a \in U$, a contradiction.

1.2 Projections and reflections

Recall that the *orthogonal projection* of a vector $v \in V$ onto the subspace U is the vector $P_U(v) = \sum_{k=1}^{m} \langle v, u_k \rangle u_k$, where $(u_1, ..., u_m)$ is an orthonormal basis for U with $1 \leq m < \dim(V)$; the *reflection* of v over U is the vector $R_U(v) = 2P_U(v) - v$.

For any $v \in V$ and $u \in U$, the vectors $P_U(v) - u$ and $v - P_U(v)$ are orthogonal so that

$$\|(P_U(v) - u) + (v - P_U(v))\| = \|(P_U(v) - u) - (v - P_U(v))\|$$

and, consequently,

$$\|v - u\| = \|R_U(v) - u\|. \tag{2}$$

2 Heron's problem in a finite dimensional inner product space.

Let U be a non-trivial, proper subspace of the real finite dimensional inner product space $(V, \langle\cdot,\cdot\rangle)$. Given points $a, b \in V \setminus U$, Heron's problem in this setting entails finding $u \in U$ for which the quantity $\|a-u\|+\|u-b\|$ is minimal.

If the line segment $\overline{ab}$ intersects the subspace U, then the point of intersection $u^\star = \overline{ab} \cap U$ is the unique solution to the problem because equation (1) and the triangle inequality ensure that

$$\|a-u^\star\| + \|u^\star - b\| = \|a-b\| < \|a-u\| + \|u-b\| \quad \text{for all } u \in U \setminus \{u^\star\}.$$

If, however, $\overline{ab} \cap U = \emptyset$ and instead $u^\star = \overline{R_U(a)b} \cap U$, then, by equation (2), for all $u \in U \setminus \{u^\star\}$,

$$\|a-u^\star\|+\|u^\star-b\| = \|R_U(a)-u^\star\|+\|u^\star-b\| = \|R_U(a)-b\| < \|R_U(a)-u\|+\|u-b\| = \|a-u\|+\|u-b\|,$$

so that $u^\star$ provides the unique solution to Heron's problem. It turns out that these are the only two possibilities when U is a hyperplane in V.

2.1 The constraint is a hyperplane.

If U is a hyperplane in the vector space V, then there is a non-zero linear functional φ on V for which $U = \ker(\varphi)$ (Proposition 8.6(i) of [10]). The following result confirms that, in this case, exactly one of the line segments $\overline{ab}$ or $\overline{R_U(a)b}$ intersects U.

Proposition 1 *In the setting detailed above, for $a, b \in V \setminus U$, the following are equivalent:*

(a) $\varphi(a)\varphi(b) < 0$;

(b) $\overline{ab} \cap U \neq \emptyset$;

(c) $\varphi\big(R_U(a)\big)\varphi(b) > 0$;

(d) $\overline{R_U(a)b} \cap U = \emptyset$.

Proof. The proof hinges on the auxiliary function $f : [0,1] \longrightarrow \mathbb{R}$ defined by

$$f(\lambda) = \varphi(a) + \lambda\varphi(b-a) \quad \text{for all } \lambda \in [0,1].$$

Evidently, the function f parametrizes the segment joining the points $\varphi(a)$ and $\varphi(b)$ on the real line. Moreover, the continuity of φ (Corollary 7.11(i) of [10]) ensures the continuity of f, so, by the intermediate value theorem, the segment passes through zero if and only if condition (a) holds. On the other hand, the segment passes through zero precisely when there is a number $\lambda \in (0,1)$ satisfying $f(\lambda) = \varphi\big(a+\lambda(b-a)\big) = 0$ which, in turn, is equivalent to the existence of $\lambda \in (0,1)$ for which $a+\lambda(b-a) \in U$. Consequently, conditions (a) and (b) are equivalent.

The equality $\varphi\big(R_U(a)\big) = -\varphi(a)$ ensures the equivalence of statements (a) and (c), and the equivalence of (c) and (d) follows from the equivalence of (a) and (b). $\square$

Proposition 1 guarantees the existence of a unique solution to Heron's problem in a finite dimensional inner product space V whenever the constraining subspace U is a hyperplane in V. Moreover, as in the classical version of the problem, it presents a strategy for locating the solution: if the points a and b are on opposite sides of U, then the unique solution is the point $\overline{ab} \cap U$, whereas if a and b are on the same side of U, then the unique solution is the point at which the line segment joining $R_U(a)$ with b intersects U.

2.2 The constraint is not a hyperplane.

In sharp contrast to the alternative provided by Proposition 1, if the subspace U is not a hyperplane in V, then the segments $\overline{ab}$ and $\overline{R_U(a)b}$ may both fail to intersect U. This possibility does not arise in the classical setting, of course, since any non-trivial, proper subspace of $\mathbb{R}^2$ is a hyperplane in $\mathbb{R}^2$.

As a typical example – in the same vein as Example 2 of [6] – let V denote the vector space $\mathbb{R}^3$ with the standard Euclidean inner product. As depicted in Figure 1, let U be the one dimensional subspace of V spanned by the vector $u_1 = (1,1,1)$ and consider the points $a = (1,1,4)$ and $b = (2,3,3)$. Then $a, b \in V \setminus U$ and one may readily verify that $\overline{ab} \cap U = \emptyset$. To compute the orthogonal projection of a onto U, we implement the matrix representation for P_U, specifically

$$P_U(a) = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}. \tag{3}$$

Thus, $R_U(a) = 2P_U(a) - a = (3,3,0)$, and it turns out that also $\overline{R_U(a)b} \cap U = \emptyset$, so the approach from Proposition 1 does not work here.

However, a slight re-contextualization of the problem will allow for an application of Proposition 1. We introduce the space $\hat{V} = U \oplus \mathbb{R}a$ and seek a vector $\hat{b} \in \hat{V} \setminus U$ that satisfies $\|u - \hat{b}\| = \|u - b\|$ for all $u \in U$. The goal then shifts to finding the vector $u \in U$ for which the quantity $\|a - u\| + \|u - \hat{b}\|$ is minimal. The problem is thus effectively recast in the ambient space $\hat{V}$ in which $U = \ker(\varphi)$ for the linear functional φ on $\hat{V}$ defined by $\varphi(\hat{v}) = \lambda$ for all $\hat{v} = u + \lambda a \in \hat{V}$. With the constraint U now represented by a hyperplane

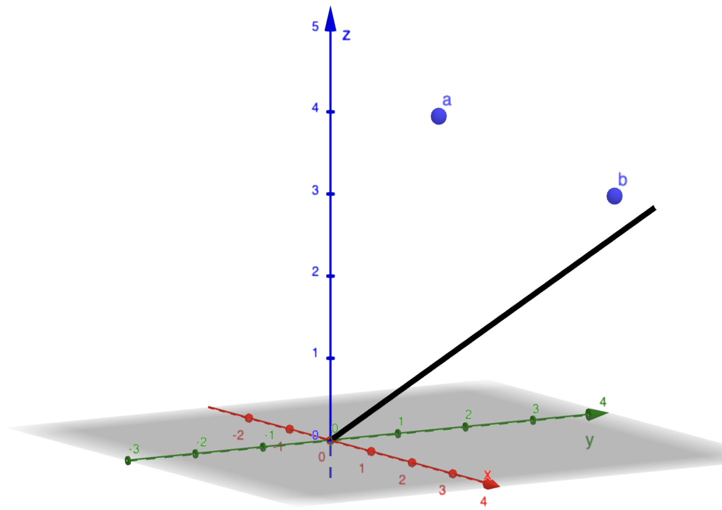


Figure 1:

in $\hat{V}$, Proposition 1 and its attendant geometric strategy for solving Heron's problem become available.

A formula for such a vector $\hat{b}$ may be obtained by first adjoining the vector a to a basis $(u_1, \ldots, u_m)$ of U, where $1 \leq m < \dim(V) - 1$, to produce the linearly independent list $(u_1, \ldots, u_m, a)$. Next, apply the Gram-Schmidt procedure (6.20 of [1]) to generate an orthonormal basis $(u_1', \ldots, u_m', a')$ for the space $\hat{V}$. Finally, define the vector $\hat{b} \in \hat{V} \setminus U$ by

$$\hat{b} = P_U(b) - \mu a', \qquad \text{where } \mu = \|b - P_U(b)\|. \tag{4}$$

To confirm the suitability of this choice, let $u \in U$ and observe that, by a couple of applications of the Pythagorean theorem,

$$\begin{aligned} \|\hat{b} - u\|^2 &= \left\|P_U(b) - \mu a' - u\right\|^2 \\ &= \left\|P_U(b) - u\right\|^2 + \|\mu a'\|^2 \\ &= \left\|P_U(b) - u\right\|^2 + \|b - P_U(b)\|^2 \\ &= \|b - u\|^2. \end{aligned}$$

The vector $\hat{b}$ plays a similarly important role in the more general context of [6].

Returning to our example, implementation of the Gram-Schmidt procedure results in the orthonormal basis vectors $u_1' = \frac{1}{\sqrt{3}}(1, 1, 1)$ and $a' = -\frac{1}{\sqrt{6}}(1, 1, -2)$ for $\hat{V}$. Again applying the matrix representation for P_U as in equation (3) reveals that $P_U(b) = \frac{8}{3}(1, 1, 1)$, from which it follows that $\mu = \|b - P_U(b)\| =$

$\|\frac{1}{3}(-2,1,1)\| = \frac{\sqrt{6}}{3}$. Formula (4) then yields

$$\hat{b} = P_U(b) - \mu a' = \frac{8}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{\sqrt{6}}{3}\cdot\frac{-1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}3\\3\\2\end{bmatrix} = \frac{10}{3}u_1 - \frac{1}{3}a.$$

Thus, $\varphi(\hat{b}) = -\frac{1}{3}$, and because $\varphi(a) = 1$, the points a and $\hat{b}$ are on the opposite sides of the hyperplane U in the space $\hat{V}$. In particular, $\varphi(a)\varphi(\hat{b}) < 0$, so that, by Proposition 1, the point $u^\star = (\frac{5}{2}, \frac{5}{2}, \frac{5}{2})$ at which the line segment joining a and $\hat{b}$ intersects the subspace U provides the unique solution to Heron's problem.

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