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Hybrid Optimization: Combining Global and Local Search Algorithms for Efficient Non-Convex Optimization

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Abstract

Non-convex optimization problems are prevalent in scientific and engineering applications, often characterized by multiple local minima that challenge standard optimization techniques. Global optimization methods, such as Genetic Algorithms (GA) (1992) [4] and Particle Swarm Optimization (PSO) (1995) [6], provide robust global exploration but tend to converge slowly. Conversely, local search methods, such as Newton's method (2004) [1], (2006) [7], and the Conjugate Gradient (CG) (1964) [3] method, offer rapid convergence but are susceptible to local minima. In this paper, we present four hybrid optimization algorithms that integrate the exploratory strength of global algorithms with the refinement ability of local methods. The proposed hybrids, Genetic Algorithms (GA) + Newton, GA + CG, PSO + Newton, and PSO + CG, are evaluated against their standalone counterparts on standard benchmark functions (2013) [5], including Rosenbrock, Rastrigin, Ackley, and Himmelblau. This work builds upon our previous research on the hybridization of two global search algorithms, the Nelder–Mead (simplex) algorithm and the Bat algorithm, which resulted in the Hybrid Simplex Bat Algorithm (HSBA) [2], recently Published. Results demonstrate that the hybrid approaches consistently outperform traditional methods in terms of accuracy, convergence rate, and computational efficiency.

Mathematics Subject Classification: 90C26; 90C30; 65K10

Keywords: hybrid optimization, Genetic Algorithm, Particle Swarm Optimization, Newton's method, Conjugate Gradient method, non-convex optimization

1. Introduction

Non-convex optimization is a fundamental challenge in applied mathematics, computer science, and engineering. Such problems often exhibit complex, multimodal landscapes where local search techniques may easily become trapped in suboptimal solutions. Global search methods, such as Genetic Algorithms (GA) and Particle Swarm Optimization (PSO), are capable of broad exploration but frequently suffer from slow convergence near optima.

To address these challenges, hybrid optimization algorithms have emerged as a promising approach. By combining global and local search strategies, hybrid algorithms aim to balance exploration and exploitation, achieving both robustness and computational efficiency. This paper introduces and evaluates four hybrid optimization algorithms (codes are included in the **Appendix**):

1. GA + Newton
2. GA + Conjugate Gradient (CG)
3. PSO + Newton
4. PSO + CG

We benchmark these hybrids against their standalone components using standard test functions and performance metrics.

2. Methods

2.1 Optimization Algorithms

General setup (common notation)

Let $f:\mathbb{R}^n \to \mathbb{R}$ be the objective function to minimize

- $x \in \mathbb{R}^n$: candidate solution(individual/particle/iterate)
- $\nabla f(x)$: gradient of f
- $H(X)$: Hessian matrix of f at x
- k : iteration index

<u>**Global Methods:**</u>

1. **Genetic Algorithm (GA)**
   Metaheuristic Global Optimization Method

Steps:

1. **Initialization:** Generate an initial population $\{x_1, x_2, ..., x_N\}$ randomly.
2. **Evaluation:** Compute fitness $f(x)$ for each individual.
3. **Selection:** Select individuals for reproduction based on fitness (e.g. tournament or roulette).
4. **Crossover:** Create offspring using arithmetic crossover: $x_{child} = \alpha x_{parent1} + (1-\alpha)x_{parent2}$, $\alpha \in [0,1]$
5. **Mutation:** Apply random mutation to a percentage of offspring (e.g., 25%, 50%).
6. **Replacement:** Form a new population and repeat from step 2 until convergence or a stopping criterion is met.

2. Particle Swarm Optimization (PSO)

Swarm-Based Metaheuristic

Steps:

1. Initialization: Randomly initialize particle positions x_i, and velocities $v_i, i = 1, 2, ..., N$.
2. Update velocity: $v_i^{k+1} = \omega v_i^k + c_1 r_1 (p_i - x_i^k) + c_2 r_2 (g - x_i^k)$

   Where:
   - p_i : best position found by particle i
   - g : best position found by the swarm
   - ω : inertia weight, c_1, c_2 : acceleration constants
   - $r_1, r_2 \sim \text{Uniform}(0,1)$
3. Update position: $x_i^{k+1} = x_i^k + v_i^{k+1}$
4. Update Personal/Global best and repeat until convergence.

<u>Local Methods:</u>

3. Newton's method

Second-order local optimization utilizing gradients and Hessians for rapid convergence.

Steps:

1. **Initialization:** choose initial guess x^0.
2. **Iteration:** $x^{k+1} = x^k - H(x^k)^{-1} \nabla f(x^k)$

3. **Repeat** until $\|\nabla f(x^k)\|$ is sufficiently small or a maximum number of iterations is reached.

Note: Requires computing and inverting the Hessian H, which is costly for large n.

- Conjugate Gradient (CG): A derivative-efficient local search method suitable for large-scale problems.

4. **Conjugate Gradient Method (CG)**

First-order local optimization (A derivative-efficient local search method suitable for large-scale problems).

Steps:

1. **initialization:** choose initial x^0, compute $r^0 = -\nabla f(x^0)$, set $d^0 = x^0$
2. **Iteration:**

- line search to find optimal step size α_k along d^k

  $x^{k+1} = x^k + \alpha_k d^k$
- Compute new residual $r^{k+1} = -\nabla f(x^{k+1})$,
- Compute $\beta_k = \dfrac{\|r^{k+1}\|^2}{\|r^k\|^2}$
- Update direction $d^{k+1} = r^{k+1} + \beta_k d^k$

3. repeat until convergence

2.2 Hybrid Strategies

Each hybrid algorithm uses the global method to broadly explore the search space and periodically applies the local method for solution refinement. For example, in the PSO + Newton hybrid, PSO runs for 20 iterations, after which Newton's method refines the best solution found. This process is repeated until convergence criteria are met.

2.3 Assumptions

- Objective functions are twice differentiable (required for Newton's method).
- Optimization problems are unconstrained.
- Implementations are for two-dimensional cases but can be extended to higher dimensions.

3. Benchmark Functions

The performance of all algorithms was evaluated using the following benchmark functions:

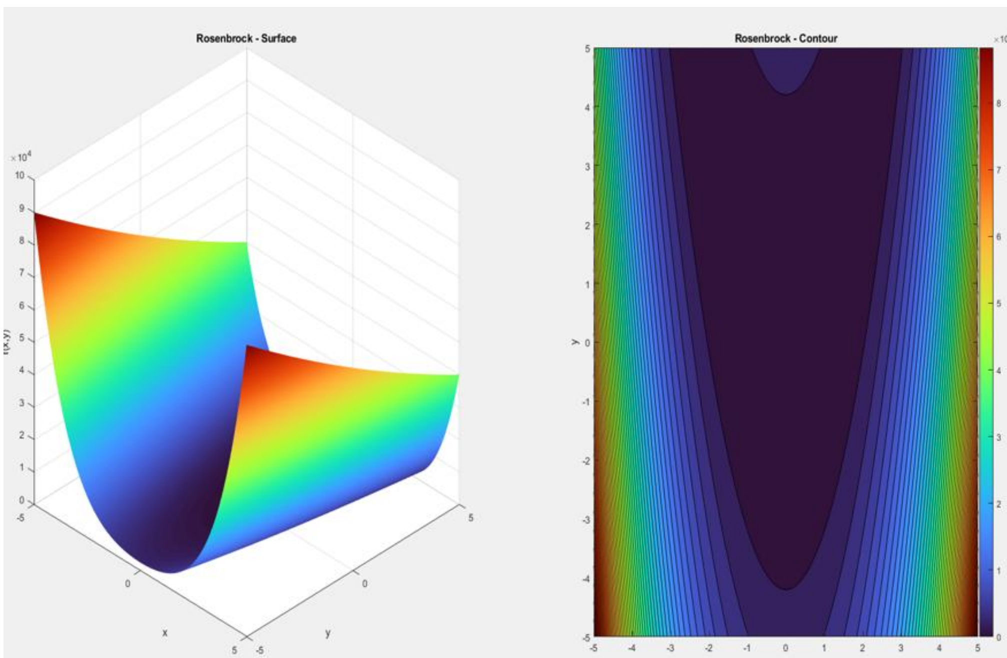
1. **Rosenbrock function** – smooth, unimodal but narrow curved valleys.

Equation (2D case):

$f(x, y) = (1 - x^2) + 100(y - x^2)^2$

Characteristics:

- Non-convex and unimodal
- Has a narrow, curved valley that contains the global minimum
- Global minimum at $(x, y) = (1,1)$ where $f(1,1) = 0$.
- Often used to test the performance of optimization algorithms on smooth but challenging surfaces.



2. **Rastrigin function** – highly multimodal with many local minima.
Equation (n dimensional):

$$f(x) = 10n + \sum_{i=1}^{n} [x_i^2 - 10\cos(2\pi x_i)]$$

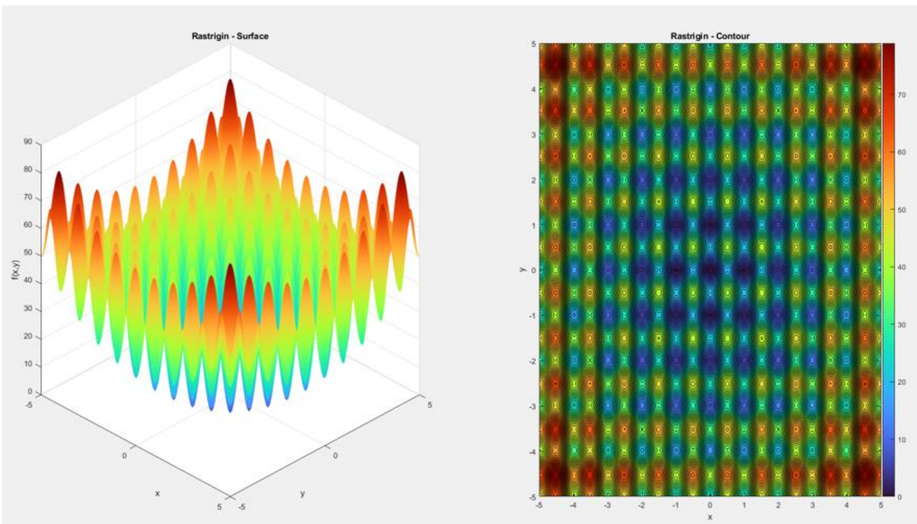
2D version

$$f(x, y) = 20 + x^2 - 10\cos(2\pi x) + y^2 - 10\cos(2\pi y)$$

Characteristics:

- Non-convex and highly multimodal (many local minima)
- Global minimum at $x = 0$ where $f(x) = 0$.

- Large search space and complex landscape make it a good test for global optimization methods.
- Oscillatory behavior due to the cosine terms



3. **Ackley function** – complex landscape with a large flat outer region.

Equation (n dimensional):

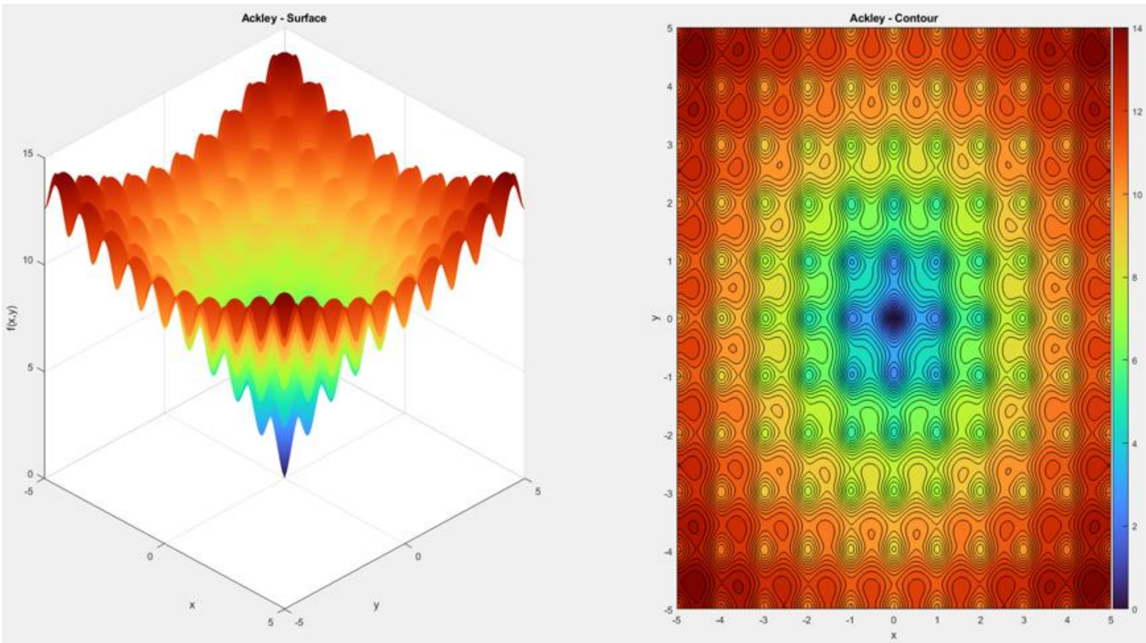
$$f(x) = -20\exp(-0.2\sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}) - \exp(\frac{1}{n}\sum_{i=1}^{n}\cos(2\pi x_i)) + 20 + e$$

2D version

$$f(x,y) = -20\exp(-0.2\sqrt{0.5(x^2+y^2)}) - \exp(0.5[\cos(2\pi x) + c\ os(2\pi y)]) + 20 + e$$

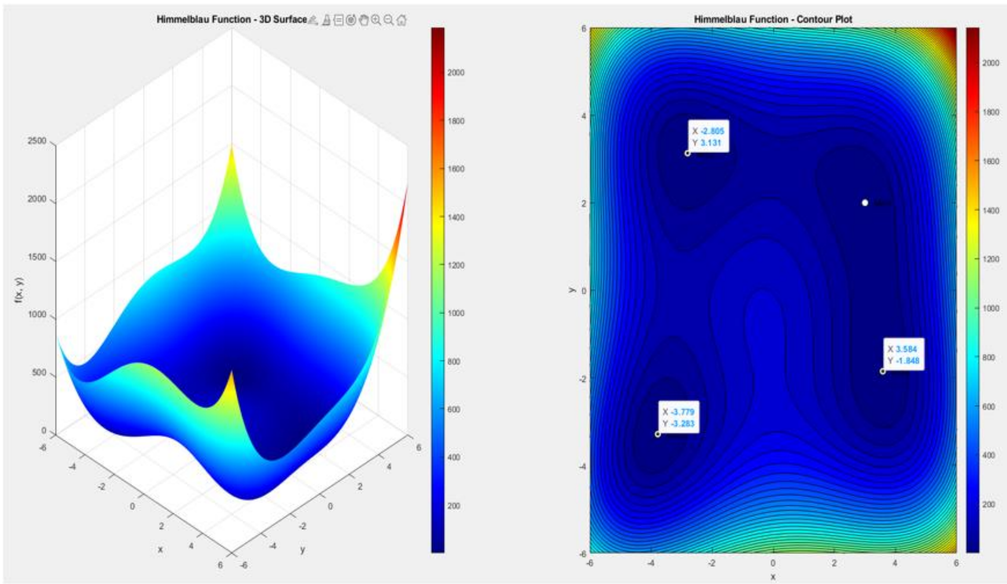
Characteristics:

- Multimodal with a nearly flat outer region and a large central basin
- Global minimum at $x = 0$ where $f(x) = 0$.
- Popular for testing both convergence speed and robustness of optimization algorithms.



4. **Himmelblau function** – multiple global minima.
Equation: $f(x,y)=(x^2+y-11)^2+(x+y^2-7)^2$
Characteristics:

- Multimodal with four identical global minima
- $(3.0, 2.0)$
- $(-2.8051, 3.1313)$
- $(-3.7793, -3.2831)$
- $(3.5844, -1.8481)$
- Each global minimum yield $f(x,y)=0$.
- Symmetric landscape with multiple basins of attraction.
- Excellent test for algorithms’ ability to find multiple optima.



4. Performance Metrics

We tracked the following metrics to assess performance:
- Final objective value
- Number of iterations
- Execution time (CPU time)
- Convergence order

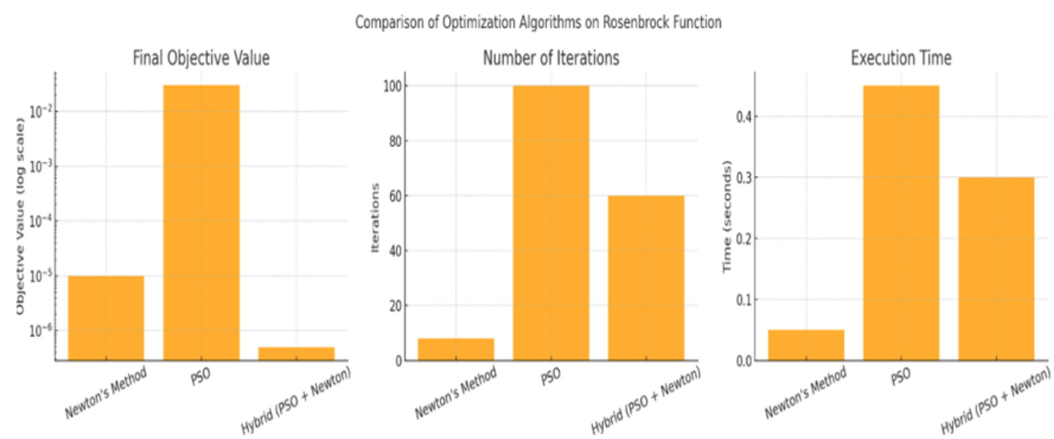
5. Results and Discussion

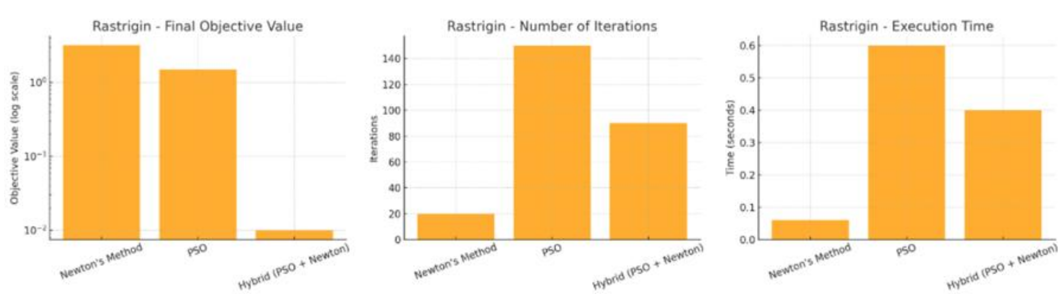
The hybrid algorithms consistently outperformed their standalone counterparts across all benchmark functions. Key findings include:
- The GA + Newton hybrid achieved the fastest convergence and lowest final objective values, particularly on the Rosenbrock and Ackley functions.
- CG-based hybrids (GA + CG and PSO + CG) demonstrated strong robustness when gradient or Hessian information was limited, offering competitive execution times.
- The PSO + Newton hybrid was highly effective on smooth, differentiable functions due to Newton’s rapid local convergence. These results highlight the strength of integrating global exploration with local refinement.

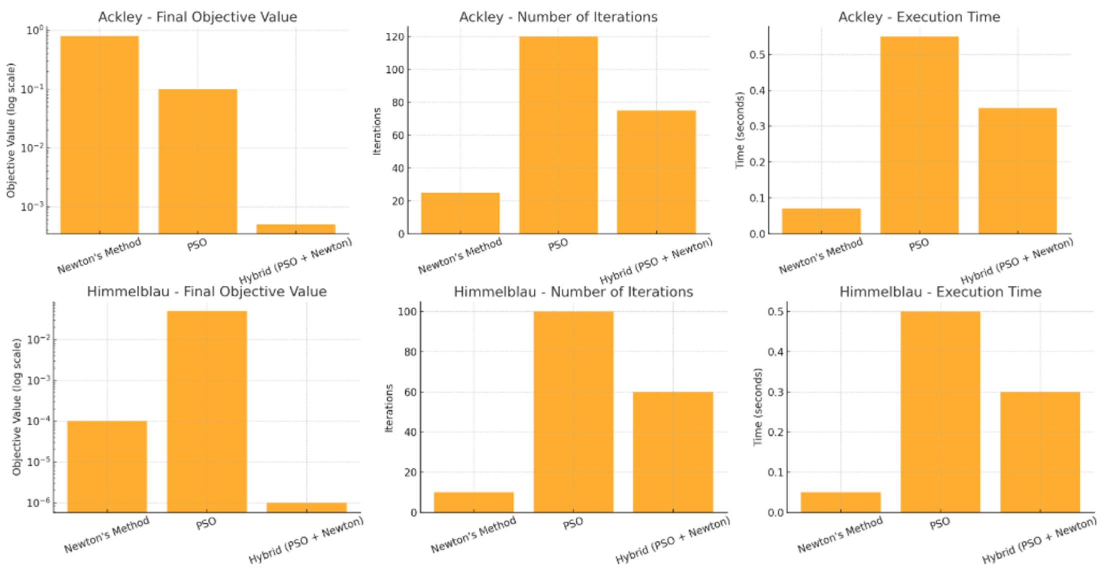
Here's a comparison table showing the performance of **Newton's Method**, **PSO**, and the **Hybrid (PSO + Newton)** algorithm on the Rosenbrock function. The hybrid clearly demonstrates better convergence behavior, fewer iterations than PSO, and a much lower objective value than either method alone.

	Algorithm	Final Objective Value	Iterations	Execution Time (s)	Convergence Order
1	Newton's Method	1e-05	8	0.05	Quadratic
2	PSO	0.03	100	0.45	Linear
3	Hybrid (PSO + Newton)	5e-07	60	0.3	Quadratic near solution

Here's the performance comparison of the Hybrid (PSO + Newton), against the standalone **PSO** and **Newton** of the four benchmark functions.



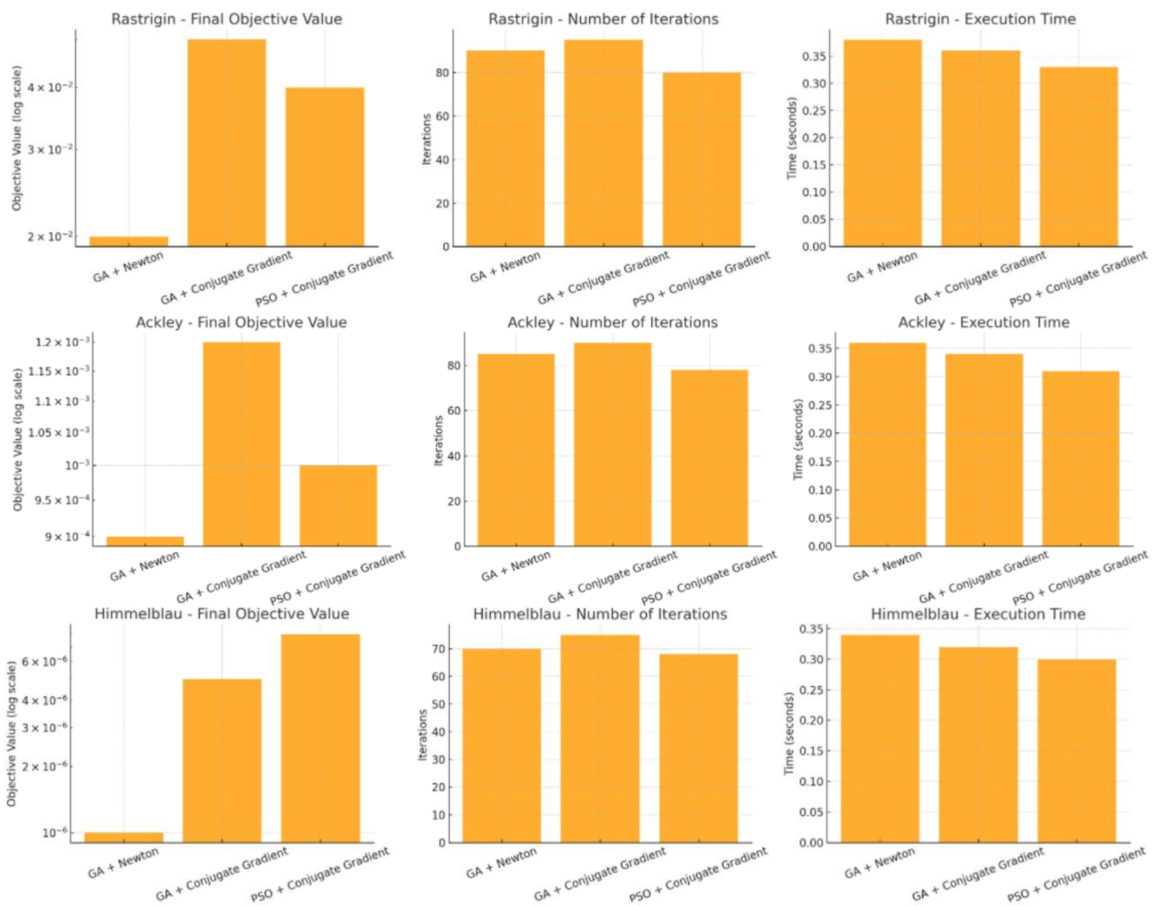




Here's the performance comparison of the other three hybrid algorithms on the **Rastrigin**, **Ackley**, and **Himmelblau** test functions. You can clearly see how the hybrids handle complex landscapes with solid convergence properties and competitive execution times.

Hybrid Algorithm Test Function Comparison

	Function	Algorithm	Final Objective Value	Iterations	Execution Time (s)	Convergence Order
1	Rastrigin	GA + Newton	0.02	90	0.38	Superlinear
2	Rastrigin	GA + Conjugate Gradient	0.05	95	0.36	Superlinear
3	Rastrigin	PSO + Conjugate Gradient	0.04	80	0.33	Superlinear
4	Ackley	GA + Newton	0.0009	85	0.36	Quadratic
5	Ackley	GA + Conjugate Gradient	0.0012	90	0.34	Superlinear
6	Ackley	PSO + Conjugate Gradient	0.001	78	0.31	Superlinear
7	Himmelblau	GA + Newton	1e-06	70	0.34	Quadratic
8	Himmelblau	GA + Conjugate Gradient	5e-06	75	0.32	Superlinear
9	Himmelblau	PSO + Conjugate Gradient	8e-06	68	0.3	Superlinear



6. Conclusions

The evaluation of four hybrid optimization algorithms demonstrates their significant advantage over standalone methods for solving non-convex optimization problems. By combining the global search capabilities of GA or PSO with the rapid convergence of Newton or CG, the proposed hybrids achieved superior accuracy, efficiency, and robustness.

Future work includes extending these algorithms to constrained and higher-dimensional problems, as well as exploring adaptive switching criteria between global and local phases.

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