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On Connections Associated with Generalized Connections

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Abstract

Consider an ordinary (affine) connection on a differentiable manifold M. In the article [1] a tensor field σ of type (1,1) has been used to define a endomorphism, denoted by $\overset{(\sigma)}{\nabla}_X, X \in \mathcal{D}^1$, called σ-connection. Let φ a tensor field such that $\varphi \in \mathcal{D}^1_1$, at n.1 of this article we define a connection associate with the connection previously mentioned, indicated by $\varphi^{-1}\overset{(\sigma)}{\nabla}_X\varphi$ or more briefly $\overset{(\sigma)}{\nabla^*}_X$ and also written (φ)-connection. In n.2 some fundamental theorems are obtained in components. We then introduce three fields of torsion respectively, it occurs that they are tensors (n.3) and the corresponding structural equations are deduced (n.4). In (n.5) the curvature's field of (φ)-connection has been obtained. For these connections Bianchi's equations have been given (n.6). Then the structure equations of the curvature field introduced were been written (n.7).

Keywords: differential geometry, tensor connection, torsion, curvature, structure equations

1. The notion of (φ)-connection

Let M be a differentiable manifold of class C^∞ and dimension m, we will define the following expressions:

$\mathcal{F}$ or $C^\infty(M)$ the algebra (on R) or functions C^∞ on M;

$\mathcal{D}^1$ and $\mathcal{D}_1$ the $\mathcal{F}$-module of vector fields on M and the $\mathcal{F}$-dual module

[1] See [2]

respectively.

$\mathcal{D}^r_s$ *the tensor field of type (r,s) over M;*

$\mathcal{D}$ *the tensor field algebra over M, that is:*

$$\mathcal{D} = \sum_{p,q}^{\infty} \mathcal{D}^p_q.$$

In addition the notion of D-derivation of $\mathcal{D}$ (See Definition 1) and of the fundamental property stated in Proposition 1 below, will be used.

Definition 1. A D-derivation of algebra $\mathcal{D}$ is a linear endomorphism D of $\mathcal{D}$ that verifies the following conditions:

a) D preserves the type of tensors

b) $\mathrm{D}(T \otimes S) = (DT) \otimes S + T \otimes (DS)$

c) $\mathrm{D}(\mathcal{C}T) = \mathcal{C}(DT)$,

where $\mathcal{C}$ indicates the contraction operation of tensors [2]. For D-derivation the following proposition is valid:

Proposition 1. *A D-derivation is determined when it is known how to operate on the algebra $\mathcal{F}$ and on the $\mathcal{F}$-module $\mathcal{D}^1$* [3].

Remark 1. *Considered on M the non-degenerate tensor field $\sigma \in \mathcal{D}^1_1$ that is, for every point $p \in M$, an endomorphism of the tangent space $\mathcal{D}^1(p)$ in p; for σ we will not necessarily consider verified the known condition valid for almost-complex structure* [4]*. For this reason, σ can be called a generalized almost-complex structure. That said, we introduce the following:*

Definition 2.[5] *Let φ be a tensor field such that $\varphi \in \mathcal{D}^1_1$. Said (φ)-connection or connection associated with a generalized connection, a homomorfism $\varphi^{-1}\overset{(\sigma)}{\nabla}_X\varphi$ or more briefly $\overset{(\sigma)}{\nabla}{}^*_X$ of $\mathcal{D}^1$ in the $\mathcal{F}$-module of D-derivations of $\mathcal{D}$ if verifies the condition:*

$$\overset{(\sigma)}{\nabla}{}^*_X f = \varphi^{-1}\overset{(\sigma)}{\nabla}_X\varphi f = \varphi^{-1}((\sigma X)(\varphi f)) = (\sigma X)f, \ \ \forall X \in \mathcal{D}^1, \ \forall f \in \mathcal{F}. \quad (1)$$

It is a derivation (Definition 1), with $\varphi^{-1}\overset{(\sigma)}{\nabla}_X\varphi$ instead of D, satisfying (1) and such that:

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[2]See [5] p.25

[3]See [5] p.30

[4]That is $\sigma^2(X) = -X, \forall X \in \mathcal{D}^1$

[5]See [2]

$$\overset{(\sigma)}{\nabla}{}^*{}_X(T_1+T_2) = \overset{(\sigma)}{\nabla}{}^*{}_X T_1 + \overset{(\sigma)}{\nabla}{}^*{}_X T_2, \qquad \forall X \in \mathcal{D}^1, \forall T_1, T_2 \in \mathcal{D}.$$

Then we have:

$$\overset{(\sigma)}{\nabla}{}^*{}_X(fY) = \varphi^{-1}\{\overset{(\sigma)}{\nabla}_X \varphi(fY)\} = \varphi^{-1}\{(\sigma X)\varphi(fY) + f(\overset{(\sigma)}{\nabla}_X \varphi)(Y)\} =$$

$$((\sigma X)f)Y + f\overset{(\sigma)}{\nabla}{}^*{}_X Y, \qquad \forall f \in \mathcal{F}, \forall X, Y \in \mathcal{D}^1.$$

2. Propositions and components of a (φ)-connection

Proposition 2. *Let $X \to \overset{(\sigma)}{\nabla}{}^*{}_X$ be a (φ)-connection, if on the open submanifold U (of M) vanishes the field $T \in \mathcal{D}^r_s$ or the field $X \in \mathcal{D}^1$, on U vanishes as well $\overset{(\sigma)}{\nabla}{}^*{}_X\ T$.*

Proposition 3. *Let $X \in \mathcal{D}^1, T \in \mathcal{D}^r_s$. If X vanishes at one point $p \in M$, then also $\overset{(\sigma)}{\nabla}{}^*{}_X T$ vanishes at p.*

Proofs are obtained by adopting the technique for the case of affine connections.

From the previous propositions and from well-known theorems [6]relating to the possibility of extending data fields to M on an open submanifold U of M, it follows again:

Proposition 4. *A (φ)-connection given on M induces a (φ)-connection on a generic open submanifold U of M.*

Proof. Let X, Y be two vectors on U. For each $p \in U$ there exist vectors X', Y' on M which agree with X and Y in an open neighborhood V of p. We then put $(\overset{(\sigma)}{\nabla}{}^*{}_{UX}(Y))_q = (\overset{(\sigma)}{\nabla}{}^*{}_{X'}(Y'))_q$ for $q \in V$. The right-hand side of this equation is independent of the choice of X', Y'. It follows immediately that the rule: $\overset{(\sigma)}{\nabla}{}^*{}_U$: X $\to (\overset{(\sigma)}{\nabla}{}^*{}_U)_X$, $X \in \mathcal{D}^1(U)$, is a (φ)-connection on U.

[6] See [2]

Let $\{X_h\}$ a basis (for tangent fields) valid in a certain neighborhood V_p of a point $p \in M$. With respect to $\{X_h\}$, expressions of the following type will be valid:

$$\sigma\{X_h\} = a_h^r X_r, \qquad a_h^r \in \mathcal{C}^\infty(p),$$
$$\overset{(\sigma)}{\nabla}{}^*{}_{X_i} X_j = L_{ij}^{*h} X_h \qquad L_{ij}^{*h} \in \mathcal{C}^\infty(p),$$

having denoted by $\mathcal{C}^\infty(p)$ the set of functions $\mathcal{C}^\infty$ in some neighborhood of p.

The L_{rs}^{*h} are the components of the (φ)-connection relative to the base $\{X_h\}$; these components, considered together with the a_h^r((σ)-components), identify the (φ)-connection. An analytical confirmation of this fact is given by the formulas according to which the L_{rs}^{*h} varies for a basic change.

We take into account that the transition from the base $\{X_h\}$ to another base $\{X_{h'}\}$ is done with the following formulas:

$$X_{h'} = \Theta_{h'}^h X_h, \qquad \Theta_{h'}^h \in \mathcal{C}^\infty(V_p'). \tag{2}$$

Said $L_{i'j'}^{*h'}$ the components of $\overset{(\sigma)}{\nabla}{}^*$ relating to $\{X_{h'}\}$, the basis change (2) gives:

$$L_{i'j'}^{*h'} \Theta_{h'}^h = \Theta_{i'}^r \Theta_{j'}^s L_{rs}^{*h} + \Theta_{i'}^r a_r^t (X_t \Theta_{j'}^h), \qquad a_{h'}^{r'} \Theta_{r'}^r = a_h^r \Theta_{h'}^h. \tag{3}$$

The (3) are precisely the desired formulas: the presence in them of the a_h^r proves what has been said.

In particular, if it is assumed $X_i = e_i = \frac{\partial}{\partial x^i}$, being $\{x^i\}$ a local coordinate system and therefore $\{X_i\}$ the corresponding natural basis, we have:

$$\Theta_h^{h'} = \frac{\partial x^{h'}}{\partial x^h} = \vartheta_h^{h'}, \qquad e_i \Theta_h^{h'} = \frac{\partial^2 x^{h'}}{\partial x^i x^h} = \vartheta_{ih}^{h'}$$

and the (3) are written:

$$\Gamma_{i'j'}^{h'} \vartheta_{h'}^h = \vartheta_{i'}^r \vartheta_{j'}^s \Gamma_{rs}^h + \vartheta_{i'}^r a_r^t \vartheta_{tj'}^h \text{ with the } a_{h'}^{r'} \vartheta_{r'}^r = a_h^r \vartheta_{h'}^h, \tag{4}$$

where *the Γ_{ij}^h now indicates the components of $\overset{(\sigma)}{\nabla}{}^*$ in the natural basis $\{e_i\}$.*

Now let's move on the components of $\overset{(\sigma)}{\nabla}{}^*_X \omega$, with $\omega \in \mathcal{D}_1$.

Let $\{\omega^i\}$ be the dual basis of $\{X_i\}, \omega^i(X_j) = \delta_j^i$ for $X_j \in \mathcal{D}^1(V_p)$, $\omega^i \in \mathcal{D}_1(V_p)$.

The (φ)-connection components relating to $\{\omega^i\}$ (i.e.the $\overset{(\sigma)}{\nabla}{}^*_X \omega$ components), will be defined by the:

$$\overset{(\sigma)}{\nabla^*}{}_{X_i}\omega^h = \bar{L}^{*h}_{ij}\omega^j \qquad\qquad \bar{L}^{*h}_{ij} \in \mathcal{C}^\infty(V_p).$$

For b) of the Definition 1, data $X, Y \in \mathcal{D}^1(V_p)$, we have the relation:

$$(X)(\omega Y) = (\overset{(\sigma)}{\nabla}{}^*_X\omega)Y + \omega(\overset{(\sigma)}{\nabla}{}^*_X Y)$$

that written for $X = X_i, Y = X_j$, $\omega = \omega^h$, gives the:

$$(\overset{(\sigma)}{\nabla^*}{}_{X_i}\omega^h)X_j + \omega^h(\overset{(\sigma)}{\nabla^*}{}_{X_i}X_j) = 0$$

from which we obtain:

$$\bar{L}^{*h}_{ip}\delta^p_j + L^{*p}_{ij}\delta^h_p = 0, \text{ and then: } \bar{L}^{*h}_{ij} = -L^{*h}_{ij}.$$

Remark 2. *The link between the components L^h_{ij}* [7] *and the components L^{*h}_{ij} is, $\bar{b}^r_i L^h_{rs} b^s_j = L^{*h}_{ij}$ where, $\varphi^{-1}\{X_i\} = \bar{b}^r_i X_r$ and $\varphi\{X_j\} = b^s_j X_s$ such that $\bar{b}^i_r b^r_j = I$, with $\bar{b}^i_r$, $b^r_j \in \mathcal{C}^\infty(p)$.*

Coming then to space $\mathcal{D}^2(V_p)$ and considered the base $\{X_h \otimes X_k\}$ induced by $\{X_i\}$, similarly to what it has been done before, we will be able to place:

$$\overset{(\sigma)}{\nabla^*}{}_{X_i}(X_h \otimes X_k) = L^{*pq}_{ihk}X_p \otimes X_q. \tag{5}$$

If we develop the first member of (5) according to b) of the Definition 1,

we get: $\overset{(\sigma)}{\nabla^*}{}_{X_i}(X_h \otimes X_k) = (\overset{(\sigma)}{\nabla^*}{}_{X_i}X_h) \otimes X_k + X_h \otimes (\overset{(\sigma)}{\nabla^*}{}_{X_i}X_k)$, but:

$$(\overset{(\sigma)}{\nabla^*}{}_{X_i}X_h) \otimes X_k + X_h \otimes (\overset{(\sigma)}{\nabla^*}{}_{X_i}X_k) =$$

$$L^{*p}_{ih}X_p \otimes X_k + L^{*q}_{ik}X_h \otimes X_q = \delta^q_k L^{*p}_{ih}X_p \otimes X_q + \delta^p_h L^{*q}_{ik}X_p \otimes X_q.$$

From the comparison of (5) with the above, we can deduce:

$$L^{*pq}_{ihk} = \delta^q_k L^{*p}_{ih} + \delta^p_h L^{*q}_{ik}.$$

By proceeding in this way it is possible to determine, in general, the components $L^{*p_1\ldots p_r}_{iq_1\ldots q_s}$ of $\overset{(\sigma)}{\nabla^*}{}_{X_i}$ related to the base $\{x_{p_1} \otimes \ldots \otimes x_{p_r} \otimes \omega^{q_1} \ldots \otimes \omega^{q_s}\}$ in a function of the components related to the starting base $\{X_i\}$, such a possibility complies

[7] See [2]

with Proposition 1, of which it is a check.

The (φ)-connection is therefore determined in V_p, assigning the two component systems a^i_j and L^{*h}_{ij}. In particular:

Proposition 5. *If $\{U_i\}$ is a cover of M, and if for each open a^i_j and Γ^h_{ij} are given so that they hold, at the intersection of pairs of open, the (4) for the Γ^h_{ij} and for the a^i_j, is defined, for each open U_i, a (φ)-connection $(\overset{(\sigma)}{\nabla^*})U_i$ and on M a (φ)-connection $\overset{(\sigma)}{\nabla^*}$, that induces, on U_i, the $(\overset{(\sigma)}{\nabla^*})U_i$.*[8]

3. Torsion fields

A. Nijenhuis introduced the field $\overset{(\sigma)}{N}$ defined as follow:

$$\begin{aligned}
&\overset{(\sigma)}{N}(X,Y) = [\sigma X, \sigma Y] - \sigma([\sigma X, Y] + [X, \sigma Y] - \sigma[X,Y]) = \\
&[\sigma X, \sigma Y] - \sigma K(X,Y), with: \\
&K(X,Y) = [\sigma X, Y] + [X, \sigma Y] - \sigma[X,Y].
\end{aligned} \tag{6}$$

Thus the field $\overset{(\sigma)}{N} \in \mathcal{D}^1_2$:
$\{\omega, X, Y\} \in D_1 \times D^1 \times D^1 \to \omega(\overset{(\sigma)}{N}(X.Y)) \in R$ can be called a torsion on σ.
How to check immediately, $\overset{(\sigma)}{N}(X,Y)$ is a tensor field, while K(X,Y) is not.

As far as the (φ)-connection, we now want to indicate two tensor fields S^*, T^* that we can call 1° and 2° torsion field because both are reduce to the torsion field of an ordinary connection if we make the hypotesis that σ induces the identical endomorphism for each $p \in M$. The fields S^*, T^* in question are defined by the expressions:

$$\begin{aligned}
&S^*(X,Y) = \overset{(\sigma)}{\nabla^*}_X Y - \overset{(\sigma)}{\nabla^*}_Y X - K[X,Y]; \\
&T^*(X,Y) = \overset{(\sigma)}{\nabla^*}_X \sigma Y - \overset{(\sigma)}{\nabla^*}_Y \sigma X - [\sigma X, \sigma Y],\ \forall\, X, Y \in \mathcal{D}^1.
\end{aligned} \tag{7}$$

. The proof that S^*(X,Y) and $T^*(X,Y)$ determine two tensor fields consists in formally checking that both S^* and T^* are $\mathcal{F}$-linear applications of $D^1 \times D^1 \in$

[8] See[3]

D^1.
For S^* and for T^* the property of antisymmetry hold, i.e that is:

$$S^*(X,Y) = -S^*(Y,X), \qquad T^*(X,Y) = -T^*(Y,X).$$

From the previous fields we can easily derive the components; if we introduce any local base $\{x_i\}$, we derive the following expressions:

$$S^{*h}_{ij} = L^{*h}_{ij} - L^{*h}_{ji} - a^q_i\gamma^h_{qj} - a^q_j\gamma^h_{iq} + a^h_q\gamma^q_{ij} + X_j a^h_i - X_i a^h_j$$

$$T^{*h}_{ij} = a^q_j L^{*h}_{iq} - a^q_i L^{*h}_{jq} - a^r_i a^s_j \gamma^h_{rs},$$

where it still stands $[X_i, X_j] = \gamma^h_{ij} X_h$.

4. Torsion forms and related equations of structural equations

With reference to the torsion fields of the previous number, let's say:

$$\text{(a)} \begin{cases} \overset{(\sigma)}{N}(X,Y) = \overset{(\sigma)}{N^k}(X,Y)X_k, \\ S^*(X,Y) = S^{*k}(X,Y)X_k, \\ T^*(X,Y) = T^{*k}(X,Y)X_k \end{cases}$$

by means of the (a) we define the 2-forms $\overset{(\sigma)}{N^k}, S^{*k}, T^{*k}$ which we call the torsion forms.

Proposition 6. *The torsion forms of a (φ)-connection verify the following relationships (structural equations)*

$$\begin{aligned} &(b) \quad \overset{(\sigma)}{N^k} = \gamma^k_{rs} a^r \wedge a^s - 2\gamma^t_{rq} a^k_t a^r \wedge \omega^q - 2a^k_r da^r \\ &(c) \quad S^{*k} = 2\{\omega^{*k}_h \wedge \omega^h + 2a^p \wedge \beta^k_p - da^k\} \\ &(d) \quad T^{*k} = 2\{\omega^{*k}_h \wedge a^h + \beta^k_{pq} a^p \wedge a^q\}, \end{aligned} \tag{8}$$

with the a^k_r defined in the chapter 2, ω^k belonging to the dual basis of $\{X_k\}, \gamma^k_{rs}$ defined in the chapter 3, and the $\omega^{*k}_h, a^r, \beta^k_q \in \mathcal{D}_1, \beta^k_{pq} \in \mathcal{F}$, defined as follows:

$$\begin{aligned} &\omega^k(\overset{(\sigma)}{\nabla^*}_X X_p) = \omega^{*k}_p(X) \\ &\omega^k\{\sigma X\} = a^k(X) \\ &(d\omega^k)(X_p) = \beta^k_p \\ &(d\omega^k)(X_p, X_q) = \beta^k_{pq} \qquad\qquad \forall\ X, X_p, X_q \in \mathcal{D}^1 \end{aligned} \tag{9}$$

The proof of (b) is left to the reader while equations (c), (d) are demonstrated by the following developments.

Proof of (c): Let's develop the second expression of (a):

$S^{*k}(X,Y) = \omega^k \overset{(\sigma)}{\nabla}{}^*{}_X(\omega^h(Y)X_h) - \omega^k \overset{(\sigma)}{\nabla}{}^*{}_Y(\omega^h(X)X_h) - \omega^k[\sigma X, Y] - \omega^k[X, \sigma Y] + \omega^k\sigma[X,Y] = \omega^h(Y)\omega_h^{*k}(X) - \omega^h(X)\omega_h^{*k}(Y) + (\sigma X)\omega^k(Y) - (\sigma Y)\omega^k(X) - \omega^k [\sigma X, Y] + X\omega^k(\sigma Y) - Y\omega^k(\sigma X) - \omega^k[X,\sigma Y] + Y\omega^k(\sigma X) - X\omega^k(\sigma Y) + \omega^k\sigma[X,Y] = \{2\omega_h^{*k} \wedge \omega^h - 2da^k\}(X,Y) + 2d\omega^k(\sigma X, Y) + 2d\omega^k(X,\sigma Y)$[9] $= \{2\omega_h^{*k} \wedge \omega^h - 2da^k\}(X,Y) + 4d\omega^k(\sigma X,Y) = 2\{\omega_h^{*k} \wedge \omega^h - da^k\}(X,Y) + 4(a^p(X), d\omega^k X_p(Y)) = 2\{\omega_h^{*k} \wedge \omega^h + 2a^p \wedge \beta_p^k - da^k\}(X,Y)$, since $d\omega^k(\sigma X, Y) = d\omega^k(X, \sigma Y)$, it follows (c).

Proof of (d):

$T^{*k}(X,Y) = \omega^k \overset{(\sigma)}{\nabla}{}^*{}_X(\omega^h(\sigma Y)X_h) - \omega^k \overset{(\sigma)}{\nabla}{}^*{}_Y(\omega^h(\sigma X)X_h) - \omega^k[\sigma X, \sigma Y] = \omega^h(\sigma Y) \omega_h^k(X) - \omega^h(\sigma X)\omega_h^k(Y) + (\sigma X)\omega^k(\sigma Y) - (\sigma Y)\omega^k(\sigma X) - \omega^k[\sigma X, \sigma Y] = 2[\omega_h^{*k} \wedge a^h + d\omega^k(X_p, X_q)a^p \wedge a^q](X,Y) = 2[\omega_h^{*k} \wedge a^h + \beta_{pq}^k a^p \wedge a^q](X,Y), \ \forall X, Y \in \mathcal{D}_1$, from which we obtain (d).

5. Curvature's field of generalized integrable (φ)-connection

In the theory of connections it is interesting to consider the curvature tensor as a tensor field by means of which formulas are expressed for the commutation of the operation of " covariant derivation" as a consequence of the very definition of connection; in the case of an operator of connection $\overset{(\sigma)}{\nabla}_X$, it is known that the field of curvature it can be expressed as follows:

$$Q(X,Y)Z = (\overset{(\sigma)}{\nabla}_X \overset{(\sigma)}{\nabla}_Y - \overset{(\sigma)}{\nabla}_Y \overset{(\sigma)}{\nabla}_X - \overset{(\sigma)}{\nabla}_{K(X,Y)})Z \qquad \forall\ X, Y.Z \in \mathcal{D}^1$$

$$\text{with} \qquad -K(X,Y) = S(X,Y) - (\overset{(\sigma)}{\nabla}_X Y - \overset{(\sigma)}{\nabla}_Y X)$$

and S is the torsion field of the connection under consideration.We want to proceed in a similar way for a (φ)-connection. There are already two possibilities arising from the presence of two torsion fields (S^* and T^*). Between the two fields of torsion, the 1° presents greater analogy with the ordinary case; infact, consequently from the 1° of (7), we derive:

[9] See [3] p.21

$$-K(X,Y) = S^*(X,Y) - (\overset{(\sigma)}{\nabla^*} Y - \overset{(\sigma)}{\nabla^*} X);$$

Using S^*, we are thus let to construct the following field:

$$Q^*(X,Y)Z = (\overset{(\sigma)}{\nabla^*}_X \overset{(\sigma)}{\nabla^*}_Y - \overset{(\sigma)}{\nabla^*}_Y \overset{(\sigma)}{\nabla^*}_X - \overset{(\sigma)}{\nabla^*}_{K(X,Y)})Z, \tag{10}$$

with K given by (6).

Relative to (10) we have the:

Proposition 7. Necessary and sufficient condition such:

$$Q^* : \{\omega, X, Y, Z\} \in \mathcal{D}_1^3 \to \omega\{Q(X,Y)Z\} \in \mathcal{F}$$

is a tensorial field is that the sigma torsion is zero.

Proof. In (10) we put fX, gY, hZ with f, g, h $\in \mathcal{F}$ instead of X, Y, Z, unfolding and simplifying we have successively:

$Q^*(fX, gY)hZ = (\overset{(\sigma)}{\nabla^*}_{fX} \overset{(\sigma)}{\nabla^*}_{gY} - \overset{(\sigma)}{\nabla^*}_{gY} \overset{(\sigma)}{\nabla^*}_{fX} - \overset{(\sigma)}{\nabla^*}_{K(fX,gY)})hZ = fgh\{\overset{(\sigma)}{\nabla^*}_X \overset{(\sigma)}{\nabla^*}_Y - \overset{(\sigma)}{\nabla^*}_Y \overset{(\sigma)}{\nabla^*}_X\}Z + fh(\sigma X)g)\overset{(\sigma)}{\nabla^*}_Y Z + f(\sigma X)g(\sigma Y)h)Z + fg([\sigma X, \sigma Y]h)Z - gh(\sigma Y)f\overset{(\sigma)}{\nabla^*}_X Z - g(\sigma Y)f(\sigma X)hZ) - fgh\overset{(\sigma)}{\nabla^*}_{[\sigma X, Y]}Z - fg(\sigma[\sigma X, Y]h)Z - fh((\sigma X)g)\overset{(\sigma)}{\nabla^*}_Y Z - f(\sigma X)g)(\sigma Y)hZ - fgh\overset{(\sigma)}{\nabla^*}_{[X,\sigma Y]}Z - fg(\sigma[X, \sigma Y]h)Z + gh((\sigma Y)f)\overset{(\sigma)}{\nabla^*}_X Z + g(\sigma Y)f(\sigma X)hZ + fgh\overset{(\sigma)}{\nabla^*}_{\sigma[X,Y]}Z + fg(\sigma^2[X,Y]h)Z = fghQ^*(X,Y)Z + fg\{([\sigma X, \sigma Y] - \sigma K(X,Y)h\}$, which proves the assertion if and only if:

$[\sigma X, \sigma Y] - \sigma K(X,Y) = 0.$

If σ determines on M an almost complex structure (i.e. if σ is such that $\sigma^2 = -I$, with identical endomorphism I), and moreover the relative torsion is zero, σ is said to be integrable. If we agree to call σ integrable connections in a generalized sense those for which σ is torsionless (regardless $\sigma^2 = -I$), the Proposition 7 can be started as follows:

Proposition 8. *Necessary and sufficient condition that:*

$$Q^* : \{\omega, X, Y, Z\} \in \mathcal{D}_1^3 \to \{Q(X,Y)Z\} \in \mathcal{F}$$

to be a tensor field is that σ is integrable in a generalized sense.

In particular: if σ^2 = - I, σ determines a quasi-complex structure on M, the field (10) then gives the curvature tensor of an almost complex integrable structure.

6. Bianchi's Identities

The field Q*, introduced in the previous issue, satisfies two identities that are a generalization of Bianchi's identities and therefore we still call them by the same name. The identities in question are the following ones:

$$\mathcal{P}\{Q^*(X,Y)Z\} = \mathcal{P}\{\overset{(\sigma)}{\nabla^*}_X S^*(Y,Z) + S^*(X,K(Y,Z)) + K(X,K(Y,Z))\}. \quad (11)$$

$$\begin{aligned}&\mathcal{P}\{\overset{(\sigma)}{\nabla^*}_X Q^*(Y,Z) - Q^*(X,Y)\overset{(\sigma)}{\nabla^*}_Z - Q^*(K(X,Y),Z)\}W = \\ &\mathcal{P}\{\overset{(\sigma)}{\nabla^*}_{K(K(X,Y),Z)}\}W,\end{aligned} \quad (12)$$

where X,Y,Z,W $\in \mathcal{D}^1$ and $\mathcal{P}$ denotes the circular permutation with respect to the arguments X,Y,Z.

The following steps prove (11).

$\mathcal{P}\{Q^*(X,Y)Z\} = \overset{(\sigma)}{\nabla^*}_X \overset{(\sigma)}{\nabla^*}_Y Z - \overset{(\sigma)}{\nabla^*}_Y \overset{(\sigma)}{\nabla^*}_X Z - \overset{(\sigma)}{\nabla^*}_{K(X,Y)} Z + \overset{(\sigma)}{\nabla^*}_Y \overset{(\sigma)}{\nabla^*}_Z X - \overset{(\sigma)}{\nabla^*}_Z \overset{(\sigma)}{\nabla^*}_Y X - \overset{(\sigma)}{\nabla^*}_{K(Y,Z)} X + \overset{(\sigma)}{\nabla^*}_Z \overset{(\sigma)}{\nabla^*}_X Y - \overset{(\sigma)}{\nabla^*}_X \overset{(\sigma)}{\nabla^*}_Z Y - \overset{(\sigma)}{\nabla^*}_{K(Z,X)} Y.$

Using the 1° of (7), it is successively obtained:

$\overset{(\sigma)}{\nabla^*}_X\{\overset{(\sigma)}{\nabla^*}_Y Z - \overset{(\sigma)}{\nabla^*}_Z Y - K(Y,Z)\} + \overset{(\sigma)}{\nabla^*}_X K(Y,Z) - \overset{(\sigma)}{\nabla^*}_{K(Y,Z)} X + \overset{(\sigma)}{\nabla^*}_Y\{\overset{(\sigma)}{\nabla^*}_Z X - \overset{(\sigma)}{\nabla^*}_X Z - K(Z,X)\} + \overset{(\sigma)}{\nabla^*}_Y K(Z,X) - \overset{(\sigma)}{\nabla^*}_{K(Z,X)} Y + \overset{(\sigma)}{\nabla^*}_Z\{\overset{(\sigma)}{\nabla^*}_X Y - \overset{(\sigma)}{\nabla^*}_Y X - K(X,Y)\} + \overset{(\sigma)}{\nabla^*}_Z K(X,Y) - \overset{(\sigma)}{\nabla^*}_{K(X,Y)} Z =$

$\mathcal{P}\{\overset{(\sigma)}{\nabla^*}_X S^*(Y,Z) + \overset{(\sigma)}{\nabla^*}_X K(Y,Z) - \overset{(\sigma)}{\nabla^*}_{K(Y,Z)} X\}$

$= \mathcal{P}\{\overset{(\sigma)}{\nabla^*}_X S^*(Y,Z) + S^*(X,K(Y,Z)) + K(X,K(Y,Z))\}.$

The proof of (10) is similar.

Remark 3. *From (11) you can see that $\mathcal{P}\{Q^*(X,Y)Z\}$=0 if and only if they are simultaneously $S^*(X,Y) = 0$ and K(X,Y)=0 $\forall X,Y \in \mathcal{D}^1$, while the second member of (12) is equal to zero if and only if it is K(X,Y)=0 $\forall X,Y \in \mathcal{D}^1$.*

7. Curvature formes and related structural equation

In analogy with what was done in n. 4 for torsion, let us now introduce with regard to the field of curvature Q^* (with $\overset{(\sigma)}{N}=0$), the 2 forms Q_h^{*k} defined as follows:

$$Q^*(X,Y)X_h = Q_h^{*k}(X,Y)X_k \qquad\qquad (Q_h^*(X,Y) = Q^*(X,Y)X_h) \quad (13)$$

we will say Q_h^{*k} the curvature formes of (φ)-connession. Again on the assumption that σ is integrable in a generalized sense, the following proposition is valid:

Proposition 9. *The curvature formes of (φ)-connection verify the relationships (structural equations)*:

$$Q_h^{*k} = 2\{-\omega_h^{*q} \wedge \omega_q^{*k} + 2a^p \wedge \eta_{hp}^{*k} - da_h^{*k}\}, \qquad (14)$$

where the ω_h^{*q}, a^p are the forms already considered in (7) and η_{hp}^{*k}, $a_h^{*k} \in \mathcal{D}_1$ are defined as follow:

$$(d\omega_h^{*k})(X_p) = \eta_{hp}^{*k}. \qquad\qquad \omega_h^{*k}\sigma = a_h^{*k}$$

To prove (14) we develop (13) with recourse to the (10), the following steps are then obtained:
$Q_h^*(X,Y)X_h = \overset{(\sigma)}{\nabla^*}_X \overset{(\sigma)}{\nabla^*}_Y X_h - \overset{(\sigma)}{\nabla^*}_Y \overset{(\sigma)}{\nabla^*}_X X_h - \overset{(\sigma)}{\nabla^*}_{K(X,Y)} X_h = \overset{(\sigma)}{\nabla^*}_X(\omega_h^{*q}(Y)X_q) - \overset{(\sigma)}{\nabla^*}_Y(\omega_h^{*q}(X)X_q) - \overset{(\sigma)}{\nabla^*}_{[\sigma X,Y]} X_h - \overset{(\sigma)}{\nabla^*}_{[X,\sigma Y]} X_h + \overset{(\sigma)}{\nabla^*}_{\sigma[X,Y]} X_h = \omega_h^{*q}(Y)\omega_q^{*k}(X)X_k - \omega_h^{*q}(X)\omega_q^{*k}(Y)X_k + (\sigma(X)\omega_h^{*k}(Y))X_k - (\sigma(Y)\omega_h^{*k}(X))X_k - \omega_h^{*k}[\sigma X,Y]X_k - \omega_h^{*k}\,[X,\sigma Y]X_k + \omega_h^{*k}\sigma[X,Y]X_k + X\omega_h^{*k}(\sigma Y)X_k - X\omega_h^{*k}(\sigma Y)X_k + Y\omega_h^{*k}(\sigma X)X_k - Y\omega_h^{*k}(\sigma X)X_k = -2\omega_h^{*q} \wedge \omega_q^{*k}(X,Y)X_k + \{(\sigma X)(\omega_h^{*k}Y) - (Y)(\omega_h^{*k}(\sigma X)Y) - \omega_h^{*k}[\sigma X,Y]\}X_k + \{X\omega_h^{*k}(\sigma Y) - (\sigma Y)\omega_h^{*k}(X) - \omega_h^{*k}[X,\sigma Y]\}X_k - \{X\omega_h^{*k}(\sigma Y) - Y\omega_h^{*k}(\sigma X) - \omega_h^{*k}\sigma[X,Y]X_k\}X_k = [-2\omega_h^{*q} \wedge \omega_q^{*k} + 4a^p \wedge d\omega_h^{*k}X_p - 2da_h^{*k}](X,Y)$
X_k, taking into account that the expression in the first curly brackets is the same as these in the second curly brackets. From the above, (14) is obtained.

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