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Initial Coefficients and Fekete-Szegö Inequality for a Subclass of Analytic Functions Associated with q-Sălăgean Differential Operator

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Abstract

In this paper, we introduce a new subclass of analytic functions associated with the q-Sălăgean differential operator. It determines the initial coefficients a_2 and a_3, and establishes the upper bound for the Fekete-Szegö inequality $|a_3 - \delta a_2^2|$ within this subclass.

Keywords: Analytic functions, Fekete-Szegö, q-Sălăgean differential operator

1 Introduction

In the field of complex analysis, analytic functions defined within the open unit disk $\mathbb{U} = \{z: z \in \mathbb{C}\ and\ |z| < 1\}$ play a fundamental role in understanding the geometric characteristics and theories of functions. As explained by [6], these functions represented as $\mathcal{A}$, are those that are differentiable at every point within the domain D and can often be expressed through a Taylor series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

where $a_n \in \mathbb{C}, n = 2,3, \cdots$.

Subordination is a key concept for analyzing the relationships between functions in this class. As described by [4], a function $f(z)$ is said to be subordinate to another function $g(z)$, symbolized as $f(z) \prec g(z)$ if Schwarz function $w(z)$ exist in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$.

Moreover, the Fekete-Szegö problem continues to be an important research topic, especially in calculating the initial coefficients a_2 and a_3, as well as determining the upper bound for the functional $|a_3 - \delta a_2^2|$. Extensive contributions by researchers like [1], [8] and [4] have provided valuable insights into these problem.

Therefore, this paper aims to present a new subclass of analytic functions and further explore the determination of the initial coefficients a_2 and a_3, alongside the upper bound for the Fekete-Szegö inequality for functions within this new subclass, which is linked to the q-Sălăgean differential operator.

The application of the q-derivative operator has opened new avenues for extending the analysis of analytic functions. As outlined by [5], the q-derivative operator for $f \in \mathcal{A}$ in the open unit disk $\mathbb{U}$ is defined as:

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}$$

where $[n]_q = \frac{1-q^n}{1-q}$.

By incorporating q-calculus and the concept of subordination, researchers have been able to refine existing bounds for the coefficients a_2 and a_3 and uncover new geometric properties of these functions. Researchers like [2] have extended this by defining the q-Sălăgean differential operator M_q^n, for given $f \in \mathcal{A}$ and $0 < q < 1$, M_q^n is defined as

$$M_q^n f(z) = zD_q\left(M_q^{n-1} f(z)\right) = z + \sum_{j=2}^{\infty} [j]_q^n a_j z^j \quad (z \in \mathbb{U}).$$

This extension of classical results is expected to provide new insights into the study of analytic functions, their coefficient result and Fekete-Szegö problem.

Definition 1 *A function $f \in \mathcal{A}$ is said to belong to the new subclass $\mathcal{L}_{q,n}(\phi)$ if it satisfies the following subordination condition:*

$$\left(\frac{zD_q\left(M_q^n f(z)\right)}{M_q^n f(z)}\right)^{\varepsilon}\left(1+\frac{zqD_q\left(D_q\left(M_q^n f(z)\right)\right)}{D_q\left(M_q^n f(z)\right)}\right)^{1-\varepsilon} \prec \phi(z), \varepsilon \geq 0.$$

2 Preliminary Results

The main findings are based on the following lemmas:

Lemma 1 ([7]) If $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ represents a function with a positive real part in the open unit disk $\mathbb{U}$, and γ is a complex number, then

$$|c_2 - \gamma c_1^2| \leq 2max\{1; |2\gamma - 1|\}.$$

Lemma 2 ([7]) If $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ is a function with positive real part in open unit disk $\mathbb{U}$ and γ is a complex number, then

$$|c_2 - \gamma c_1^2| \leq \begin{cases} -4\gamma + 2, & \gamma < 0 \\ 2, & 0 \leq \gamma \leq 1 \\ 4\gamma - 2, & \gamma \geq 1 \end{cases}.$$

Remarks. The previously mentioned upper bound is sharp and can be adjusted as shown below, provided the conditions $0 < \gamma < 1$ are met.

3 Main Results

Theorem 1 Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ with $B_1 \neq 0$, and f given by (1) belongs to $\mathcal{L}_{q,n}(\phi)$, then

$$\left|a_3 - \delta a_2^2\right| \leq \frac{|B_1|}{[3]_q^n\left[(1-\varepsilon)q[3]_q^2 + \varepsilon\left([3]_q - 1\right)\right]} max\left\{1; \left|\frac{B_2}{B_1} + \frac{B_1}{\left((1-\varepsilon)q[2]_q^2 + \varepsilon\left([2]_q - 1\right)\right)^2}\left(\rho - \delta\frac{[3]_q^n\left[(1-\varepsilon)q[3]_q^2 + \varepsilon\left([3]_q - 1\right)\right]}{[2]_q^{2n}}\right)\right|\right\}$$

where

$$\rho = \left(\frac{\varepsilon^2 + \varepsilon - 2}{2}\right)q[2]_q^3 + \varepsilon(1-\varepsilon)q[2]_q^2\left([2]_q - 1\right) + \left(\frac{\varepsilon^2 - 3\varepsilon}{2}\right)\left([2]_q - 1\right).$$

Proof. Let $f \in \mathcal{L}_{q,n}(\phi)$, by definition there exist a function $w(z)$ with $w(0) = 0$, and $|w(z)| < 1$ in $\mathbb{U}$ such that

$$\left(\frac{zD_q\left(M_q^n f(z)\right)}{M_q^n f(z)}\right)^{\varepsilon}\left(1+\frac{zqD_q\left(D_q\left(M_q^n f(z)\right)\right)}{D_q\left(M_q^n f(z)\right)}\right)^{1-\varepsilon} = \phi\left(w(z)\right). \qquad (2)$$

Now, define the function $p(z)$ by

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + \cdots. \tag{3}$$

Since $w(z)$ is a schwarz function $Re\big(p(z)\big) > 0$ and $p(0) = 1$. Let

$$g(z) = \left(\frac{zD_q\left(M_q^n f(z)\right)}{M_q^n f(z)}\right)^{\varepsilon}\left(1 + \frac{zqD_q\left(D_q\left(M_q^n f(z)\right)\right)}{D_q\left(M_q^n f(z)\right)}\right)^{1-\varepsilon}$$

$$= 1 + d_1 z + d_2 z^2 + \cdots. \tag{4}$$

From equation (2), (3) and (4), we obtain

$$g(z) = \ \phi\big(w(z)\big)$$

and from equation (3) we know that

$$g(z) = 1 + \frac{1}{2}B_1 p_1 z + \left(\frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2 p_1^2\right)z^2 + \cdots. \tag{5}$$

Hence, from equation (4) and (5) we get

$$d_1 = \frac{1}{2}B_1 p_1 \tag{6}$$

$$d_2 = \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2 p_1^2 \tag{7}$$

Therefore, computation shows that

$$\left(\frac{zD_q\left(M_q^n f(z)\right)}{M_q^n f(z)}\right)^{\varepsilon}\left(1 + \frac{zqD_q\left(D_q\left(M_q^n f(z)\right)\right)}{D_q\left(M_q^n f(z)\right)}\right)^{1-\varepsilon}$$

$$= 1 + \Big((1-\varepsilon)q[2]_q^{n+2} + \varepsilon[2]_q^n\big([2]_q - 1\big)\Big)a_2 z + \Bigg[\Big((1-\varepsilon)q[3]_q^{n+2} + \varepsilon[3]_q^n\big([3]_q - 1\big)\Big)a_3 + \left(\left(\frac{\varepsilon^2+\varepsilon-2}{2}\right)q[2]_q^{2n+3} + \varepsilon(1-\varepsilon)q[2]_q^{2n+2}\big([2]_q - 1\big) + \left(\frac{\varepsilon^2-3\varepsilon}{2}\right)[2]_q^{2n}\big([2]_q - 1\big)\right)a_2^2\Bigg]z^2 + \cdots \tag{8}$$

Hence, from equation (8) and (6)

$$a_2 = \frac{B_1 p_1}{2[2]_q^n\left[(1-\varepsilon)q[2]_q^2 + \varepsilon\big([2]_q - 1\big)\right]} \tag{9}$$

Also, from equation (8) and (7), we have

$$a_3 = \frac{B_1}{2[3]_q^n\left[(1-\varepsilon)q[3]_q^2 + \varepsilon\big([3]_q - 1\big)\right]}\left[p_2 - \left(\frac{1}{2} - \frac{B_2}{2B_1} - \frac{\rho B_1}{2\left[(1-\varepsilon)q[2]_q^2 + \varepsilon\big([2]_q - 1\big)\right]^2}\right)p_1^2\right]. \tag{10}$$

where

$$\rho = \left(\frac{\varepsilon^2 + \varepsilon - 2}{2}\right)q[2]_q^3 + \varepsilon(1-\varepsilon)q[2]_q^2\big([2]_q - 1\big) + \left(\frac{\varepsilon^2 - 3\varepsilon}{2}\right)\big([2]_q - 1\big). \tag{11}$$

By using equation (9) and equation (10), we define

$$a_3 - \delta a_2^2 = \frac{B_1}{2[3]_q^n\left[(1-\varepsilon)q[3]_q^2 + \varepsilon\big([3]_q - 1\big)\right]}\left[p_2 - \gamma p_1^2\right]$$

where

$$\gamma = \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{B_1}{\left[(1-\varepsilon)q[2]_q^2 + \varepsilon\big([2]_q - 1\big)\right]^2}\rho - \delta\frac{[3]_q^n\left[(1-\varepsilon)q[3]_q^2 + \varepsilon\big([3]_q - 1\big)\right]}{[2]_q^{2n}}\right].$$

Our result now follows an application of Lemma 1. Hence, Theorem 1 has been proved.

Theorem 2 Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ with $B_1 > 0$ and $B_2 \geq 0$, and f given by equation (1) belongs to $\mathcal{L}_{q,n}(\phi)$, and

$$\varepsilon_1 = \frac{[2]_q^{2n}}{[3]_q^n\left[(1-\varepsilon)q[3]_q^2+\varepsilon\left([3]_q-1\right)\right]}\left(\frac{\rho B_1^2+\left((1-\varepsilon)q[2]_q^2+\varepsilon\left([2]_q-1\right)\right)^2(B_2-B_1)}{B_1^2}\right),$$

$$\varepsilon_2 = \frac{[2]_q^{2n}}{[3]_q^n\left[(1-\varepsilon)q[3]_q^2+\varepsilon\left([3]_q-1\right)\right]}\left(\frac{\rho B_1^2+\left((1-\varepsilon)q[2]_q^2+\varepsilon\left([2]_q-1\right)\right)^2(B_2+B_1)}{B_1^2}\right).$$

Then

$$|a_3 - \delta a_2^2| \leq \begin{cases} \dfrac{B_2}{[3]_q^n\left[(1-\varepsilon)q[3]_q^2+\varepsilon\left([3]_q-1\right)\right]} \\ \quad + \dfrac{\rho B_1^2}{[3]_q^n\left[(1-\varepsilon)q[3]_q^2+\varepsilon q[2]_q\right]\left[(1-\varepsilon)q[2]_q^2+\varepsilon q\right]^2} & \text{, if } \delta \leq \varepsilon_1 \\ \quad -\delta\dfrac{B_1^2}{[2]_q^{2n}\left[(1-\varepsilon)q[2]_q^2+\varepsilon q\right]^2} \\ \dfrac{B_1}{[3]_q^n\left[(1-\varepsilon)q[3]_q^2+\varepsilon\left([3]_q-1\right)\right]} & \text{, if } \varepsilon_1 \leq \delta \leq \varepsilon_2 \\ \delta\dfrac{B_1^2}{[2]_q^{2n}\left[(1-\varepsilon)q[2]_q^2+\varepsilon q\right]^2} - \dfrac{B_2}{[3]_q^n\left[(1-\varepsilon)q[3]_q^2+\varepsilon\left([3]_q-1\right)\right]} & \text{, if } \delta \geq \varepsilon_2 \\ \quad - \dfrac{\rho B_1^2}{[3]_q^n\left[(1-\varepsilon)q[3]_q^2+\varepsilon\left([3]_q-1\right)\right]\left[(1-\varepsilon)q[2]_q^2+\varepsilon q\right]^2} \end{cases}$$

where

$$\rho = \left(\frac{\varepsilon^2+\varepsilon-2}{2}\right)q[2]_q^3 + \varepsilon(1-\varepsilon)q[2]_q^2\left([2]_q-1\right) + \left(\frac{\varepsilon^2-3\varepsilon}{2}\right)\left([2]_q-1\right).$$

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