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On Oppermann Conjecture

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Abstract

Oppermann conjecture states that there exists a prime between n^2 and $n(n+1)$ and between $n(n+1)$ and $(n+1)^2$, respectively. In this paper, on the basis of the characteristic function of odd primes, we introduce some conditional extreme values problems related to above conjecture and use the method of Lagrange multiplies and the induction to confirm the conjecture. Here the technique of "adding a new variable" and the infinitude of odd primes are used.

Mathematics Subject Classification: 11N05

Keywords: Oppermann conjecture, conditional extreme values problem

1 Introduction

Oppermann in 1882 proposed the following conjecture (see [1], [3]):
there exists a prime between n^2 and $n(n+1)$ and between $n(n+1)$ and $(n+1)^2$, respectively.

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The conjecture involves actually two statements and contacts closely with Legendre conjecture (i.e., there exists a prime between n^2 and $(n+1)^2$). We will confirm the conjecture.

Let $\delta(i)$ be the characteristic function of odd primes, i.e.,

$$\delta(i) = 1, \text{ if } i \text{ is an odd prime;}$$

$$\delta(i) = 0, \text{ if } i = 1, 2 \text{ or composite number.}$$

For example, $\delta(1) = 0$, $\delta(2) = 0$, $\delta(3) = 1$, $\delta(4) = 0$, $\delta(5) = 1$, $\delta(6) = 0$, $\delta(7) = 1$, $\delta(8) = 0$, and $\delta(9) = 0, \cdots$. It sees easily

$$\delta(i)^2 = \delta(i).$$

The main result of the paper is

Theorem 1.1. *Oppermann conjectur is true.*

To prove Theorem 1.1, we apply the induction and on the basis of the characteristic function of odd primes, introduce some conditional extreme values problems to obtain the conlusion. In the process, the technique of "adding a new variable" and the infinitude of odd primes are used.

The plan of the paper is as follows. The proof of Theorem 1.1 (there exists a prime between n^2 and $n(n+1)$) is in Section 2. The proof of Theorem 1.1 (there exists a prime between $n(n+1)$ and $(n+1)^2$) is in Section 3.

2 Proof of Theorem 1.1 (there exists a prime between n^2 and $n(n+1)$)

Let us first relate the method of Lagrange multipliers (e.g., refer to [2]). For seeking the maximum and minimum values of $f(x)(x \in R^n)$ subject to constraints

$$g_i(x) = 0 \ (i = 1, 2, \cdots, k, k < n)$$

(assuming that these extreme values exist and the rank of Jacobian matrix

$$\frac{\partial(g_1, \cdots, g_k)}{\partial(x_1, \cdots, x_n)}$$

of $g_i(x)$ $(i = 1, 2, \cdots, k)$ is k):

(a) find all $x \in R^n, \lambda_1, \cdots, \lambda_k \in R$ such that

$$\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \cdots + \lambda_k \frac{\partial g_k}{\partial x_i} = 0, \ i = 1, \cdots, n,$$

$$g_i(x) = 0, \ i = 1, 2, \cdots, k,$$

where x is the stationary point and $\lambda_1, \cdots, \lambda_k$ are multipliers;
(b) evaluate f at all the points x that result from (a). The largest of these values is the maximum value of f and the smallest is the minimum value of f.

Proof of Theorem 1.1(there exists a prime between n^2 and $n(n+1)$)

For the positive integer n,
when $n = 1$, there exists a prime 2 between 1^2 and $1 \cdot 2$;
when $n = 2$, there exists a prime 5 between 2^2 and $2 \cdot 3$;
when $n = 3$, there exists a prime 11 between 3^2 and $3 \cdot 4$;
when $n = 4$, there exist primes 17, 19 between 4^2 and $4 \cdot 5$;
when $n = 5$, there exists a prime 29 between 5^2 and $5 \cdot 6$;
when $n = 6$, there exist primes 37, 41 between 6^2 and $6 \cdot 7$;
when $n = 7$, there exists a prime 53 between 7^2 and $7 \cdot 8$;
when $n = 8$, there exist primes 67, 71 between 8^2 and $8 \cdot 9$;
$\cdots\cdots$.

Now we apply the induction and suppose that the conclusion is true for $k(k > 8)$, then there exists a prime between k^2 and $k(k+1)$. Let us prove that there exists a prime between $(k+1)^2$ and $(k+1)(k+2)$. Denote odd primes between 1 and k^2 by

$$k_1, k_2, \cdots, k_l,$$

clearly, $l > 1$ from $k > 8$. Denote odd primes between k^2 and $k(k+1)$ by

$$p_1, p_2, \cdots, p_{l_1},$$

we know $l_1 \geq 1$ from the assumption above. Denote odd primes between $k(k+1)$ and $(k+1)^2$ by

$$r_1, r_2, \cdots, r_{l_2},$$

then $l_2 \geq 0$. We note that odd integers between $(k+1)^2$ and $(k+1)(k+2)$ have two forms. The first form is

$$(k+1)^2 + (p_j + 1 - k^2) = 2k + 2 + p_j,$$

(where $2k + 2 + p_j < 2k + 2 + k(k+1) = (k+1)(k+2)$, $\delta(2k + 2 + p_j) = 0$ or 1), and the second form is

$$(k+1)^2 + t \ (\text{where } (k+1)^2 + t \neq 2k + 2 + p_j)$$

with writting odd primes in the second form as

$$q_1, q_2, \cdots, q_{l_3}, \ l_3 \geq 0.$$

Then
(2.1)

$$\sum_{i=1}^{l} \delta(k_i) + \sum_{j=1}^{l_1} \delta(p_j) + \sum_{s=1}^{l_2} \delta(r_s) + \sum_{j=1}^{l_1} \delta(2k + 2 + p_j) + \sum_{\alpha=1}^{l_3} \delta(q_\alpha) = \pi((k+1)(k+2)),$$

where $\pi((k+1)(k+2))$ means the number of odd primes not excedding $(k+1)(k+2)$.

Take a large odd prime $N>(k+1)(k+2)$, i.e., $\delta(N)=1$. Such N can be chosen from the infinitude of odd primes.

If $l_3>0$, then it shows that there exists a prime between $(k+1)^2$ and $(k+1)(k+2)$, the result is obtained. If $l_3=0$, we will prove that there exists some p_j, such that $2k+2+p_j$ is an prime.

Denote the point with the components

$$\delta(k_i)(i=1,\cdots,l),\delta(p_j)(i=1,\cdots,l_1),\delta(r_s)(s=1,\cdots,l_2),\delta(2k+2+p_j)(j=1,\cdots,l_1),\delta(N)$$

by $P\in \mathrm{R}^{2l+l_1+l_2+1}$, where $\mathrm{R}^{2l+l_1+l_2+1}$ is the $2l+l_1+l_2+1$ dimensional Euclidean space. Denote

$$x=(x_1,x_2,\cdots,x_l),y=(y_1,y_2,\cdots,y_{l_1}),u=(u_1,u_2,\cdots,u_{l_2}),v=(v_1,v_2,\cdots,v_{l_1}),z=z.$$

Introduce an objective function on $\mathrm{R}^{2l+l_1+l_2+1}$:

$$f(x,y,u,v,z)=\sum_{j=1}^{l}\left(v_j^2+v_j\right). \tag{2.2}$$

Noting (2.1), properties of $\delta(i)$ and P satisfies

$$\sum_{i=1}^{l}\delta(k_i)^2+\sum_{j=1}^{l_1}\delta(p_j)^2+\sum_{s=1}^{l_2}\delta(r_s)^2+\sum_{j=1}^{l_1}\delta(2k+2+p_j)^2=\pi((k+1)(k+2))\delta(N)$$

and

$$\sum_{i=1}^{l}\left(\delta(k_i)^2+\delta(k_i)\right)+\sum_{i=1}^{l_1}\left(\delta(p_j)^2+\delta(p_j)\right)+\sum_{s=1}^{l_2}\left(\delta(r_s)^2+\delta(\delta(r_s))\right)+\sum_{j=1}^{l_1}\delta(2k+2+p_j)^2$$
$$=\pi((k+1)(k+2)+l+l_1+l_2,$$

we define two functions

$$g(x,y,u,v,z)=\sum_{i=1}^{l}x_i^2+\sum_{j=1}^{l_1}y_j^2+\sum_{s=1}^{l_2}u_s^2+\sum_{j=1}^{l_1}v_j^2-\pi((k+1)(k+2))z, \tag{2.3}$$

$$\begin{aligned}h(x,y,u,v,z)=&\sum_{i=1}^{l}\left(x_i^2+x_i\right)+\sum_{i=1}^{l_1}\left(y_j^2+y_j\right)+\sum_{s=1}^{l_2}\left(u_s^2+u_s\right)+\sum_{j=1}^{l_1}v_j^2\\&-\pi((k+1)(k+2)-l-l_1-l_2.\end{aligned} \tag{2.4}$$

Consider the extreme values of $f(x,y,u,v,z)$ subject to constraits

$$g(x,y,u,v,z)=0 \text{ and } h(x,y,u,v,z)=0.$$

Denote

$$A=\{(x,y,u,v,z)\in \mathrm{R}^{2l+l_1+l_2+1}|g(x,y,u,v,z)=0,h(x,y,u,v,z)=0\}.$$

Obviously,

$$P\in A.$$

Since $g(x,y,u,v,z)=0$ is the rotating paraboloid in $\mathrm{R}^{2l+l_1+l_2+1}$ and $h(x,y,u,v,z)=0$ is the ellipse cylinder in $\mathrm{R}^{2l+l_1+l_2+1}$, we see that A is a bounded closed set in $\mathrm{R}^{2l+l_1+l_2+1}$, and the rank of Jacobian matrix on A of $g(x,y,u,v,z)$ and $h(x,y,u,v,z)$ is 2. Then $f(x,y,u,v,z)$ allows the maximum value and minimum value on A.

Define the Lagrange function

$$Q(x,y,u,v,z,\lambda,\mu)=f(x,y,u,v,z)+\lambda g(x,y,u,v,z)+\mu h(x,y,u,v,z). \tag{2.5}$$

We will use the method of Lagrange multipliers to solve all stationary points of $f(x,y,u,v,z)$ on A.

Because of

$$Q_z=-\pi((k+1)(k+2))\lambda=0,$$

we have

$$\lambda=0.$$

Using

$$\begin{cases} Q_{x_i}=2\lambda x_i+2\mu x_i+\mu=0,\\ Q_{y_j}=2\lambda y_j+2\mu y_j+\mu=0,\\ Q_{u_s}=2\lambda u_s+2\mu u_s+\mu=0,\\ Q_{v_j}=2v_j+1+2\lambda v_j+2\mu v_j=0,\end{cases}$$

we combine $\lambda=0$ to derive

$$\begin{cases} \mu(2x_i+1)=0,\\ \mu(2y_j+1)=0,\\ \mu(2u_s+1)=0,\\ (2+2\mu)v_j=-1.\end{cases} \tag{2.6}$$

Now we deal with cases $\mu=0$ and $\mu\neq 0$.

If $\mu=0$, it follows from (2.6) that

$$x_i,y_j,u_s \text{ are arbitrary, } v_j=-\tfrac{1}{2},$$

so

$$f=\left(\frac{1}{4}-\frac{1}{2}\right)l_1<0,$$

and

$$f_{\max}=-\frac{1}{4}l_1<0,$$

hence $f(P)=\sum_{j=1}^{l_1}(\delta(2k+2+p_j)^2+\delta(2k+2+p_j))\leq f_{\max}<0$, but it contradicts to $f(P)\geq 0$.

If $\mu\neq 0$, then

$$x_i=-\frac{1}{2},y_j=-\frac{1}{2},u_s=-\frac{1}{2},v_j=-\frac{1}{2+2\mu}$$

(where $\mu\neq -1$, otherwise, it will yield $0\cdot v_j=-1$ from $\mu=-1$, a contradiction), and

$$0=h(x,y,u,v,z)=-\frac{1}{4}l-\frac{1}{4}l_1-\frac{1}{4}l_2+\frac{l_1}{(2+2\mu)^2}-(\pi((k+1)(k+1))+l+l_1+l_2)\,,$$

i.e.,

$$\frac{1}{2+2\mu}=\pm\sqrt{\frac{5(l+l_1+l_2)}{4l_1}+\frac{\pi(n^2+2n+1)}{l_1}},$$
$$v_j=\mp\sqrt{\frac{5(l+l_1+l_2)}{4l_1}+\frac{\pi(n^2+2n+1)}{l_1}},$$

therefore,

$$f_{\min}=l_1\left(\frac{5(l+l_1+l_2)}{4l_1}+\frac{\pi((k+1)(k+2))}{l_1}-\sqrt{\frac{5(l+l_1+l_2)}{4l_1}+\frac{\pi((k+1)(k+2))}{l_1}}\right)>0,$$

and

$$f(P)=\sum_{j=1}^{l_1}\left(\delta(2k+2+p_j)^2+\delta(2k+2+p_j)\right)\geq f_{\min}>0.$$

It shows that there exists some p_i such that $2k+2+p_i$ is an odd prime. The conclusion is proved.

3 Proof of Theorem 1.1 (there exists a prime between $n(n+1)$ and $(n+1)^2$)

For the positive integer n,
when $n=1$, there exist primes 2, 3 between $1\cdot 2$ and 2^2;
when $n=2$, there exists a prime 7 between $2\cdot 3$ and 3^2;
when $n=3$, there exists a prime 13 between $3\cdot 4$ and 4^2;
when $n=4$, there exists a prime 23 between $4\cdot 5$ and 5^2;
when $n=5$, there exists a prime 31 between $5\cdot 6$ and 6^2;
when $n=6$, there exist primes 43, 47 between $6\cdot 7$ and 7^2;

when $n = 7$, there exist primes 59, 61 between $7 \cdot 8$ and 8^2;
when $n = 8$, there exist primes 73, 79 between $8 \cdot 9$ and 9^2;
$\cdots\cdots$.

We apply the induction and assume that the conclusion holds for $k(k > 8)$, i.e., there exists a prime between $k(k+1)$ and $(k+1)^2$. We will prove that there exists a prime between $(k+1)(k+2)$ and $(k+2)^2$. Let us denote odd primes between 1 and $k(k+1)$ by

$$k_1, k_2, \cdots, k_l,$$

then $l > 1$ from $k > 8$. Denote odd primes between $k(k+1)$ and $(k+1)^2$ by

$$p_1, p_2, \cdots, p_{l_1},$$

so $l_1 \geq 1$ from the assumption above. Denote odd primes between $(k+1)^2$ and $(k+1)(k+2)$ by

$$r_1, r_2, \cdots, r_{l_2},$$

then $l_2 \geq 0$. The odd integers between $(k+1)(k+2)$ and $(k+2)^2$ have the first form

$$(k+1)(k+2) + p_j - k(k+1) = 2k + 2 + p_j,$$

(note $2k + 2 + p_j < 2k + 2 + (k+1)^2 < (k+2)^2$, $\delta(2k + 2 + p_j) = 0$ or 1), and the second form is $(k+1)(k+2) + t$ (where $(k+1)(k+2) + t \neq 2k + 2 + p_j$). Denote odd primes in the second form by

$$q_1, q_2, \cdots, q_{l_3},\ l_3 \geq 0.$$

Then

$$\sum_{i=1}^{l} \delta(k_i) + \sum_{j=1}^{l_1} \delta(p_j) + \sum_{s=1}^{l_2} \delta(r_s) + \sum_{j=1}^{l_1} \delta(2k + 2 + p_j) + \sum_{\alpha=1}^{l_3} \delta(q_\alpha) = \pi((k+1)(k+2)).$$

From now on we can do as in the prevous section, and omit details.

Conflicts of Interest. The authors declare that there is no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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