

Formulas for Calculating ζ_3

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Abstract

In this article we want to show an application of the Huygens-Steiner theorem to number theory and in particular to the calculation of ζ_3 .

Introduction and Main results

Given that:

Given a material point P of mass m and a generic straight line r distant d from P, the moment of inertia of P with respect to r is defined as the product md^2 .

In general, if we have a finite system of material points $P_i (i = 1, 2, \dots)$, the sum of the moments of inertia of its individual points will be called the moment of inertia of the system with respect to r, the sum of the moments of inertia of its individual points:

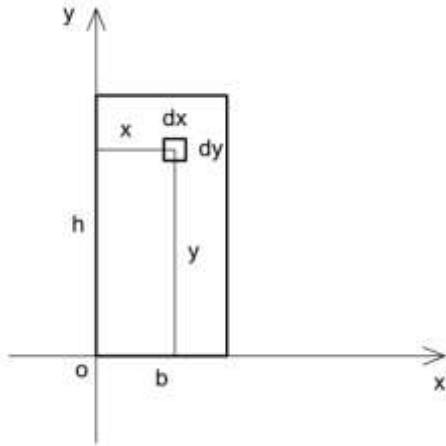
by definition $I = \sum_{i=1}^n m_i d_i^2$.

Having said this, the moment of inertia of a system varies as the axis with respect to which it is calculated varies and for parallel axes the Huygens-Steiner theorem holds:

The moment of inertia of a system with respect to an axis r is equal to the moment of inertia I_0 with respect to the parallel axis r_0 , passing through the center of gravity, increased by the product of the total mass m by the square of the d of these two axes:

$$I = I_0 + \left(\sum_{i=1}^n m_i \right) d^2$$

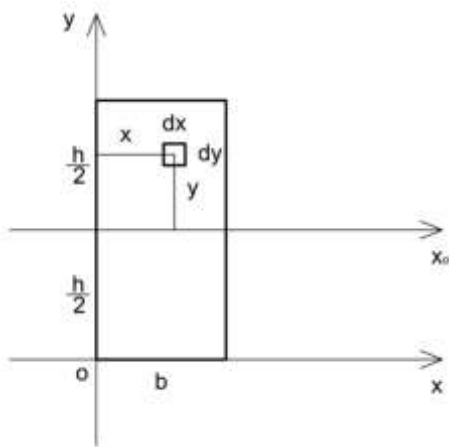
Now consider a rectangular plane figure as shown below



With respect to the x axis the moment of inertia will be

$$I_x = \int_0^b \int_0^h y^2 dx dy = \frac{bh^3}{3};$$

Considering the axis passing through the center of gravity and parallel to the base side we have

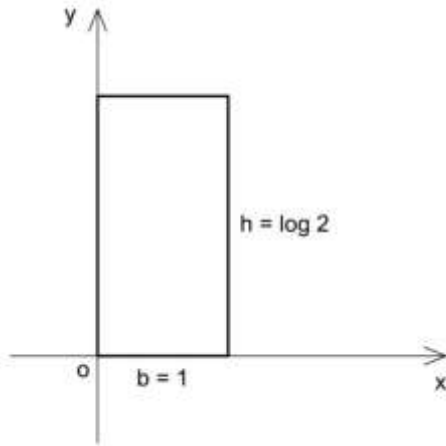


$$I_{x_0} = \int_0^b \int_{-\frac{h}{2}}^{+\frac{h}{2}} y^2 dx dy = \frac{bh^3}{12}.$$

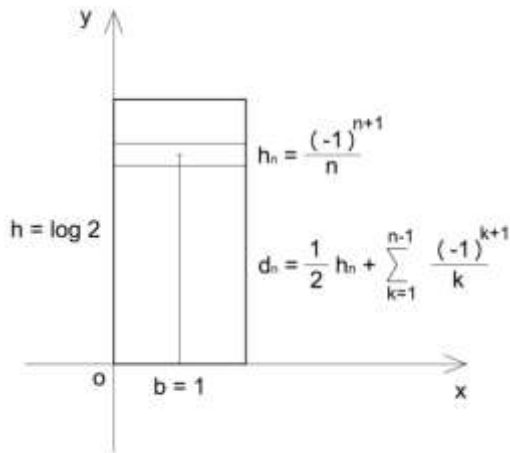
Now consider a rectangle with base $b = 1$ and height $h = \log 2$.

Such a rectangle has a moment of inertia with respect to the base X equal to:

$$I_x = \frac{1}{3} \log^3 2.$$



Furthermore, since $\log 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ (P. Mengoli 1647) one can think of $h = \log 2$ as the sum of infinite segments each with an amplitude equal to $\frac{(-1)^{n+1}}{n}$ so that the rectangle with base $b = 1$ and height $h = \log 2$ is made up of infinite rectangles all having base $b = 1$ and heights equal to $h_n = \frac{(-1)^{n+1}}{n}$.



A rectangle composed in this way has, with respect to the base (x-axis), a moment of inertia given by the sum of the moments of inertia of the individual rectangles with respect to the same axis.

The moment of inertia of each single rectangle with base $b = 1$ and height $h_n = \frac{(-1)^{n+1}}{n}$ with respect to the x axis is given by the moment of inertia with respect to its center of gravity increased by the product of the area times the squared distance of its center of gravity from the x axis.

$$I_{x_n} = \frac{1}{12} \frac{(-1)^{n+1}}{n^3} + \frac{(-1)^{n+1}}{n} \left[\frac{(-1)^{n+1}}{2n} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \right]^2$$

For the whole rectangle we will have:

$$I_x = \sum_{n=1}^{\infty} I_{x_n} = \frac{1}{12} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\frac{(-1)^{n+1}}{2n} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \right]^2$$

So we have it

$$\frac{1}{3} \log^3 2 = \frac{1}{12} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\frac{(-1)^{n+1}}{2n} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \right]^2$$

being that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} = \frac{3}{4} \zeta_3$ you get it

$$\zeta_3 = \frac{16}{3} \log^3 2 - 16 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\frac{(-1)^{n+1}}{2n} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \right]^2$$

For $\log 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n}$ you get it:

$$\begin{aligned} \zeta_3 = & \frac{4}{3} \sum_{n=1}^{\infty} \frac{1}{n^3 2^{3n}} \\ & + 16 \left[\sum_{n=1}^{\infty} \frac{1}{n2^n} \left(\frac{1}{n2^{n+1}} + \sum_{k=1}^{n-1} \frac{1}{k2^k} \right)^2 \right. \\ & \left. - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{(-1)^{n+1}}{2n} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \right)^2 \right] \end{aligned}$$

For $\log 2 = \frac{2}{3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)9^n}$ you get it:

$$\begin{aligned} \zeta_3 = & \frac{32}{81} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3 9^{3n}} \\ & + \frac{32}{3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)9^n} \left[\frac{1}{3(2n+1)9^n} + \frac{2}{3} \sum_{k=0}^{n-1} \frac{1}{(2k+1)9^k} \right]^2 \\ & - 16 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\frac{(-1)^{n+1}}{2n} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \right]^2 \end{aligned}$$

For $\log 2 = \frac{3}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)(3n+2)}$ you get it

$$\begin{aligned}\zeta_3 = & \frac{9}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)^3(3n+2)^3} \\ & + 24 \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)(3n+2)} \left[\frac{3}{4} \frac{(-1)^n}{(3n+1)(3n+2)} \right. \\ & \left. + \sum_{k=0}^{n-1} \frac{3}{2} \frac{(-1)^k}{(3k+1)(3k+2)} \right]^2 \\ & - 16 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\frac{(-1)^{n+1}}{2n} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \right]^2\end{aligned}$$

For $\log 2 = \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)}$ you get it:

$$\begin{aligned}\zeta_3 = & \frac{1}{6} \sum_{n=1}^{\infty} \frac{1}{n^3(2n-1)^3} \\ & + 16 \left[\sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} \left(\frac{1}{4n(2n-1)} + \sum_{k=1}^{n-1} \frac{1}{2k(2k-1)} \right)^2 \right. \\ & \left. - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{(-1)^{n+1}}{2n} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \right)^2 \right]\end{aligned}$$

For $\log 2 = \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)}$ you get it:

$$\begin{aligned}\zeta_3 = & \frac{4}{3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3(2n+2)^3} \\ & + 16 \left[\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} \left(\frac{1}{2(2n+1)(2n+2)} \right. \right. \\ & \left. \left. + \sum_{k=0}^{n-1} \frac{1}{(2k+1)(2k+2)} \right)^2 \right. \\ & \left. - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{(-1)^{n+1}}{2n} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \right)^2 \right]\end{aligned}$$

For $\log 2 = 2 \sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)^{2n+1}}{2n+1}$ (L. Euler 1748) you get it:

$$\begin{aligned}\zeta_3 = & \frac{32}{3} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)^{6n+3}}{(2n+1)^3} + 32 \sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)^{2n+1}}{(2n+1)} \left[\frac{\left(\frac{1}{3}\right)^{2n+1}}{2n+1} + 2 \sum_{k=0}^{n-1} \frac{\left(\frac{1}{3}\right)^{2k+1}}{2k+1} \right]^2 \\ & - 16 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\frac{(-1)^{n+1}}{2n} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \right]^2\end{aligned}$$

For $\log 2 = \frac{1}{6} \sum_{n=0}^{\infty} \frac{\left(9 - \frac{1}{9^n}\right)}{(2n+1)(2n+3)}$ (C.S. Siracusa 2021) you get it:

$$\begin{aligned}\zeta_3 = & \frac{1}{162} \sum_{n=0}^{\infty} \frac{\left(9 - \frac{1}{9^n}\right)^3}{(2n+1)^3(2n+3)^3} \\ & + \frac{8}{3} \sum_{n=0}^{\infty} \frac{\left(9 - \frac{1}{9^n}\right)}{(2n+1)(2n+3)} \left[\frac{1}{12} \frac{\left(9 - \frac{1}{9^n}\right)}{(2n+1)(2n+3)} \right. \\ & \left. + \sum_{k=0}^{n-1} \frac{\left(9 - \frac{1}{9^k}\right)}{(2k+1)(2k+3)} \right]^2 \\ & - 16 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\frac{(-1)^{n+1}}{2n} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \right]^2\end{aligned}$$

For $\log 2 = \sum_{n=1}^{\infty} \frac{\left(1 - \frac{1}{2^n}\right)}{n(n+1)}$ you get it:

$$\begin{aligned}\zeta_3 = & \frac{4}{3} \sum_{n=1}^{\infty} \frac{\left(1 - \frac{1}{2^n}\right)^3}{n^3(n+1)^3} + 16 \sum_{n=1}^{\infty} \frac{\left(1 - \frac{1}{2^n}\right)}{n(n+1)} \left[\frac{1}{2} \frac{\left(1 - \frac{1}{2^n}\right)}{n(n+1)} + \sum_{k=1}^{n-1} \frac{\left(1 - \frac{1}{2^k}\right)}{k(k+1)} \right]^2 \\ & - 16 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\frac{(-1)^{n+1}}{2n} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \right]^2\end{aligned}$$

For $\log 2 = \sum_{n=1}^{\infty} \frac{\zeta_{(2n)} - 1}{n}$ you get it:

$$\frac{1}{3} \log^3 2 = \frac{1}{12} \sum_{n=1}^{\infty} \frac{(\zeta_{(2n)} - 1)^3}{n^3} + \sum_{n=1}^{\infty} \frac{(\zeta_{(2n)} - 1)}{n} \left[\frac{(\zeta_{(2n)} - 1)}{2n} + \sum_{k=1}^{n-1} \frac{(\zeta_{(2k)} - 1)}{k} \right]^2$$

$$\begin{aligned}\zeta_3 = & \frac{4}{3} \sum_{n=1}^{\infty} \frac{(\zeta_{(2n)} - 1)^3}{n^3} \\ & + 16 \sum_{n=1}^{\infty} \frac{(\zeta_{(2n)} - 1)}{n} \left[\frac{(\zeta_{(2n)} - 1)}{2n} + \sum_{k=1}^{n-1} \frac{(\zeta_{(2k)} - 1)}{k} \right]^2 \\ & - 16 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\frac{(-1)^{n+1}}{2n} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \right]^2\end{aligned}$$

For $\log 2 = \sum_{n=1}^{\infty} \frac{2^{2n}-1}{n3^{2n}} \zeta_{(2n)}$ you get it:

$$\begin{aligned}\frac{1}{3} \log^3 2 = & \frac{1}{12} \sum_{n=1}^{\infty} \frac{(2^{2n} - 1)^3}{n^3 3^{6n}} \zeta_{(2n)}^3 \\ & + \sum_{n=1}^{\infty} \frac{(2^{2n} - 1)}{n3^{2n}} \zeta_{(2n)} \left[\frac{(2^{2n} - 1)}{2n3^{2n}} \zeta_{(2n)} + \sum_{k=1}^{n-1} \frac{(2^{2k} - 1)}{k3^{2k}} \zeta_{(2k)} \right]^2\end{aligned}$$

$$\begin{aligned}\zeta_3 = & \frac{4}{3} \sum_{n=1}^{\infty} \frac{(2^{2n} - 1)^3}{n^3 3^{6n}} \zeta_{(2n)}^3 \\ & + 16 \sum_{n=1}^{\infty} \frac{(2^{2n} - 1)}{n3^{2n}} \zeta_{(2n)} \left[\frac{(2^{2n} - 1)}{2n3^{2n}} \zeta_{(2n)} + \sum_{k=1}^{n-1} \frac{(2^{2k} - 1)}{k3^{2k}} \zeta_{(2k)} \right]^2 \\ & - 16 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\frac{(-1)^{n+1}}{2n} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \right]^2\end{aligned}$$

For $\log 2 = 2 \sum_{n=1}^{\infty} \frac{2^{2n}-1}{n4^{2n}} \zeta_{(2n)}$ you get it:

$$\begin{aligned}\frac{1}{3} \log^3 2 = & \frac{8}{12} \sum_{n=1}^{\infty} \frac{(2^{2n} - 1)^3}{n^3 4^{6n}} \zeta_{(2n)}^3 \\ & + 2 \sum_{n=1}^{\infty} \frac{(2^{2n} - 1)}{n4^{2n}} \zeta_{(2n)} \left[\frac{(2^{2n} - 1)}{n4^{2n}} \zeta_{(2n)} + 2 \sum_{k=1}^{n-1} \frac{(2^{2k} - 1)}{k4^{2k}} \zeta_{(2k)} \right]^2\end{aligned}$$

$$\begin{aligned}
\zeta_3 = & \frac{32}{3} \sum_{n=1}^{\infty} \frac{(2^{2n} - 1)^3}{n^3 4^{6n}} \zeta_{(2n)}^3 \\
& + 32 \sum_{n=1}^{\infty} \frac{(2^{2n} - 1)}{n 4^{2n}} \zeta_{(2n)} \left[\frac{(2^{2n} - 1)}{n 4^{2n}} \zeta_{(2n)} + 2 \sum_{k=1}^{n-1} \frac{(2^{2k} - 1)}{k 4^{2k}} \zeta_{(2k)} \right]^2 \\
& - 16 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\frac{(-1)^{n+1}}{2n} + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \right]^2
\end{aligned}$$

References

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