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# Convergence of Solutions of a Kind of Convex Feasibility Problem on Hadamard Manifolds

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**Abstract**

In this paper, a kind of convex feasibility problem on pesudomonotone variational inequality problem and fixed point problem of demicontraction mapping in Hadamard manifold is considered. For solving the kind of convex feasibility problem, two inertial viscosity-type extragradient algorithms are proposed and two strong convergence theorems are established when some conditions are satisfied. Moreover, a convex minimization problem is solved by the main results of this paper. Finally, the convergence of the two algorithms are demonstrated by numerical experiments.

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# 1 Introduction

Let $\{C_i\}_{i=1}^{\mathbb{N}}$ be some nonempty closed convex sets of a space $H$ with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Convex feasibility problem (in short, CFP) is to find a common element of $\{C_i\}_{i=1}^{\mathbb{N}}$, that is, to find $x^* \in C_1 \cap C_2 \cap \cdots \cap C_{\mathbb{N}}$.

CFPs arise from various problems of nonlinear analysis fields. In fact, many common solution problems are CFPs, the details can be found in [1–9]. In terms of application, CFP is a universal problem appeared in diverse application areas, such as radiation therapy treatment planning, image recovery, crystallography, and so on [10–13].

In recent years, many important problems in nonlinear analysis field on the spaces with linear structure have been extended to Hadamard manifolds, which is a space without linear structure. Since some nonconvex problems and constrained problems in the spaces with linear structure may be transformed into convex problems and unconstrained problems in the spaces without linear structure.

In 2012, Bento et al. [14] introduced subgradient type algorithm to solve CFP in Riemannian manifolds. In order to solve the CFP presented by Bento et al. in [14], Wang et al. [15] proposed cyclic subgradient projection algorithm and solved partially the open problem proposed in [14]. Further, Wang et al. [16] modified the subgradient algorithm presented by [14] in 2015 to solve CFP without Slater condition assumption in Riemannian manifolds.

On the other hand, some scholars focus on many nonliner problems on Hadamard manifolds, such as optimization problem, variational inequality problem, inclusion problem, fixed point problem, and so on. In 2019, Al-Homidan et al. [17] proposed Halpern-type and Mann-type algorithms to find a common point of the fixed point set of a nonexpansive mapping and the solution set of the maximal monotone variational inclusion problem on Hadamard manifolds. In 2021, Filali et al. [18] proposed a kind of viscosity type method to find a common point of solution set of monotone variational inclusion problem and fixed point set of a nonexpansive mapping on Hadamard manifolds. In 2021, Chang et al. [19] introduced proximal point method to find a common element of the common fixed point set of a quasi-pseudocontractive mapping and a demicontraction mapping and the zero point set of monotone inclusion problem on Hadamard manifolds. In the same year, Chang et al. [20] solved a finite family of quasi-variational inclusion problems on Hadamard manifolds by shrinking projection method. In 2022, using inertial technique, Khammahawong et al. [21] proposed two kinds of inertial type algorithms to solve the common solution problems of monotone variational inequality problems and fixed point problems of nonexpansive mapping on Hadamard manifolds.

In fact, the CFP on Hadamard manifold is to solve a common element of some nonempty closed geodesic convex sets, that is, to find a point $x \in M$, such that $x \in K_1 \cap K_2 \cap \cdots \cap K_{\mathbb{N}}$, where $\{K_i\}_{i=1}^{\mathbb{N}}$ are some nonempty closed geodesic convex subsets of Hadamard manifold $M$.

Let $K$ be a nonempty closed geodesic convex subset of $M$ and $\exp : TM \to M$ be an exponential vector field, where $TM$ is the tangent bundle of $M$. The variational inequality problem on Hadamard manifolds was introduced by Nmeth [22], which is to find $q \in K$, such that

$$\left\langle A(q), \exp_q^{-1} y \right\rangle \geq 0, \ \forall y \in K, \tag{1.1}$$

where $A : K \to TM$ is a vector field, denote the solution set of (1.1) by $VI(A, K)$. In addition, the fixed point set of mapping $F : K \to K$ is denoted by $Fix(F)$.

For solving variational inequality problems on Hadamard manifolds, many scholars proposed various algorithms, the details can be found in [23–25]. In 2021, Chen et al. [26] proposed two kind of Tsengs extragradient algorithms to sovle variational inequality problem on Hadamard manifolds. For avoiding the computation cost of Lipschitzian constant, Ma et al. in 2023 [27] proposed viscosity type subgradient extragradient algorithm with Armijo-like linear search technique to solve variational inequality problem on Hadamard manifolds.

Inspired by the work above, we focus our attention on a kind of convex feasibility problem, that is, find a point $x \in M$ such that

$$x \in Fix(U) \cap VI(M, A) \cap VI(M, B), \tag{1.2}$$

where $M$ is a Hadamard manifold, $VI(M, A)$ and $VI(M, B)$ are solution sets of variational inequality problems with respect to pesudomonotone vector fields $A$ and $B$, respectively, $Fix(U)$ is the fixed point set of demicontractive mapping $U$. We introduce two inertial extragradient algorithms whose step sizes do not depend on the Lipschitzian constant of the vector fields to solve (1.2). Finally, we establish two convergence theorems for (1.2).

The rest of the paper is organized as follows: In Section 2, we provide some useful lemmas and definitions on Riemannian manifolds. In Section 3, we present the details of our two algorithms and prove the convergence of our algorithms. In Section 4, we use the main results obtained in section 3 to solve a convex minimization problem on Hadamard manifolds. In Section 5, a numerical example is provided to illustrate the numerical behavior of our algorithms. In Section 6, we present a summary for the work in this paper.

# 2 Preliminaries

Let $M$ be a finite-dimensional Riemannian manifold, $T_pM$ be the tangent space of $M$ at $p$, where $p \in M$, and the tangent bundle $TM := \bigcup_{p\in M} T_pM$. Since $M$ is a Riemannian manifold, Riemannian metric $\langle \cdot, \cdot \rangle_p : T_pM \times T_pM \to R$, can be equipped for any $p \in M$. Moreover, $\| \cdot \|_p$ is the norm corresponding to the Riemannian metric $\langle \cdot, \cdot \rangle_p$ on $T_pM$, where the subscript $p$ can be omitted if there is no confusion.

The length of a piecewise smooth curve $\omega : [a, b] \to M$ joining $\omega(a) = p$ to $\omega(b) = q$, where $p, q \in M$, is defined as follow

$$L(\omega) := \int_b^a \|\omega'(t)\| \, dt,$$

where $\omega'(t)$ stands for the tangent vector. The Riemannian distance $d(p, q)$ is the minimum length of all such curves joining $p$ to $q$.

Let $\nabla$ be the LeviCivita connection associated with Riemannian manifold $M$, $\omega$ be a smooth curve and $E$ be a smooth vector field along $\omega$. If $\nabla_{\omega'(s)}E = 0$, then the vector field $E$ is called parallel. If $\omega'$ is parallel to itself, then $\omega$ is said to be geodesic. The graph of a geodesic to a closed bounded interval is called a geodesic segment. If $\omega$ is a minimal geodesic joining $p$ to $q$, then the length of geodesic joining $p$ to $q$ in $M$ is equal to $d(p, q)$.

The parallel transport $P_{\omega,\omega(b),\omega(a)} : T_{\omega(a)}M \longrightarrow T_{\omega(b)}M$ on the tangent bundle $TM$ along $\omega : [a, b] \to M$ is defined by

$$P_{\omega,\omega(b),\omega(a)}(v) = V(\omega(b)), \forall a, b \in \mathbb{R}, v \in T_{\omega(a)}M,$$

where $V$ is the unique vector field such that $\nabla_{\omega'(t)}V = 0,\ V(\omega(a)) = v$.

If any geodesics of a Riemannian manifold $M$ are well defined for all $t \in \mathbb{R}$, then $M$ is said to be complete. HopfRinow's theorem [28] asserts that $(M, d)$ is a complete metric space and any two points in $M$ can be joined by a minimal geodesic if $M$ is complete. A simply connected complete Riemannian manifold with nonpositive sectional curvature is called Hadamard manifold.

Let $M$ be a complete Riemannian manifold. The exponential vector field $\exp_q : T_qM \longrightarrow M$ is defined as $\exp_q(v) = \omega_v(1, q)$, where $v \in T_qM$, $\omega(\cdot) =$

$\omega_v(\cdot, q)$ is the geodesic starting from $q$ with velocity $v$, i.e., $\omega(0) = 0$ and $\omega'(0) = v$. For each real number $t$, $\exp_q(tv) = \omega_v(t, q)$ and $\exp_q(0) = \omega_v(0, q) = q$ hold. Furthermore, $\exp_q$ is differentiable on $T_qM$ for any $q \in M$.

**Definition 2.1** *[28] Let $K$ be a subset of Hadamard manifold $M$. If for any $p, q \in K$, the geodesic segment $\omega$ joining $p$ to $q$ is contained in $K$, where $\omega : [a, b] \to K$ satisfies $p = \omega(a)$, $q = \omega(b)$, that is, $\omega((1-t)a + tb) \in K$, $t \in [0, 1]$, then the subset $K$ is called geodesic convex.*

**Definition 2.2** *[29] For a given nonempty closed geodesic convex subset $K$ of Hadamard manifold $M$, the metric projection $\pi_K$ from $M$ onto $K$ is defined by*

$$\pi_K(x) = \{y \in K : d(x, y) \leq d(x, z), \forall z \in K\}, \forall x \in M.$$

**Definition 2.3** *[19] Suppose that $K$ is a nonempty closed geodesic convex subset of Hadamard manifold $M$, a mapping $S : K \longrightarrow K$ is said to be contractive if there exists a constant $k \in (0, 1)$ such that*

$$d(Sx, Sy) \leq kd(x, y), \ \forall x, y \in K.$$

*If $k = 1$, then $S$ is called nonexpansive.*

**Definition 2.4** *[19, 30] Let $M$ be a Hadamard manifold, $A$ be a vector field from $M$ into $TM$ satisfying $A(x) \in T_xM$, $x \in M$ and $U$ be a mapping from $M$ into $M$, then*

1. *$A$ is said to be pesudomonotone if*

$$\left\langle A(x), \exp_x^{-1} y \right\rangle \geq 0 \Rightarrow \left\langle A(y), \exp_y^{-1} x \right\rangle \leq 0, \ \forall x, y \in M;$$

2. *$A$ is said to be $L$-Lipschitz continuous with $L > 0$ if*

$$\|P_{x,y}A(y) - A(x)\| \leq Ld(x, y), \ \forall x, y \in M;$$

3. *$U$ is said to be $\lambda$-demicontractive with $\lambda \in [0, 1)$ if $Fix(U) \neq \varnothing$ and the following inequality holds*

$$d^2(Ux, z) \leq d^2(x, z) + \lambda d^2(x, Ux), \forall z \in Fix(U), x \in M.$$

**Lemma 2.5** *[29] Let $K$ be a nonempty closed geodesic convex subset of Hadamard manifold $M$. For any $x \in M$, $z = \pi_K(x)$ if and only if $\langle \exp_z^{-1} x, \exp_z^{-1} y \rangle \leq 0$, for all $y \in K$.*

**Lemma 2.6** *[31] The following statements are equivalent:*

1. *$x$ is a solution of variational inequality problem* (1.1);
2. *for all $\mu > 0, x = \pi_K(\exp_x(-\mu A(x)))$;*
3. *$r(x,\lambda) = 0$, where $r(x,\lambda)$ is defined by $r(x,\lambda) := \exp_x^{-1}[\pi_K(\exp_x(-\lambda A(x)))]$.*

**Lemma 2.7** *[32] Let $\triangle(p_1, p_2, p_3)$ be a geodesic triangle in Hadamard manifold $M$. Then there exists a comparison triangle $\triangle(\overline{p_1}, \overline{p_2}, \overline{p_3})$ for $\triangle(p_1, p_2, p_3)$, such that $d(p_i, p_{i+1}) = \|\overline{p_i} - \overline{p_{i+1}}\|$, the indices $i$ taken modulo 3, and it is unique up to the isometry of $R^2$.*

**Lemma 2.8** *[28] Let $\triangle(p_1, p_2, p_3)$ be a geodesic triangle in Hadamard manifold $M$ and $\alpha$ be the angle of $\triangle(p_1, p_2, p_3)$ at the vertex $p_1$.Then the following results hold:*

1. *$d^2(p_1, p_2) + d^2(p_2, p_3) - 2\left\langle \exp_{p_2}^{-1} p_1, \exp_{p_2}^{-1} p_3 \right\rangle \leq d^2(p_3, p_1)$;*
2. *$d^2(p_1, p_2) \leq \left\langle \exp_{p_1}^{-1} p_3, \exp_{p_1}^{-1} p_2 \right\rangle + \left\langle \exp_{p_2}^{-1} p_3, \exp_{p_2}^{-1} p_1 \right\rangle$;*
3. *$\left\langle \exp_{p_1}^{-1} p_3, \exp_{p_1}^{-1} p_2 \right\rangle = d(p_2, p_1) d(p_1, p_3) \cos\alpha$.*

**Lemma 2.9** *[28] Let $\triangle(\overline{p_1}, \overline{p_2}, \overline{p_3})$ be the comparison triangle of a geodesic triangle $\triangle(p_1, p_2, p_3)$ in Hadamard manifold $M$. The following conclusions hold :*

1. *Let $\alpha_1, \alpha_2, \alpha_3$ be the angles of $\triangle(p_1, p_2, p_3)$ at the vertices $p_1, p_2, p_3$, $\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}$ be the angles of $\triangle(\overline{p_1}, \overline{p_2}, \overline{p_3})$ at the vertices $\overline{p_1}, \overline{p_2}, \overline{p_3}$. Then $\alpha_1 \leq \overline{\alpha_1}$, $\alpha_2 \leq \overline{\alpha_2}$ and $\alpha_3 \leq \overline{\alpha_3}$;*
2. *Let $p$ be a point on the geodesic segment joining $p_1$ to $p_2$ and $\overline{p}$ its comparison point in the interval $[\overline{p_1}, \overline{p_2}]$. Moreover, if $d(p_1, p) = \|\overline{p_1} - \overline{p}\|$ and $d(p_2, p) = \|\overline{p_2} - \overline{p}\|$, then $d(p_3, p) \leq \|\overline{p_3} - \overline{p}\|$.*

**Lemma 2.10** *[33] Let $M$ be a Hadamard manifold, $T_yM$ be a tangent space of $M$ at $y \in M$. If $x \in M$ and $v \in T_yM$, then*

$$\left\langle v, -\exp_y^{-1} x \right\rangle = \left\langle v, P_{y,x} \exp_x^{-1} y \right\rangle = \left\langle P_{x,y} v, \exp_x^{-1} y \right\rangle.$$

**Lemma 2.11** *[21] Let $M$ be a Hadamard manifold and $x_1, x_2, x_3 \in M$, then the following inequality holds:*

$$\left\| \exp_{x_1}^{-1} x_3 - P_{x_1,x_2} \exp_{x_2}^{-1} x_3 \right\| \leq d(x_1, x_2).$$

**Lemma 2.12** *[19] Let $M$ be a Hadamard manifold, $\omega : [0,1] \to M$ be a geodesic joining $x$ to $y$, then*

$$d\left(\omega\left(t_1\right), \omega\left(t_2\right)\right) = |t_1 - t_2| d\left(x, y\right), \ \forall t_1, t_2 \in [0,1],$$

*where $d(\cdot,\cdot)$ is Riemannian distance.*

*In addition, for any $x, y, z, u, w \in M$ and $t \in [0,1]$ , the following inequalities hold:*

1. $d\left(\exp_x t \exp_x^{-1} y, z\right) \le (1-t) d(x,z) + t d(y,z)$;
2. $d^2\left(\exp_x t \exp_x^{-1} y, z\right) \le (1-t) d^2(x,z) + t d^2(y,z) - t(1-t) d^2(x,y)$;
3. $d\left(\exp_x t \exp_x^{-1} y, \exp_u t \exp_u^{-1} w\right) \le (1-t) d(x,u) + t d(y,w)$.

**Lemma 2.13** *[29] Suppose that $K$ is a nonempty closed geodesic convex subset of Hadamard manifold $M$ and $S$ is a mapping from $K$ into $K$. Then these statements are equivalent:*

1. *$S$ is firmly nonexpansive*;
2. *For any $x, y \in K$,*
$$\left\langle \exp_{S(x)}^{-1} S(y), \exp_{S(x)}^{-1} x \right\rangle + \left\langle \exp_{S(y)}^{-1} S(x), \exp_{S(y)}^{-1} y \right\rangle \le 0;$$
3. *For any $x, y \in K$ and $t \in [0,1]$,*
$$d(S(x), S(y)) \le d(\exp_x t \exp_x^{-1} S(x), \exp_y t \exp_y^{-1} S(y)).$$

**Remark 2.14** *It is well known that the metric projection for a nonempty closed geodesic convex subset of Hadamard manifold is firmly nonexpansive mapping.*

**Lemma 2.15** *[34] Let $\{p_n\}$ be a nonnegative sequence, $\{q_n\}$ be a sequence of real numbers and $\{\sigma_n\}$ be a sequence in $[0,1]$ such that $\sum_{n=1}^{\infty} \sigma_n = \infty$. Suppose that*

$$p_{n+1} \le (1-\sigma_n) p_n + \sigma_n q_n, \ \forall n \ge 1.$$

*If $\limsup_{k\to\infty} q_{n_k} \le 0$ for every subsequence $\{p_{n_k}\}$ of $\{p_n\}$ satisfying $\liminf_{k\to\infty}(p_{n_{k+1}} - p_{n_k}) \ge 0$, then $\lim_{n\to\infty} p_n = 0$.*

**Lemma 2.16** *[33] Let $M$ be a Hadamard manifold, $p, q \in M$ and $\{x_n\}$, $\{y_n\} \subset M$ satisfying $x_n \to p$, $y_n \to q$, then the following conclusions hold.*

1. $\exp_{x_n}^{-1} z \to \exp_p^{-1} z$ , $\exp_z^{-1} x_n \to \exp_z^{-1} p$ *and* $\exp_{y_n}^{-1} x_n \to \exp_q^{-1} p$, $\forall z \in M$*;*

2. *If* $\{w_n\}$ *is a sequence in* $T_{x_n}M$ *and* $w_n \to w$*, then* $w \in T_pM$*;*

3. *If the sequences* $\{w_n\}$ *and* $\{v_n\}$ *are two sequences in* $T_{x_n}M$*, and* $\{w_n\} \to w \in T_pM$*,* $\{v_n\} \to v \in T_pM$*, then* $\langle w_n, v_n\rangle \to \langle w, v\rangle$ *.*

**Lemma 2.17** *[19] Let* $M$ *be a Hadamard manifold, if* $U$ *is a* $\lambda$*-demicontractive mapping from* $M$ *into* $M$*, where* $\lambda \in (0,1)$*, mapping* $W : M \to M$ *defined by*

$$W(x) := exp_x(1-\mu)exp_x^{-1}Ux, \ x \in M, \ 0 < \lambda \le \mu < 1,$$

*then* $W$ *is demiclosed at zero, that is, for any bounded sequence* $\{x_n\}$ *in* $M$ *such that*

$$\lim_{n\to\infty} x_n = p \text{ and } \lim_{n\to\infty} d(x_n, Wx_n) = 0,$$

*then* $Wp = p$*.*

**Lemma 2.18** *[35] Let* $M$ *be a Riemannian manifold with constant curvature. For given* $x \in M$ *and* $x' \in T_xM$*, then the set* $\{y \in M : \langle exp_x^{-1}y, x'\rangle \le 0\}$ *is geodesic convex.*

# 3 Main results

In this section, before we introduce our inertial extragradient algorithms to solve (1.2) on Hadamard manifolds, we need to make the following assumptions:

(A1) $M$ is a finite dimensional Hadamard manifold;

(A2) The solution set of (1.2) is nonempty, that is, $\Omega := Fix(U) \cap VI(M,A) \cap VI(M,B) \neq \varnothing$;

(A3) $A$ and $B$ are two pesudomonotone and $L$-Lipschitz continuous vector fields from $M$ into $TM$;

(A4) $U$ is a $\lambda$-demicontractive mapping from $M$ into $M$, where $\lambda \in (0,1)$;

(A5) $f$ is a $\rho$-contraction mapping from $M$ into $M$, where $\rho \in [0,1)$;

(A6) $\{\varepsilon_n\}$ and $\{\alpha_n\}$ are two positive sequences such that $\lim_{n\to\infty} \alpha_n = 0$ , $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\varepsilon_n = \circ(\alpha_n)$;

(A7) $\{\delta_n\} \subset (0,1)$ and $\{\beta_n\} \subset (a, 1-\lambda)$, where $a > 0$.

Now, we introduce our first algorithm.

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**Algorithm 1:** The viscosity-type inertial subgradient extragradient algorithm

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**Initialization:** Take $\theta > 0, \varepsilon_1 > 0, \kappa_1 > 0, \tau_1 > 0, \mu \in (0,1)$. Let $x_0, x_1$ be arbitrarily chosen two points in $M$.

**Step 1.** Input the iterates $x_{n-1}$ and $x_n$ , $(n \geq 1)$. Set $\omega_n = \exp_{x_n}(-\theta_n \exp_{x_n}^{-1} x_{n-1})$, where

$$\theta_n = \begin{cases} \min\left\{ \dfrac{\varepsilon_n}{d(x_n, x_{n-1})}, \theta \right\}, & \text{if } d(x_n, x_{n-1}) \neq 0, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

**Step 2.** Compute $y_n$ satisfying

$$\left\langle \tau_n P_{y_n,\omega_n} A(\omega_n) - \exp_{y_n}^{-1} \omega_n, \exp_{y_n}^{-1} x \right\rangle \geq 0, \ \forall x \in M. \tag{3.2}$$

**Step 3.** Compute $z_n$ satisfying

$$\left\langle \kappa_n P_{z_n,\omega_n} B(\omega_n) - \exp_{z_n}^{-1} \omega_n, \exp_{z_n}^{-1} x \right\rangle \geq 0, \ \forall x \in M. \tag{3.3}$$

**Step 4.** Compute $u_n$ satisfying

$$\left\langle \tau_n P_{u_n,y_n} A(y_n) - \exp_{u_n}^{-1} \omega_n, \exp_{u_n}^{-1} y \right\rangle \geq 0, \ \forall y \in T_n, \tag{3.4}$$

where the half-space $T_n$ is defined by

$$T_n := \left\{ a \in M \mid \left\langle exp_{y_n}^{-1} a, exp_{y_n}^{-1} \omega_n - \tau_n P_{y_n,\omega_n} A(\omega_n) \right\rangle \leq 0 \right\}. \tag{3.5}$$

**Step 5.** Compute $v_n$ satisfying

$$\left\langle \kappa_n P_{v_n,z_n} B(z_n) - \exp_{v_n}^{-1} \omega_n, \exp_{v_n}^{-1} y \right\rangle \geq 0, \ \forall y \in F_n,$$

where the half-space $F_n$ is defined by

$$F_n := \left\{ b \in M \mid \left\langle exp_{z_n}^{-1} b, exp_{z_n}^{-1} \omega_n - \kappa_n P_{z_n,\omega_n} B(\omega_n) \right\rangle \leq 0 \right\}.$$

**Step 6.** Compute $s_n = \exp_{u_n} \delta_n \exp_{u_n}^{-1} v_n$.

**Step 7.** Compute $x_{n+1} = \exp_{t_n} \alpha_n \exp_{t_n}^{-1} f(x_n)$, where $t_n = \exp_{s_n} \beta_n \exp_{s_n}^{-1} U s_n$, and update

$$\tau_{n+1} = \begin{cases} \min\left\{ \frac{\mu d(y_n,\omega_n)}{d(A(y_n),A(\omega_n))}, \tau_n \right\}, & \text{if } d(A(y_n), A(\omega_n)) \neq 0, \\ \tau_n, & \text{otherwise.} \end{cases} \tag{3.6}$$

$$\kappa_{n+1} = \begin{cases} \min\left\{ \frac{\mu d(y_n,\omega_n)}{d(B(y_n),B(\omega_n))}, \kappa_n \right\}, & \text{if } d(B(y_n), B(\omega_n)) \neq 0, \\ \kappa_n, & \text{otherwise.} \end{cases} \tag{3.7}$$

Set $n := n + 1$ and go to **Step 1**.

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**Remark 3.1** *By (3.1) and (A6), it is obvious that* $\theta_n d(x_n, x_{n-1}) \leq \varepsilon_n, \forall n \geq 1$, *so,*

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} d(x_n, x_{n-1}) \leq \lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0.$$

*Due to Lemma 2.18, the half-spaces* $T_n$ *and* $F_n$ *are geodesic convex, so the Algorithm 1 is well defined.*

**Lemma 3.2** *Let the sequences* $\{\tau_n\}$ *and* $\{\kappa_n\}$ *be generated by* (3.6) *and* (3.7), *respectively, then the sequences* $\{\tau_n\}$ *and* $\{\kappa_n\}$ *are nonincreasing sequences and* $\lim_{n\to\infty} \tau_n = \tau \geq \min\left\{\tau_1, \frac{\mu}{L}\right\}$, $\lim_{n\to\infty} \kappa_n = \kappa \geq \min\left\{\kappa_1, \frac{\mu}{L}\right\}$.

**Proof** *It is obvious that* $\{\tau_n\}$ *is nondecreasing from* (3.6). *In addition, since* $A$ *is L-Lipschitz continuous, we have*

$$\mu \frac{d(\omega_n, y_n)}{d(A(\omega_n), A(y_n))} \geq \frac{\mu}{L}, \text{ if } d(A(\omega_n), A(y_n)) \neq 0,$$

*which together with* (3.6) *implies that*

$$\tau_n \geq \min\left\{\tau_1, \frac{\mu}{L}\right\}.$$

*So there exists* $\tau > 0$ *such that* $\lim_{n\to\infty} \tau_n = \tau \geq min\left\{\tau_1, \frac{\mu}{L}\right\}$. *Similarly, we can show that there exists* $\kappa > 0$ *such that* $\lim_{n\to\infty} \kappa_n = \kappa \geq min\left\{\kappa_1, \frac{\mu}{L}\right\}$. *The proof is completed.*

**Lemma 3.3** *Let the sequences* $\{\omega_n\}$, $\{y_n\}$, $\{u_n\}$, $\{z_n\}$ *and* $\{v_n\}$ *be generated by Algorithm 1 and the assumptions* $(A1-A7)$ *hold, then the following inequalities hold:*

$$d^2(u_n, p) \leq d^2(\omega_n, p) - \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right) d^2(y_n, \omega_n) - \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right) d^2(u_n, y_n), \tag{3.8}$$

$$d^2(v_n, p) \leq d^2(\omega_n, p) - \left(1 - \mu\frac{\kappa_n}{\kappa_{n+1}}\right) d^2(z_n, \omega_n) - \left(1 - \mu\frac{\kappa_n}{\kappa_{n+1}}\right) d^2(v_n, z_n), \tag{3.9}$$

*where* $p \in \Omega$.

**Proof** *It follows from Lemma 2.16, Lemma 2.18 and the definition of* $T_n$ *that* $T_n$ *is closed geodesic convex and* $T_n \subset M$. *Let* $\triangle(u_n, \omega_n, p)$ *and* $\triangle(u_n, \omega_n, y_n)$

*be two geodesic triangles. By Lemma 2.7, there exist comparison triangles* $\triangle(\overline{u_n}, \overline{\omega_n}, \overline{p})$ *and* $\triangle(\overline{u_n}, \overline{\omega_n}, \overline{y_n})$ *such that*

$$\begin{aligned} &d(u_n, \omega_n) = \|\overline{u_n} - \overline{\omega_n}\|\,,\ d(u_n, p) = \|\overline{u_n} - \overline{p}\|\,,\ d(\omega_n, p) = \|\overline{\omega_n} - \overline{p}\|\,, \\ &d(u_n, y_n) = \|\overline{u_n} - \overline{y_n}\|\,,\ d(\omega_n, y_n) = \|\overline{\omega_n} - \overline{y_n}\|\,. \end{aligned}$$

*This implies that,*

$$\begin{aligned} d^2(u_n, p) &= \|\overline{u_n} - \overline{p}\|^2 \\ &= \|\overline{u_n} - \overline{y_n}\|^2 + \|\overline{y_n} - \overline{p}\|^2 + 2\left\langle \overline{u_n} - \overline{y_n}, \overline{y_n} - \overline{p} \right\rangle \\ &= \|\overline{u_n} - \overline{y_n}\|^2 + \|\overline{y_n} - \overline{\omega_n}\|^2 + \|\overline{\omega_n} - \overline{p}\|^2 + 2\left\langle \overline{y_n} - \overline{\omega_n}, \overline{\omega_n} - \overline{p} \right\rangle + 2\left\langle \overline{u_n} - \overline{y_n}, \overline{y_n} - \overline{p} \right\rangle \\ &= \|\overline{u_n} - \overline{y_n}\|^2 + \|\overline{y_n} - \overline{\omega_n}\|^2 + \|\overline{\omega_n} - \overline{p}\|^2 + 2\left\langle \overline{y_n} - \overline{\omega_n}, \overline{\omega_n} - \overline{y_n} \right\rangle + 2\left\langle \overline{y_n} - \overline{\omega_n}, \overline{y_n} - \overline{p} \right\rangle \\ &\quad + 2\left\langle \overline{u_n} - \overline{y_n}, \overline{y_n} - \overline{u_n} \right\rangle + 2\left\langle \overline{u_n} - \overline{y_n}, \overline{u_n} - \overline{p} \right\rangle \\ &= \|\overline{\omega_n} - \overline{p}\|^2 - \|\overline{u_n} - \overline{y_n}\|^2 - \|\overline{y_n} - \overline{\omega_n}\|^2 + 2\left\langle \overline{u_n} - \overline{y_n}, \overline{u_n} - \overline{p} \right\rangle + 2\left\langle \overline{y_n} - \overline{\omega_n}, \overline{y_n} - \overline{p} \right\rangle \\ &= \|\overline{\omega_n} - \overline{p}\|^2 - \|\overline{u_n} - \overline{y_n}\|^2 - \|\overline{y_n} - \overline{\omega_n}\|^2 + 2\left\langle \overline{u_n} - \overline{\omega_n}, \overline{u_n} - \overline{p} \right\rangle + 2\left\langle \overline{\omega_n} - \overline{y_n}, \overline{u_n} - \overline{p} \right\rangle \\ &\quad + 2\left\langle \overline{y_n} - \overline{\omega_n}, \overline{y_n} - \overline{p} \right\rangle \\ &= \|\overline{\omega_n} - \overline{p}\|^2 - \|\overline{u_n} - \overline{y_n}\|^2 - \|\overline{y_n} - \overline{\omega_n}\|^2 + 2\left\langle \overline{u_n} - \overline{\omega_n}, \overline{u_n} - \overline{p} \right\rangle + 2\left\langle \overline{y_n} - \overline{\omega_n}, \overline{y_n} - \overline{u_n} \right\rangle. \end{aligned} \tag{3.10}$$

*Let* $\gamma$ *and* $\overline{\gamma}$ *be the angles at the vertices* $u_n$ *and* $\overline{u_n}$*, respectively. It follows from Lemma 2.8 and Lemma 2.9 that* $\gamma \le \overline{\gamma}$ *and*

$$\begin{aligned} \left\langle \overline{u_n} - \overline{\omega_n}, \overline{u_n} - \overline{p} \right\rangle &= \|\overline{u_n} - \overline{\omega_n}\|\,\|\overline{u_n} - \overline{p}\|\,cos\overline{\gamma} \\ &= d(u_n, \omega_n) d(u_n, p) cos\overline{\gamma} \\ &\le d(u_n, \omega_n) d(u_n, p) cos\gamma \\ &= \left\langle \exp_{u_n}^{-1} \omega_n, \exp_{u_n}^{-1} p \right\rangle. \end{aligned} \tag{3.11}$$

*Similarly, we have*

$$\left\langle \overline{y_n} - \overline{\omega_n}, \overline{y_n} - \overline{u_n} \right\rangle \ \le \left\langle \exp_{y_n}^{-1} \omega_n, \exp_{y_n}^{-1} u_n \right\rangle. \tag{3.12}$$

*Substituting (3.11) and (3.12) into (3.10), we have*

$$d^2(u_n, p) \le d^2(\omega_n, p) - d^2(u_n, y_n) - d^2(y_n, \omega_n) + 2\left\langle \exp_{u_n}^{-1} \omega_n, \exp_{u_n}^{-1} p \right\rangle + 2\left\langle \exp_{y_n}^{-1} \omega_n, \exp_{y_n}^{-1} u_n \right\rangle. \tag{3.13}$$

*According to (3.4) and (3.13), we obtain*

$$\begin{aligned} d^2(u_n, p) \le& d^2(\omega_n, p) - d^2(u_n, y_n) - d^2(y_n, \omega_n) + 2\left\langle \exp_{y_n}^{-1} \omega_n, \exp_{y_n}^{-1} u_n \right\rangle \\ &+ 2\left\langle \tau_n P_{u_n, y_n} A(y_n), \exp_{u_n}^{-1} p \right\rangle. \end{aligned} \tag{3.14}$$

*Since $p \in \Omega$, we have $\left\langle A(p), exp_p^{-1} y_n \right\rangle \geq 0$. By the pseudomonotonicity of $A$, we know that*

$$\left\langle A(y_n), exp_{y_n}^{-1} p \right\rangle \leq 0.$$

*Further, since $\tau_n > 0$ and (3.14), we have*

$$\begin{aligned} d^2(u_n,p) \leq & d^2(\omega_n,p) - d^2(u_n,y_n) - d^2(y_n,\omega_n) + 2\left\langle \exp_{y_n}^{-1}\omega_n, \exp_{y_n}^{-1}u_n \right\rangle + 2\left\langle \tau_n P_{u_n,y_n}A(y_n), \exp_{u_n}^{-1}p \right\rangle \\ & - 2\left\langle \tau_n A(y_n), exp_{y_n}^{-1}p \right\rangle \\ \leq & d^2(\omega_n,p) - d^2(u_n,y_n) - d^2(y_n,\omega_n) + 2\left\langle \exp_{y_n}^{-1}\omega_n, \exp_{y_n}^{-1}u_n \right\rangle \\ & + 2\left\langle \tau_n A(y_n), P_{y_n,u_n}\exp_{u_n}^{-1}p - \exp_{y_n}^{-1}p \right\rangle. \end{aligned} \tag{3.15}$$

*From (3.5), (3.6), (3.15) and Lemma 2.11, we obtain*

$$\begin{aligned} d^2(u_n,p) \leq & d^2(\omega_n,p) - d^2(u_n,y_n) - d^2(y_n,\omega_n) + 2\left\langle \exp_{y_n}^{-1}\omega_n, \exp_{y_n}^{-1}u_n \right\rangle \\ & - 2\left\langle \tau_n A(y_n), \exp_{y_n}^{-1}p - P_{y_n,u_n}\exp_{u_n}^{-1}p \right\rangle \\ \leq & d^2(\omega_n,p) - d^2(u_n,y_n) - d^2(y_n,\omega_n) + 2\left\langle \exp_{y_n}^{-1}\omega_n, \exp_{y_n}^{-1}u_n \right\rangle \\ & + 2\left\langle \tau_n P_{y_n,\omega_n}A(\omega_n) - \tau_n A(y_n), \exp_{y_n}^{-1}p - P_{y_n,u_n}\exp_{u_n}^{-1}p \right\rangle \\ & - 2\left\langle \tau_n P_{y_n,\omega_n}A(\omega_n), \exp_{y_n}^{-1}p - P_{y_n,u_n}\exp_{u_n}^{-1}p \right\rangle \\ \leq & d^2(\omega_n,p) - d^2(u_n,y_n) - d^2(y_n,\omega_n) + 2\left\langle \exp_{y_n}^{-1}\omega_n, \exp_{y_n}^{-1}u_n \right\rangle \\ & + 2\left\| \tau_n P_{y_n,\omega_n}A(\omega_n) - \tau_n A(y_n) \right\| \left\| \exp_{y_n}^{-1}p - P_{y_n,u_n}\exp_{u_n}^{-1}p \right\| \\ & - 2\left\langle \tau_n P_{y_n,\omega_n}A(\omega_n), \exp_{y_n}^{-1}u_n \right\rangle \\ \leq & d^2(\omega_n,p) - d^2(u_n,y_n) - d^2(y_n,\omega_n) + 2\tau_n d(A(\omega_n),A(y_n))d(y_n,u_n) \\ & + 2\left\langle \exp_{y_n}^{-1}\omega_n - \tau_n P_{y_n,\omega_n}A(\omega_n), \exp_{y_n}^{-1}u_n \right\rangle \\ \leq & d^2(\omega_n,p) - d^2(u_n,y_n) - d^2(y_n,\omega_n) + 2\frac{\tau_n\mu}{\tau_{n+1}}d(\omega_n,y_n)d(y_n,u_n) \\ \leq & d^2(\omega_n,p) - d^2(u_n,y_n) - d^2(y_n,\omega_n) + \frac{\tau_n\mu}{\tau_{n+1}}(d^2(\omega_n,y_n) + d^2(y_n,u_n)) \\ \leq & d^2(\omega_n,p) - (1 - \frac{\tau_n\mu}{\tau_{n+1}})d^2(u_n,y_n) - (1 - \frac{\tau_n\mu}{\tau_{n+1}})d^2(y_n,\omega_n). \end{aligned}$$

*The proof of (3.9) is similar to the proof of (3.8). So we omit the proof of (3.9). The proof is completed.*

**Theorem 3.4** *Assume that the assumptions $(A1)-(A7)$ hold. Then the sequence $\{x_n\}$ generated by $Algorithm$ 1 converges to $q \in \Omega$, where $q = \pi_\Omega(f(q))$.*

**Proof** *It is well kown that $\Omega$ is closed geodesic convex and the mapping $\pi_\Omega(f) : M \longrightarrow M$ is contractive. Therefore there exists a unique point $q \in M$ such that $q = \pi_\Omega(f(q))$ by the Banach contraction principle. Further we have*

$$\left\langle \exp_q^{-1} f(q), \exp_q^{-1} z \right\rangle \leq 0, \ \forall z \in \Omega.$$

*Next, we divide the proof into four parts.*

***Claim 1.*** *The sequence $\{x_n\}$ is bounded.*

*From Lemma 2.12 and Definition 2.4, we have*

$$\begin{aligned} d^2(t_n,q) \leq &\beta_n d^2(Us_n,q)+(1-\beta_n)d^2(s_n,q)-\beta_n(1-\beta_n)d^2(Us_n,s_n)\\ \leq&\beta_n d^2(s_n,q)+\beta_n\lambda d^2(s_n,Us_n)+(1-\beta_n)d^2(s_n,q)-\beta_n(1-\beta_n)d^2(Us_n,s_n)\\ =&d^2(s_n,q)+\beta_n(\lambda-1+\beta_n)d^2(Us_n,s_n). \end{aligned} \tag{3.16}$$

*According to (3.16) and Lemma 2.12, we obtain*

$$\begin{aligned} d^2(t_n,q) \leq &d^2(s_n,q)+\beta_n(\lambda-1+\beta_n)d^2(Us_n,s_n)\\ \leq&\delta_n d^2(v_n,q)+(1-\delta_n)d^2(u_n,q)-\delta_n(1-\delta_n)d^2(v_n,u_n)\\ &+\beta_n(\lambda-1+\beta_n)d^2(Us_n,s_n). \end{aligned} \tag{3.17}$$

*By Lemma 3.3, (3.17), $\delta_n\subset(0,1)$ and $\beta_n\subset(a,1-\lambda)$, we have*

$$\begin{aligned} d^2(t_n,q) \leq &\delta_n\left[d^2(\omega_n,p)-\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)d^2(y_n,\omega_n)-\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)d^2(u_n,y_n)\right]\\ &+(1-\delta_n)\left[d^2(\omega_n,p)-\left(1-\mu\frac{\kappa_n}{\kappa_{n+1}}\right)d^2(z_n,\omega_n)-\left(1-\mu\frac{\kappa_n}{\kappa_{n+1}}\right)d^2(v_n,z_n)\right]\\ &-\delta_n(1-\delta_n)d^2(v_n,u_n)+\beta_n(\lambda-1+\beta_n)d^2(Us_n,s_n) \qquad (3.18)\\ \leq&\delta_n\left[d^2(\omega_n,p)-\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)d^2(y_n,\omega_n)-\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)d^2(u_n,y_n)\right]\\ &+(1-\delta_n)\left[d^2(\omega_n,p)-\left(1-\mu\frac{\kappa_n}{\kappa_{n+1}}\right)d^2(z_n,\omega_n)-\left(1-\mu\frac{\kappa_n}{\kappa_{n+1}}\right)d^2(v_n,z_n)\right]. \end{aligned} \tag{3.19}$$

*By Lemma 3.2, we have* $\lim_{n\to\infty}(1-\mu\frac{\tau_n}{\tau_{n+1}})=1-\mu>0$ *and* $\lim_{n\to\infty}(1-\mu\frac{\kappa_n}{\kappa_{n+1}})=1-\mu>0$*, which implies that there exists $n_1\in\mathbb{N}$ such that $(1-\mu\frac{\tau_n}{\tau_{n+1}})>0$ and $(1-\mu\frac{\kappa_n}{\kappa_{n+1}})>0$, for all $n\geq n_1$. It follows from (3.19) that*

$$d(t_n,q)\leq d(\omega_n,q),\ \forall n\geq n_1. \tag{3.20}$$

*For geodesic triangle $\triangle(x_n,x_{n-1},q)$, it follows from Lemma 2.7 that there exists a corresponding comparison triangle $\triangle(\overline{x_n},\overline{x_{n-1}},\overline{q})$ such that*

$$d(x_n,x_{n-1})=\|\overline{x_n}-\overline{x_{n-1}}\|,d(x_{n-1},q)=\|\overline{x_{n-1}}-\overline{q}\|,d(x_n,q)=\|\overline{x_n}-\overline{q}\|.$$

*Further, from Lemma 2.9, we have*

$$\begin{aligned} d(\omega_n,q)&\leq\|\overline{\omega_n}-\overline{q}\|\\ &\leq\|\overline{\omega_n}-\overline{x_n}\|+\|\overline{x_n}-\overline{q}\|\\ &=\left\|-\theta_n\exp_{x_n}^{-1}x_{n-1}\right\|+\left\|\exp_q^{-1}x_n\right\|\\ &\leq d(x_n,q)+\theta_n d(x_n,x_{n-1})\\ &=d(x_n,q)+\alpha_n\frac{\theta_n}{\alpha_n}d(x_n,x_{n-1}). \end{aligned} \tag{3.21}$$

*In view of Remark 3.1, we have* $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}d(x_n,x_{n-1})=0$*. Thus, there exists a constant* $M_1>0$ *such that*

$$\frac{\theta_n}{\alpha_n}d(x_n,x_{n-1})\le M_1,\forall n\ge 1. \tag{3.22}$$

*Combining (3.20), (3.21) and (3.22), we have*

$$d(t_n,q)\le d(\omega_n,q)\le d(x_n,q)+\alpha_n M_1,\forall n\ge n_1. \tag{3.23}$$

*For geodesic triangles* $\triangle(f(x_n),f(x),q)$ *and* $\triangle(f(x_n),t_n,q)$*, there exist corresponding comparison triangles* $\triangle(\overline{f(x_n)},\overline{f(x)},\overline{q})$ *and* $\triangle(\overline{f(x_n)},\overline{t_n},\overline{q})$ *such that*

$$d(f(x_n),f(q))=\left\|\overline{f(x_n)}-\overline{f(q)}\right\|,d(f(q),q)=\left\|\overline{f(q)}-\overline{q}\right\|,\ d(f(x_n),q)=\left\|\overline{f(x_n)}-\overline{q}\right\|,$$
$$d(f(x_n),t_n)=\left\|\overline{f(x_n)}-\overline{t_n}\right\|,\ d(t_n,q)=\left\|\overline{t_n}-\overline{q}\right\|.$$

*Let* $\overline{x_{n+1}}=\alpha_n\overline{f(x_n)}+(1-\alpha_n)\overline{t_n}$ *be the comparison point of* $x_{n+1}$*. From Lemma 2.9 and (3.23), we have*

$$\begin{aligned}
d(x_{n+1},q)\le&\left\|\overline{x_{n+1}}-\overline{q}\right\|\\
=&\left\|\alpha_n\overline{f(x_n)}+(1-\alpha_n)\overline{t_n}-\overline{q}\right\|\\
\le&\alpha_n\left\|\overline{f(x_n)}-\overline{f(q)}\right\|+\alpha_n\left\|\overline{f(q)}-\overline{q}\right\|+(1-\alpha_n)\left\|\overline{t_n}-\overline{q}\right\|\\
=&\alpha_n d(f(x_n),f(q))+\alpha_n d(f(q),q)+(1-\alpha_n)d(t_n,q)\\
\le&\alpha_n\rho d(x_n,q)+\alpha_n d(f(q),q)+(1-\alpha_n)d(\omega_n,q)\\
\le&\alpha_n\rho d(x_n,q)+\alpha_n d(f(q),q)+(1-\alpha_n)d(x_n,q)+(1-\alpha_n)\alpha_n M_1\\
\le&\left[1-\alpha_n(1-\rho)\right]d(x_n,q)+\alpha_n(1-\rho)\frac{d(f(q),q)+M_1}{1-\rho}\\
\le&\max\left\{d(x_n,q),\frac{d(f(q),q)+M_1}{1-\rho}\right\}\\
\le&\cdots\le\max\left\{d(x_{n_1},q),\frac{d(f(q),q)+M_1}{1-\rho}\right\},\ \forall n\ge n_1,
\end{aligned}$$

*which implies that the sequence* $\{x_n\}$ *is bounded. So, the sequences* $\{\omega_n\}$*,* $\{f(x_n)\}$*,* $\{y_n\}$*,* $\{z_n\}$*,* $\{u_n\}$ *and* $\{v_n\}$ *are also bounded.*

***Claim 2.***

$$\begin{aligned}
&(1-\alpha_n)\delta_n\left[\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)d^2(y_n,\omega_n)+\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)d^2(u_n,y_n)\right]\\
&+(1-\alpha_n)(1-\delta_n)\left[\left(1-\mu\frac{\kappa_n}{\kappa_{n+1}}\right)d^2(z_n,\omega_n)+\left(1-\mu\frac{\kappa_n}{\kappa_{n+1}}\right)d^2(v_n,z_n)\right]\\
&+(1-\alpha_n)\delta_n(1-\delta_n)d^2(v_n,u_n)+(1-\alpha_n)\beta_n(1-\lambda-\beta_n)d^2(Us_n,s_n)\\
&\le d^2(x_n,q)-d^2(x_{n+1},q)+\alpha_n d^2(f(x_n),q)+\alpha_n M_2,
\end{aligned}$$

*where* $M_2 > 0$ *is a constant, such that* $M_2 \geq \sup\{2M_1 d(x_n, q) + \alpha_n M_1^2\}$.

*According to (3.23), we have*

$$\begin{aligned} d^2(\omega_n, q) \leq & (d(x_n, q) + \alpha_n M_1)^2 \\ = & d^2(x_n, q) + 2\alpha_n M_1 d(x_n, q) + \alpha_n^2 M_1^2 \\ = & d^2(x_n, q) + \alpha_n (2M_1 d(x_n, q) + \alpha_n M_1^2) \\ \leq & d^2(x_n, q) + \alpha_n M_2 \end{aligned} \tag{3.24}$$

*where* $M_2 > 0$ *be a constant, such that* $M_2 \geq \sup\{2M_1 d(x_n, q) + \alpha_n M_1^2\}$.

*It follows from Lemma 2.12,* (3.18) *and* (3.21) *that*

$$\begin{aligned} d^2(x_{n+1}, q) \leq & \alpha_n d^2(f(x_n), q) + (1-\alpha_n) d^2(t_n, q) \\ \leq & \alpha_n d^2(f(x_n), q) + (1-\alpha_n) d^2(\omega_n, p) - (1-\alpha_n)\delta_n(1-\delta_n) d^2(v_n, u_n) \\ & - (1-\alpha_n)\delta_n \left[ \left(1 - \mu \frac{\tau_n}{\tau_{n+1}}\right) d^2(y_n, \omega_n) + \left(1 - \mu \frac{\tau_n}{\tau_{n+1}}\right) d^2(u_n, y_n) \right] \\ & - (1-\alpha_n)(1-\delta_n) \left[ \left(1 - \mu \frac{\kappa_n}{\kappa_{n+1}}\right) d^2(z_n, \omega_n) + \left(1 - \mu \frac{\kappa_n}{\kappa_{n+1}}\right) d^2(v_n, z_n) \right] \\ & - (1-\alpha_n)\beta_n(1-\lambda-\beta_n) d^2(Us_n, s_n) \\ \leq & \alpha_n d^2(f(x_n), q) + d^2(\omega_n, p) - (1-\alpha_n)\delta_n(1-\delta_n) d^2(v_n, u_n) \\ & - (1-\alpha_n)\delta_n \left[ \left(1 - \mu \frac{\tau_n}{\tau_{n+1}}\right) d^2(y_n, \omega_n) + \left(1 - \mu \frac{\tau_n}{\tau_{n+1}}\right) d^2(u_n, y_n) \right] \\ & - (1-\alpha_n)(1-\delta_n) \left[ \left(1 - \mu \frac{\kappa_n}{\kappa_{n+1}}\right) d^2(z_n, \omega_n) + \left(1 - \mu \frac{\kappa_n}{\kappa_{n+1}}\right) d^2(v_n, z_n) \right] \\ & - (1-\alpha_n)\beta_n(1-\lambda-\beta_n) d^2(Us_n, s_n) \\ \leq & \alpha_n d^2(f(x_n), q) + d^2(x_n, q) + \alpha_n M_2 - (1-\alpha_n)\delta_n(1-\delta_n) d^2(v_n, u_n) \\ & - (1-\alpha_n)\delta_n \left[ \left(1 - \mu \frac{\tau_n}{\tau_{n+1}}\right) d^2(y_n, \omega_n) + \left(1 - \mu \frac{\tau_n}{\tau_{n+1}}\right) d^2(u_n, y_n) \right] \\ & - (1-\alpha_n)(1-\delta_n) \left[ \left(1 - \mu \frac{\kappa_n}{\kappa_{n+1}}\right) d^2(z_n, \omega_n) + \left(1 - \mu \frac{\kappa_n}{\kappa_{n+1}}\right) d^2(v_n, z_n) \right] \\ & - (1-\alpha_n)\beta_n(1-\lambda-\beta_n) d^2(Us_n, s_n). \end{aligned}$$

*By an easy deformation, we can obtain the result desired.*

***Claim 3.***

$$\begin{aligned} d^2(x_{n+1}, q) \leq & (1-\rho)\alpha_n \left[ \frac{2}{1-\rho} \left\langle \exp_q^{-1} f(q), \exp_q^{-1} x_{n+1} \right\rangle + \frac{3M\theta_n}{(1-\rho)\alpha_n} d(x_n, x_{n-1}) \right] \\ & + (1-(1-\rho)\alpha_n) d^2(x_n, q), \ \forall n \geq n_1, \end{aligned}$$

*where* $M := \sup_{n\in\mathbb{N}} \{d(x_n, q), \theta_n d(x_n, x_{n-1})\} > 0$.

*By* (3.21), *we obtain*

$$
\begin{aligned}
d^2(\omega_n, q) \leq& \left[\theta_n d(x_n, x_{n+1}) + d(x_n, q)\right]^2\\
=&d^2(x_n, q) + 2\theta_n d(x_n, q)d(x_n, x_{n-1}) + \theta_n^2 d^2(x_n, x_{n-1})\\
\leq& d^2(x_n, q) + (2d(x_n, q) + \theta_n d(x_n, x_{n-1}))\theta_n d(x_n, x_{n-1})\\
\leq& d^2(x_n, q) + 3M\theta_n d(x_n, x_{n-1}). \qquad (3.25)
\end{aligned}
$$

*where* $M := \sup_{n\in\mathbb{N}} \{d(x_n, q), \theta_n d(x_n, x_{n-1})\} > 0$.

*For geodesic triangles* $\triangle(f(x_n), f(q), q)$, $\triangle(f(x_n), t_n, q)$ *and* $\triangle(x_{n+1}, f(q), q)$, *there exist corresponding comparison triangles* $\triangle(\overline{f(x_n)}, \overline{f(q)}, \overline{q})$, $\triangle(\overline{x_{n+1}}, \overline{f(q)}, \overline{q})$ *and* $\triangle(\overline{f(x_n)}, \overline{t_n}, \overline{q})$ *satisfying*

$$
d(f(x_n), f(q)) = \left\|\overline{f(x_n)} - \overline{f(q)}\right\|,\ d(f(x_n), q) = \left\|\overline{f(x_n)} - \overline{q}\right\|,\ d(f(x_n), t_n) = \left\|\overline{f(x_n)} - \overline{t_n}\right\|,
$$

$$
d(f(q), q) = \left\|\overline{f(q)} - \overline{q}\right\|,\ d(t_n, q) = \left\|\overline{t_n} - \overline{q}\right\|, d(x_{n+1}, f(q)) = \left\|\overline{x_{n+1}} - \overline{f(q)}\right\|, d(x_{n+1}, q) = \left\|\overline{x_{n+1}} - \overline{q}\right\|.
$$

*Let* $\overline{x_{n+1}} = \alpha_n\overline{f(x_n)} + (1 - \alpha_n)\overline{t_n}$ *be the comparison point of* $x_{n+1}$, *then*

$$
\begin{aligned}
d^2(x_{n+1}, q) \leq& \left\|\overline{x_{n+1}} - \overline{q}\right\|^2\\
=& \left\|\alpha_n\overline{f(x_n)} + (1 - \alpha_n)\overline{t_n} - \overline{q}\right\|^2\\
=& \left\|\alpha_n(\overline{f(x_n)} - \overline{f(q)}) + (1 - \alpha_n)(\overline{t_n} - \overline{q}) + \alpha_n(\overline{f(q)} - \overline{q})\right\|^2\\
\leq& \left\|\alpha_n(\overline{f(x_n)} - \overline{f(q)}) + (1 - \alpha_n)(\overline{t_n} - \overline{q})\right\|^2 + 2\alpha_n \left\langle \overline{f(q)} - \overline{q}, \overline{x_{n+1}} - \overline{q}\right\rangle\\
\leq& \alpha_n \left\|\overline{f(x_n)} - \overline{f(q)}\right\|^2 + (1 - \alpha_n) \left\|\overline{t_n} - \overline{q}\right\|^2 + 2\alpha_n \left\langle \overline{f(q)} - \overline{q}, \overline{x_{n+1}} - \overline{q}\right\rangle.
\end{aligned}
\qquad (3.26)
$$

*Let* $\gamma$ *and* $\overline{\gamma}$ *be the angles at the vertices* $q$ *and* $\overline{q}$, *respectively, It follows from Lemma 2.8 and Lemma 2.9 that* $\gamma \leq \overline{\gamma}$ *and*

$$
\begin{aligned}
\left\langle \overline{f(q)} - \overline{q}, \overline{x_{n+1}} - \overline{q}\right\rangle =& \left\|\overline{f(q)} - \overline{q}\right\| \left\|\overline{x_{n+1}} - \overline{q}\right\| cos\overline{\gamma}\\
=& d(f(q), q)d(x_{n+1}, q)cos\overline{\gamma}\\
\leq& d(f(q), q)d(x_{n+1}, q)cos\gamma\\
=& \left\langle \exp_q^{-1} f(q), \exp_q^{-1} x_{n+1}\right\rangle. \qquad (3.27)
\end{aligned}
$$

*It follows from* (3.20), (3.25), (3.26) *and* (3.27) *that*

$$
\begin{aligned}
d^2(x_{n+1}, q) \leq& \alpha_n d^2(f(x_n), f(q)) + (1 - \alpha_n)d^2(t_n, q) + 2\alpha_n \left\langle \exp_q^{-1} f(q), \exp_q^{-1} x_{n+1}\right\rangle\\
\leq& \alpha_n\rho d^2(x_n, q) + (1 - \alpha_n)d^2(\omega_n, q) + 2\alpha_n \left\langle \exp_q^{-1} f(q), \exp_q^{-1} x_{n+1}\right\rangle\\
\leq& \alpha_n\rho d^2(x_n, q) + (1 - \alpha_n)d^2(x_n, q) + 3M\theta_n d(x_n, x_{n-1}) + 2\alpha_n \left\langle \exp_q^{-1} f(q), \exp_q^{-1} x_{n+1}\right\rangle\\
=& (1 - \rho)\alpha_n \left[\frac{2}{1 - \rho} \left\langle \exp_q^{-1} f(q), \exp_q^{-1} x_{n+1}\right\rangle + \frac{3M\theta_n}{(1 - \rho)\alpha_n} d(x_n, x_{n-1})\right]\\
&+ (1 - (1 - \rho)\alpha_n)d^2(x_n, q).
\end{aligned}
$$

***Claim 4.*** *The sequence $\{x_n\}$ converges to $q = \pi_\Omega f(q) \in \Omega$.*

*To obtain the result desired, we firstly show that* $\limsup_{x\to\infty} \left\langle \exp_q^{-1} f(q), \exp_q^{-1} x_{n_{k+1}} \right\rangle \leq 0$ *for every subsequence* $\{d(x_{n_k}, q)\}$ *of* $\{d(x_n, q)\}$ *satisfying*

$$\liminf_{k\to\infty}(d(x_{n_{k+1}}, q) - d(x_{n_k}, q)) \geq 0. \tag{3.28}$$

*By Claim 2, Assumption* (*A*6) *and (3.28), we have*

$$\begin{aligned}
\limsup_{k\to\infty} \Bigg\{ &(1-\alpha_{n_k})\delta_{n_k} \left[ \left(1 - \mu \frac{\tau_{n_k}}{\tau_{n_k+1}}\right) d^2(y_{n_k}, \omega_{n_k}) + \left(1 - \mu \frac{\tau_{n_k}}{\tau_{n_k+1}}\right) d^2(u_{n_k}, y_{n_k}) \right] \\
&+ (1-\alpha_{n_k})(1-\delta_{n_k}) \left[ \left(1 - \mu \frac{\kappa_{n_k}}{\kappa_{n_k+1}}\right) d^2(z_{n_k}, \omega_{n_k}) + \left(1 - \mu \frac{\kappa_{n_k}}{\kappa_{n_k+1}}\right) d^2(v_{n_k}, z_{n_k}) \right] \\
&+ (1-\alpha_{n_k})\delta_{n_k}(1-\delta_{n_k}) d^2(v_{n_k}, u_{n_k}) + (1-\alpha_{n_k})\beta_{n_k}(1-\lambda-\beta_{n_k}) d^2(Us_{n_k}, s_{n_k}) \Bigg\} \\
&\leq \limsup_{k\to\infty} \left\{ d^2(x_{n_k}, q) - d^2(x_{n_k+1}, q) + \alpha_{n_k} d^2(f(x_{n_k}), q) + \alpha_{n_k} M_2 \right\} \\
&= -\liminf_{k\to\infty} \left\{ d^2(x_{n_k+1}, q) - d^2(x_{n_k}, q) \right\} \leq 0,
\end{aligned}$$

*which implies that*

$$\begin{aligned}
&\lim_{k\to\infty} d(\omega_{n_k}, y_{n_k}) = 0, \ \lim_{k\to\infty} d(u_{n_k}, y_{n_k}) = 0 \ , \ \lim_{k\to\infty} d(s_{n_k}, Us_{n_k}) = 0, \\
&\lim_{k\to\infty} d(\omega_{n_k}, z_{n_k}) = 0, \ \lim_{k\to\infty} d(v_{n_k}, z_{n_k}) = 0 \ , \ \lim_{k\to\infty} d(u_{n_k}, v_{n_k}) = 0.
\end{aligned} \tag{3.29}$$

*By* (3.29) *and Lemma 2.12, we may get* $\lim_{k\to\infty} d(s_{n_k}, \omega_{n_k}) = 0$.

*Since* $\omega_n = \exp_{x_n}(-\theta_n \exp_{x_n}^{-1} x_{n-1})$, *then* $\exp_{x_n}^{-1} \omega_n = -\theta_n \exp_{x_n}^{-1} x_{n-1}$. *Further, we have*

$$\begin{aligned}
d(\omega_{n_k}, x_{n_k}) &= \left\| \exp_{x_{n_k}}^{-1} \omega_{n_k} \right\| \\
&= \theta_{n_k} \left\| \exp_{x_{n_{k-1}}}^{-1} x_{n_k} \right\| \\
&= \alpha_{n_k} \frac{\theta_{n_k}}{\alpha_{n_k}} \left\| \exp_{x_{n_{k-1}}}^{-1} x_{n_k} \right\| \\
&= \alpha_{n_k} \frac{\theta_{n_k}}{\alpha_{n_k}} d(x_{n_k}, x_{n_{k-1}}) \to 0, \ (\text{as } k \to \infty).
\end{aligned} \tag{3.30}$$

*This together with* $\lim_{k\to\infty} d(\omega_{n_k}, s_{n_k}) = 0$ *yields that*

$$\lim_{k\to\infty} d(x_{n_k}, s_{n_k}) = 0. \tag{3.31}$$

*Since* $t_n = \exp_{s_n} \beta_n \exp_{s_n}^{-1} U s_n$, *then* $\exp_{s_n}^{-1} t_n = \beta_n \exp_{s_n}^{-1} U s_n$, *we can see that*

$$d(t_{n_k}, s_{n_k}) = \left\|\exp_{s_{n_k}}^{-1} t_{n_k}\right\| = \beta_{n_k} \left\|\exp_{s_{n_k}}^{-1} U s_{n_k}\right\| \le (1-\lambda) \left\|\exp_{s_{n_k}}^{-1} U s_{n_k}\right\| = (1-\lambda) d(U s_{n_k}, s_{n_k}).$$

*In view of* (3.29), *we get*

$$\lim_{k\to\infty} d(t_{n_k}, s_{n_k}) = 0. \tag{3.32}$$

*For geodesic triangles* $\triangle(t_{n_k}, x_{n_k}, s_{n_k})$ *and* $\triangle(f(x_{n_k}), t_{n_k}, x_{n_k})$, *there exist corresponding comparison triangles* $\triangle(\overline{t_{n_k}}, \overline{x_{n_k}}, \overline{s_{n_k}})$ *and* $\triangle(\overline{f(x_{n_k})}, \overline{t_{n_k}}, \overline{x_{n_k}})$ *such that*

$$d(t_{n_k}, x_{n_k}) = \left\|\overline{t_{n_k}} - \overline{x_{n_k}}\right\|, d(t_{n_k}, s_{n_k}) = \left\|\overline{t_{n_k}} - \overline{s_{n_k}}\right\|, d(x_{n_k}, s_{n_k}) = \left\|\overline{x_{n_k}} - \overline{s_{n_k}}\right\|,$$
$$d(t_{n_k}, f(x_{n_k})) = \left\|\overline{t_{n_k}} - \overline{f(x_{n_k})}\right\|, \; d(x_{n_k}, f(x_{n_k})) = \left\|\overline{x_{n_k}} - \overline{f(x_{n_k})}\right\|.$$

*Let* $\overline{x_{n_{k+1}}} = \alpha_{n_k} \overline{f(x_{n_k})} + (1-\alpha_{n_k}) \overline{t_{n_k}}$ *be the comparison point of* $x_{n_{k+1}}$. *By Lemma 2.9,* (3.31) *and* (3.32), *we can get that*

$$\begin{aligned}
d(x_{n_{k+1}}, x_{n_k}) \le & \left\|\overline{x_{n_{k+1}}} - \overline{x_{n_k}}\right\| \\
= & \left\|\alpha_{n_k} \overline{f(x_{n_k})} + (1-\alpha_{n_k}) \overline{t_{n_k}} - \overline{x_{n_k}}\right\| \\
\le & \alpha_{n_k} \left\|\overline{f(x_{n_k})} - \overline{x_{n_k}}\right\| + (1-\alpha_{n_k}) \left\|\overline{t_{n_k}} - \overline{x_{n_k}}\right\| \\
\le & \alpha_{n_k} \left\|\overline{f(x_{n_k})} - \overline{x_{n_k}}\right\| + \left\|\overline{t_{n_k}} - \overline{x_{n_k}}\right\| \\
\le & \alpha_{n_k} \left\|\overline{f(x_{n_k})} - \overline{x_{n_k}}\right\| + \left\|\overline{t_{n_k}} - \overline{s_{n_k}}\right\| + \left\|\overline{s_{n_k}} - \overline{x_{n_k}}\right\| \\
= & \alpha_{n_k} d(f(x_{n_k}), x_{n_k}) + d(t_{n_k}, s_{n_k}) + d(s_{n_k}, x_{n_k}) \to 0, \; (as\ k \to \infty).
\end{aligned}$$

*Since the sequence* $\{x_{n_k}\}$ *is bounded, so there exists a subsequence* $\left\{x_{n_{k_j}}\right\}$ *of* $\{x_{n_k}\}$ *such that* $\lim_{j\to\infty} x_{n_{k_j}} \to z$. *Hence, we have*

$$\begin{aligned}
\limsup_{k\to\infty} \left\langle \exp_q^{-1} f(q), \exp_q^{-1} x_{n_k} \right\rangle &= \lim_{j\to\infty} \left\langle \exp_q^{-1} f(q), \exp_q^{-1} x_{n_{k_j}} \right\rangle \\
&= \left\langle \exp_q^{-1} f(q), \exp_q^{-1} z \right\rangle.
\end{aligned} \tag{3.33}$$

*Since* $x_{n_{k_j}} \to z$, *so from (3.30), (3.31), (3.32) and (3.29), we know that* $\omega_{n_{k_j}} \to z$, $y_{n_{k_j}} \to z$, $z_{n_{k_j}} \to z$ *and* $s_{n_{k_j}} \to z$, *respectively. Further, from (3.2), (3.3) and Lemma 2.16, we have* $\langle A(z), \exp_z^{-1} x\rangle \ge 0$ *and* $\langle B(z), \exp_z^{-1} y\rangle \ge 0$ *for any* $x, y \in M$. *This implies that* $z \in VI(M, A) \cap VI(M, B)$. *From (3.31),* *(3.32) and Lemma 2.17, we get that $z \in Fix(U)$. So, we have that $z \in \Omega$.*

*Since $z \in \Omega$ and $q = \pi_\Omega f(q)$, it follows from (3.33) and Lemma 2.5 that*

$$\limsup_{k\to\infty} \left\langle \exp_q^{-1} f(q), \exp_q^{-1} x_{n_k} \right\rangle = \left\langle \exp_q^{-1} f(q), \exp_q^{-1} z \right\rangle \leq 0. \tag{3.34}$$

*which, together with $\lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) = 0$, (3.33) and (3.34) yields,*

$$\begin{aligned}
&\limsup_{k\to\infty} \left\langle \exp_q^{-1} f(q), \exp_q^{-1} x_{n_k+1} \right\rangle \\
\leq &\limsup_{k\to\infty} \left\langle \exp_q^{-1} f(q), \exp_{x_{n_k}}^{-1} x_{n_k+1} \right\rangle + \limsup_{k\to\infty} \left\langle \exp_q^{-1} f(q), \exp_q^{-1} x_{n_k} \right\rangle \\
= &\left\langle \exp_q^{-1} f(q), \exp_q^{-1} z \right\rangle \leq 0.
\end{aligned} \tag{3.35}$$

*From Lemma 3.2, Lemma 2.16, (3.35) and Claim 3, we have*

$$p_{n+1} \leq \sigma_n q_n + (1-\sigma_n)\, p_n, \ \forall n \geq 1,$$

*where*

$$\begin{aligned}
p_n &= d^2(x_n, q), \ \sigma_n = (1-\rho)\alpha_n, \\
q_n &= \frac{2}{1-\rho} \left\langle \exp_q^{-1} f(q), \exp_q^{-1} x_{n+1} \right\rangle + \frac{3M\theta_n}{(1-\rho)\alpha_n} d(x_n, x_{n-1}).
\end{aligned}$$

*Then it follows from Lemma 2.15 that $\{x_n\}$ converges to $q$. The proof of Theorem 3.1 is completed.*

Next, we introduce our Algorithm 2 to solve (1.2) on Hadamard manifolds. The Algorithm 2 is as follows:

Motivated by the proof of Lemma 3.3 of Khammahawong et al. [21], we get a similar conclusion under more general assumptions.

**Lemma 3.5** *Assume that Assumptions (A1-A7) holds, and the sequences $\{u_n\}$ and $\{v_n\}$ are generated by Algorithm 2. Then*

$$d^2(u_n, p) \leq d^2(\omega_n, p) - (1 - \mu^2 \frac{\tau_n^2}{\tau_{n+1}^2}) d^2(\omega_n, y_n), \ \forall p \in \Omega, \tag{3.38}$$

$$d^2(v_n, p) \leq d^2(\omega_n, p) - (1 - \mu^2 \frac{\kappa_n^2}{\kappa_{n+1}^2}) d^2(\omega_n, z_n), \ \forall p \in \Omega, \tag{3.39}$$

*and*

$$d(u_n, y_n) \leq (\mu \frac{\tau_n}{\tau_{n+1}}) d(\omega_n, y_n), \tag{3.40}$$

$$d(v_n, z_n) \leq (\mu \frac{\kappa_n}{\kappa_{n+1}}) d(\omega_n, z_n). \tag{3.41}$$

**Algorithm 2:** The viscosity-type inertial Tseng's extragradient algorithm

**Initialization:** Take $\theta > 0, \tau_1 > 0, \kappa_1 > 0, \mu \in (0,1)$. Let $x_0, x_1 \in M$ be arbitrarily choose.

**Step 1.** Input the iterates $x_{n-1}$ and $x_n$, $(n > 1)$. Set $\omega_n = \exp_{x_n} -\theta_n \exp_{x_n}^{-1} x_{n-1}$, where $\theta_n$ is defined in (3.1).

**Step 2.** Compute $y_n$ satisfying

$$\left\langle \tau_n P_{y_n,\omega_n} A(\omega_n) - \exp_{y_n}^{-1} \omega_n, \exp_{y_n}^{-1} p \right\rangle \geq 0,\ \forall x \in M. \tag{3.36}$$

**Step 3.** Compute $z_n$ satisfying

$$\left\langle \kappa_n P_{z_n,\omega_n} B(\omega_n) - \exp_{z_n}^{-1} \omega_n, \exp_{z_n}^{-1} p \right\rangle \geq 0,\ \forall x \in M.$$

**Step 4.** Compute $u_n$ satisfying

$$u_n = \exp_{y_n} \tau_n (P_{y_n,\omega_n} A(\omega_n) - A(y_n)). \tag{3.37}$$

**Step 5.** Compute $v_n$ satisfying

$$v_n = \exp_{z_n} \kappa_n (P_{z_n,\omega_n} B(\omega_n) - B(z_n)).$$

**Step 6.** Compute $s_n = \exp_{u_n} \delta_n \exp_{u_n}^{-1} v_n$.

**Step 7.** Compute $x_{n+1} = \exp_{t_n} \alpha_n \exp_{t_n}^{-1} f(x_n)$,
where $t_n = \exp_{s_n} \beta_n \exp_{s_n}^{-1} U s_n$, update $\tau_{n+1}$ by (3.6) and update $\kappa_{n+1}$ by (3.7).

Set $n := n + 1$ and go to **Step 1**.

**Proof** *For geodesic triangles $\triangle(u_n, y_n, p)$ and $\triangle(y_n, \omega_n, p)$, there exist comparison triangles $\triangle(\overline{u_n}, \overline{y_n}, \overline{p})$ and $\triangle(\overline{y_n}, \overline{\omega_n}, \overline{p})$ such that*

$$d(u_n, y_n) = \|\overline{u_n} - \overline{y_n}\|\,,\ d(u_n, p) = \|\overline{u_n} - \overline{p}\|\,,\ d(y_n, p) = \|\overline{y_n} - \overline{p}\|\,,$$
$$d(\omega_n, p) = \|\overline{\omega_n} - \overline{p}\|\,,\ d(\omega_n, y_n) = \|\overline{\omega_n} - \overline{y_n}\|\,.$$

*This implies that*

$$\begin{aligned}
d^2(u_n, p) &= \|\overline{u_n} - \overline{p}\|^2\\
&= \|\overline{u_n} - \overline{y_n}\|^2 + \|\overline{y_n} - \overline{p}\|^2 + 2\left\langle \overline{u_n} - \overline{y_n}, \overline{y_n} - \overline{p}\right\rangle\\
&= \|\overline{u_n} - \overline{y_n}\|^2 + \|\overline{y_n} - \overline{\omega_n}\|^2 + \|\overline{\omega_n} - \overline{p}\|^2 + 2\left\langle \overline{u_n} - \overline{y_n}, \overline{y_n} - \overline{p}\right\rangle + 2\left\langle \overline{y_n} - \overline{\omega_n}, \overline{\omega_n} - \overline{p}\right\rangle\\
&= \|\overline{u_n} - \overline{y_n}\|^2 + \|\overline{y_n} - \overline{\omega_n}\|^2 + \|\overline{\omega_n} - \overline{p}\|^2 + 2\left\langle \overline{u_n} - \overline{y_n}, \overline{y_n} - \overline{p}\right\rangle + 2\left\langle \overline{y_n} - \overline{\omega_n}, \overline{\omega_n} - \overline{y_n}\right\rangle\\
&\quad + 2\left\langle \overline{y_n} - \overline{\omega_n}, \overline{y_n} - \overline{p}\right\rangle\\
&= \|\overline{\omega_n} - \overline{p}\|^2 + \|\overline{u_n} - \overline{y_n}\|^2 - \|\overline{y_n} - \overline{\omega_n}\|^2 - 2\left\langle \overline{y_n} - \overline{u_n}, \overline{y_n} - \overline{p}\right\rangle + 2\left\langle \overline{y_n} - \overline{\omega_n}, \overline{y_n} - \overline{p}\right\rangle. \qquad (3.42)
\end{aligned}$$

*Let $\gamma$ and $\overline{\gamma}$ be the angles at the vertices $y_n$ and $\overline{y_n}$, respectively, It follows from Lemma 2.8 and Lemma 2.9 that $\gamma \leq \overline{\gamma}$ and*

$$\begin{aligned}
\left\langle \overline{y_n} - \overline{\omega_n}, \overline{y_n} - \overline{p}\right\rangle &= \|\overline{y_n} - \overline{\omega_n}\|\,\|\overline{y_n} - \overline{p}\|\, cos\overline{\gamma}\\
&= d(y_n, \omega_n) d(y_n, p) cos\overline{\gamma}\\
&\leq d(y_n, \omega_n) d(y_n, p) cos\gamma\\
&= \left\langle \exp_{y_n}^{-1} \omega_n, \exp_{y_n}^{-1} p\right\rangle. \qquad (3.43)
\end{aligned}$$

*Similarily, we have*

$$\left\langle \overline{y_n} - \overline{u_n}, \overline{y_n} - \overline{p}\right\rangle \ \leq \left\langle \exp_{y_n}^{-1} u_n, \exp_{y_n}^{-1} p\right\rangle. \qquad (3.44)$$

*Since $u_n = exp_{y_n}\tau_n(P_{y_n,\omega_n}A(\omega_n) - A(y_n))$, then $exp_{y_n}^{-1}u_n = \tau_n(P_{y_n,\omega_n}A(\omega_n) - A(y_n))$. Substituting (3.43) and (3.44) into (3.42), we obtain*

$$\begin{aligned}
d^2(u_n, p) \leq& d^2(\omega_n, p) - d^2(y_n, \omega_n) + \tau_n^2 d^2(A(\omega_n), A(y_n)) - 2\tau_n\left\langle P_{y_n,\omega_n}A(\omega_n) - A(y_n), exp_{y_n}^{-1}p\right\rangle \qquad (3.45)\\
&+ 2\left\langle \exp_{y_n}^{-1} \omega_n, \exp_{y_n}^{-1} p\right\rangle.
\end{aligned}$$

*From (3.36), we get*

$$\left\langle \exp_{y_n}^{-1} \omega_n, \exp_{y_n}^{-1} p\right\rangle \leq \left\langle \tau_n P_{y_n,\omega_n}A(\omega_n), \exp_{y_n}^{-1} p\right\rangle. \qquad (3.46)$$

*According to (3.45) and (3.46), we may obtain*

$$\begin{aligned}
d^2(u_n, p) \leq& d^2(\omega_n, p) - d^2(y_n, \omega_n) + \tau_n^2 d^2(A(\omega_n), A(y_n)) - 2\tau_n\left\langle P_{y_n,\omega_n}A(\omega_n) - A(y_n), exp_{y_n}^{-1}p\right\rangle\\
&+ 2\tau_n\left\langle P_{y_n,\omega_n}A(\omega_n), \exp_{y_n}^{-1} p\right\rangle\\
=& d^2(\omega_n, p) - d^2(y_n, \omega_n) + \tau_n^2 d^2(A(\omega_n), A(y_n)) + 2\tau_n\left\langle A(y_n), \exp_{y_n}^{-1} p\right\rangle.
\end{aligned}$$

*Since $A$ is a pesudomonotone vector field, from (3.4), we have*

$$\begin{aligned} d^2(u_n,p) \leq & d^2(\omega_n,p) - d^2(y_n,\omega_n) + \tau_n^2 d^2(A(\omega_n),A(y_n)) \\ \leq & d^2(\omega_n,p) - d^2(y_n,\omega_n) + \tau_n^2 \frac{\mu^2}{\tau_{n+1}^2} d^2(y_n,\omega_n) \\ \leq & d^2(\omega_n,p) - (1-\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}) d^2(y_n,\omega_n). \end{aligned}$$

*Further, from definition of $u_n$ and (3.6), we obtian*

$$d(u_n,y_n) = \left\|\exp_{y_n}^{-1} u_n\right\| = \|\tau_n(P_{y_n,\omega_n}A(\omega_n) - A(y_n))\| \leq (\mu\frac{\tau_n}{\tau_{n+1}}) d(\omega_n,y_n).$$

*Using the similar proofs of* (3.38) *and* (3.40)*, we may easily get* (3.39) *and* (3.41)*, respectively. The proof is completed.*

**Theorem 3.6** *Suppose that Assumptions $(A1)-(A7)$ hold. Then the sequence $\{x_n\}$ generated by Algorithm 2 converges to $q \in \Omega$, where $q = \pi_\Omega(f(q))$.*

**Proof** *We also divide the proof into four parts.*
***Claim 1.*** *The sequence $\{x_n\}$ is bounded.*
*It follows from Lemma 3.5 and (3.17) that*

$$\begin{aligned} d^2(t_n,q) \leq & \delta_n d^2(v_n,q) + (1-\delta_n) d^2(u_n,q) - \delta_n(1-\delta_n) d^2(v_n,u_n) + \beta_n(\lambda - 1 + \beta_n) d^2(Us_n,s_n) \\ \leq & \delta_n \left[ d^2(\omega_n,p) - (1-\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}) d^2(\omega_n,y_n) \right] + (1-\delta_n)\left[ d^2(\omega_n,p) - (1-\mu^2\frac{\kappa_n^2}{\kappa_{n+1}^2}) d^2(\omega_n,z_n) \right] \\ & - \delta_n(1-\delta_n) d^2(v_n,u_n) + \beta_n(\lambda - 1 + \beta_n) d^2(Us_n,s_n). \end{aligned} \tag{3.47}$$

*In view of Lemma3.2, there exists $n_1 \in \mathbb{N}$ such that $1-\mu^2\frac{\tau_n^2}{\tau_{n+1}^2} > 0$ and $1-\mu^2\frac{\tau_n^2}{\tau_{n+1}^2} > 0$ for any $n \geq n_1$. Since $\delta_n \in (0,1)$ and $\beta_n \in (a, 1-\lambda)$, we have*

$$d(t_n,q) \leq d(\omega_n,q),\ \forall n \geq n_1 \tag{3.48}$$

*Using the similar proof of the Claim 1 in the Theorem 3.4, we may show that the sequences $\{x_n\}$ is bounded. Further, the sequences $\{\omega_n\}$, $\{f(x_n)\}$, $\{y_n\}$, $\{z_n\}$, $\{u_n\}$ and $\{v_n\}$ are also bounded.*
***Claim 2.***

$$\begin{aligned} & (1-\alpha_n)\delta_n \left[ (1-\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}) d^2(\omega_n,y_n) \right] + (1-\alpha_n)(1-\delta_n)\left[ (1-\mu^2\frac{\kappa_n^2}{\kappa_{n+1}^2}) d^2(\omega_n,z_n) \right] \\ & + (1-\alpha_n)\delta_n(1-\delta_n) d^2(v_n,u_n) + (1-\alpha_n)\beta_n(1-\lambda-\beta_n) d^2(Us_n,s_n) \\ & \leq d^2(x_n,q) - d^2(x_{n+1},q) + \alpha_n d^2(f(x_n),q) + \alpha_n M_2, \end{aligned}$$

*where* $M_2 > 0$ *is a constant, such that* $M_2 \geq \sup\{2M_1 d(x_n, q) + \alpha_n M_1^2\}$.

*In view of Lemma 2.12, (3.24) and (3.47), we get*

$$\begin{aligned} d^2(x_{n+1}, q) \leq & \alpha_n d^2(f(x_n), q) + (1-\alpha_n) d^2(t_n, q) \\ \leq & \alpha_n d^2(f(x_n), q) + (1-\alpha_n) d^2(\omega_n, q) - (1-\alpha_n)\delta_n(1-\delta_n) d^2(v_n, u_n) \\ & - (1-\alpha_n)\delta_n \left[ (1 - \mu^2 \frac{\tau_n^2}{\tau_{n+1}^2}) d^2(\omega_n, y_n) \right] \\ & - (1-\alpha_n)(1-\delta_n) \left[ (1 - \mu^2 \frac{\kappa_n^2}{\kappa_{n+1}^2}) d^2(\omega_n, z_n) \right] \\ & + (1-\alpha_n)\beta_n(\lambda - 1 + \beta_n) d^2(U s_n, s_n) \\ \leq & \alpha_n d^2(f(x_n), q) + d^2(x_n, q) + \alpha_n M_2 - (1-\alpha_n)\delta_n(1-\delta_n) d^2(v_n, u_n) \\ & - (1-\alpha_n)(1-\delta_n) \left[ (1 - \mu^2 \frac{\kappa_n^2}{\kappa_{n+1}^2}) d^2(\omega_n, z_n) \right] \\ & - (1-\alpha_n)\delta_n \left[ (1 - \mu^2 \frac{\tau_n^2}{\tau_{n+1}^2}) d^2(\omega_n, y_n) \right] \\ & + (1-\alpha_n)\beta_n(\lambda - 1 + \beta_n) d^2(U s_n, s_n). \end{aligned}$$

*So, we may conclude Claim 2.*

***Claim 3.***

$$\begin{aligned} d^2(x_{n+1}, q) \leq & (1-\rho)\alpha_n \left[ \frac{2}{1-\rho} \left\langle \exp_q^{-1} f(q), \exp_q^{-1} x_{n+1} \right\rangle + \frac{3M\theta_n}{(1-\rho)\alpha_n} d(x_n, x_{n-1}) \right] \\ & + (1 - (1-\rho)\alpha_n) d^2(x_n, q), \ \forall n \geq n_1, \end{aligned}$$

*where* $M := \sup_{n\in\mathbb{N}} \{d(x_n, q), \theta_n d(x_n, x_{n-1})\} > 0$.

*The proof is similar to the proof of Claim 3 of Theorem 3.4. So we omit it here.*

***Claim 4.*** *The sequence* $\{x_n\}$ *converges to* $q = \pi_\Omega(f(q)) \in \Omega$.

*The proof is also similar to the proof of Claim 4 in the Theorem 3.4, we omit it here. The proof is completed.*

# 4 Application

In this section, we use our main results to find a solution of constrained convex minimization problems on Hadamard manifolds. Let $M$ be a Hadamard manifold and $g : M \to \mathbb{R}$ be a differentiable function. The directional derivative of $g$ at $q$ in direction $v \in T_q M$ is defined by Bento et al. in [36] as follows:

$$g'(q; v) := \lim_{t\to 0^+} \frac{g(\exp_q tv) - g(q)}{t}.$$

For any $v \in T_qM$, the gradient of $g$ at $q \in M$ is defined by $\langle gradg(q), v\rangle := g'(q; v)$.

**Lemma 4.1** *[37] Let $g : M \to \mathbb{R}$ be a differentiable function, where $M$ is a Riemannian manifold. Then, $gradg$ is a monotone vector field if and only if $g$ is geodesic convex.*

Assume that $g : M \to R$ is a twice differentiable function, then the Hessian of $g$ at $q \in M$ [38], denoted by $Hessg$, is defined by

$$Hessg(q) := \nabla_v(gradg(q)),\ \forall v \in T_qM,$$

where $\nabla$ stands for the Riemannian connection of $M$.

**Lemma 4.2** *[39] Let $g$ be a differentiable function from Hadamard manifold $M$ into $\mathbb{R}$. Then, $gradg$ is a $\Gamma$-Lipschitz continuous vector field if $Hessg$ is bounded.*

Here, we consider the following constrained convex minimization problem:

$$\min_{x\in K} g(x), \tag{4.1}$$

where $K$ is a subset of Hadamard manifold $M$ and $g : K \to \mathbb{R}$ is a differentiable geodesic convex function. The minimizer set of (4.1) is denoted by $S(g)$, in other words, $S(g) := \{x \in K : g(x) \le g(y), \forall y \in K\}$. In [22], it has been shown that the solutions of (4.1) are equivalent to the solutions of the variational inequality problem $VI(gradg, K)$, that is,

$$x^* \in S(g) \Longleftrightarrow \left\langle gradg(x^*), exp_{x^*}^{-1}y\right\rangle \ge 0,\ \forall y \in K. \tag{4.2}$$

It follows from Lemma 4.1 and Lemma 4.2 that $gradg$ is a $\Gamma$-Lipschitzian monotone vector field when $g$ is a twice continuously differentiable and geodesic convex function, and $Hessg$ is bounded. Therefore, replacing $A$ and $B$ in Algorithm 1 with $gradg$ and $I$, respectively, where $I$ is identity vector field, we can directly get Theorem 4.3 from our Theorem 3.4.

**Theorem 4.3** *Let $g : M \to \mathbb{R}$ be a twice continuously differentiable and geodesic convex function, and $Hessg$ be bounded, $U : M \longrightarrow M$ be a $\lambda$-demicontractive mapping. Given $x_0 \in M,\ x_1 \in M$, calculate $x_n$ by*

$$\begin{cases} \omega_n = \exp_{x_n}(-\theta_n \exp_{x_n}^{-1} x_{n-1}), \\ \left\langle P_{y_n,\omega_n} gradg(\omega_n) - \frac{1}{\tau_n}\exp_{y_n}^{-1}\omega_n, \exp_{y_n}^{-1} p\right\rangle \ge 0,\ \forall x \in M, \\ \left\langle P_{z_n,y_n} gradg(y_n) - \frac{1}{\tau_n}\exp_{z_n}^{-1}\omega_n, \exp_{z_n}^{-1} q\right\rangle \ge 0,\ \forall y \in T_n, \\ t_n = \exp_{z_n}\beta_n \exp_{z_n}^{-1} Uz_n, \\ x_{n+1} = \exp_{t_n}\alpha_n \exp_{t_n}^{-1} f(x_n), \end{cases} \tag{4.3}$$

where the half-space $T_n := \{a \in M \mid \langle \exp_{y_n}^{-1} \omega_n - \tau_n P_{y_n,\omega_n} gradg(\omega_n), \exp_{y_n}^{-1} x \rangle \leq 0\}$, $\{\theta_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are defined in (3.1), (A6), (A7), respectively, and step size $\tau_n$ is updated by following rule:

$$\tau_{n+1} = \begin{cases} \min\left\{\frac{\mu d(y_n,\omega_n)}{d(gradg(y_n),gradg(\omega_n))}, \tau_n\right\}, & \text{if } d(gradg(y_n), gradg(\omega_n)) \neq 0, \\ \tau_n, & \text{otherwise.} \end{cases}$$

If $Fix(U) \cap S(g) \neq \varnothing$, then the iterative sequence $\{x_n\}$ generated by (4.3) converges to a point of $Fix(U) \cap S(g)$.

Moreover, replacing $A$ and $B$ in Algorithm 2 with $gradg$ and $I$, respectively, we can directly get Theorem 4.4 from our Theorem 3.6, where $I$ is identity vector field and $\{\delta_n\} = 0$.

**Theorem 4.4** *Let $g : M \to \mathbb{R}$ be a twice continuously differentiable and geodesic convex function, and $Hessg$ be bounded, $U : M \longrightarrow M$ be a $\lambda$-demicontractive mapping. Given $x_0 \in M,\ x_1 \in M$, calculate $x_n$ by*

$$\begin{cases} \omega_n = \exp_{x_n}(-\theta_n \exp_{x_n}^{-1} x_{n-1}), \\ \left\langle P_{y_n,\omega_n} gradg(\omega_n) - \frac{1}{\tau_n} \exp_{y_n}^{-1} \omega_n, \exp_{y_n}^{-1} p \right\rangle \geq 0,\ \forall x \in M, \\ z_n = \exp_{y_n} \tau_n (P_{y_n,\omega_n} gradg(\omega_n) - gradg(y_n)), \\ t_n = \exp_{z_n} \beta_n \exp_{z_n}^{-1} U z_n, \\ x_{n+1} = \exp_{t_n} \alpha_n \exp_{t_n}^{-1} f(x_n), \end{cases} \tag{4.4}$$

where $\{\theta_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are defined in (3.1), (A6), (A7), respectively, and step size $\tau_n$ is updated by following rule:

$$\tau_{n+1} = \begin{cases} \min\left\{\frac{\mu d(y_n,\omega_n)}{d(gradg(y_n),gradg(\omega_n))}, \tau_n\right\}, & \text{if } d(gradg(y_n), gradg(\omega_n)) \neq 0, \\ \tau_n, & \text{otherwise.} \end{cases}$$

If $Fix(U) \cap S(g) \neq \varnothing$, then the iterative sequence $\{x_n\}$ generated by (4.4) converges to a point of $Fix(U) \cap S(g)$.

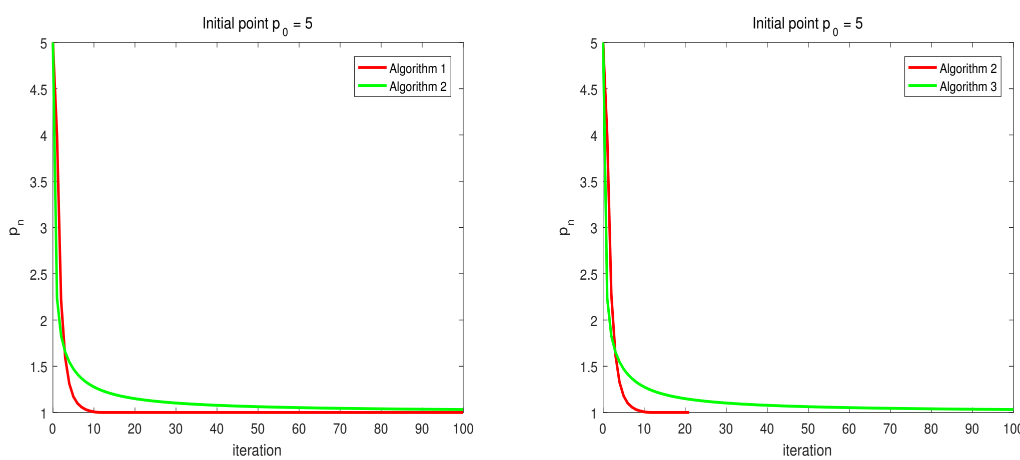
# 5 Numerical examples

In this section we provide a numerical example to illustrate the numerical behavior of Algorithms 1 and 2 on Hadamard manifolds and compare our algorithms with the Halpern-type algorithm [17] named by Algorithms 3. All the programs are performed in Matlab R2016a and computed on Intel(R) Core(TM) i7-7700HQ CPU 2.80 GHz with RAM 16GB.

***Example*** Let $M := R^{++} = \{x \in \mathbb{R} : x > 0\}$, the Riemannian metric $\langle \cdot, \cdot \rangle$ defined by $\langle a, b \rangle := \frac{1}{x^2} ab$, where $a, b \in T_x M$. Obviously, the tangent space $T_x M$ is $\mathbb{R}$ for any $x \in M$. So, the parallel transport $P_{y,x} : T_y M \to T_x M$ is the identity vector field, the Riemannian distance $d(\cdot, \cdot)$ is defined by $d(x, y) := \left| \ln \frac{x}{y} \right|$, for $x, y \in M$, then $M$ is a Hadamard manifold and the unique geodesic $\chi : \mathbb{R} \to M$ is $\chi(t) := x e^{(vt/x)}$, where $v = \chi^{'}(0) \in T_x M$. Moreover, the exponential mapping is $\exp_x tv = x e^{(vt/x)}$ and the inverse of exponential mapping is $\exp_x^{-1} y = x \ln \frac{y}{x}$.

Let $K := [1, +\infty)$. Given the contraction mapping $f(x) = \sqrt{x}$, the demicontractive mapping $U(x) = \sqrt[3]{x}$ and the monotone vector field $A(x) = x \ln x$, where $x \in M$.

Since $d^2(U(x), 1) = |\ln \sqrt[3]{x}|^2$, $d^2(x, 1) = |\ln x|^2$, where $\beta > 0$, we have $d^2(U(x), 1) \leq d^2(x, 1) + \beta d^2(x, U(x))$, then assumptions (A3)-(A5) hold and the unique solution of (1.2) is 1, the detail in [21]. Suppose the vector field $B$ is an identity vector field and the parameter $\delta_n = 0$, $n \geq 1$. We choose the initial point $x_0 = 5$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\mu = 0.5$, $\beta_n = \frac{n}{2n+1}$, $\alpha_n = \frac{1}{n+1}$ and $\theta_1 = \tau_1 = 1$. Then the numerical results are reported in Figure 1 and Table 1 as follows:




(a) Iterative process of Algorithms 1 and 3 in Example

(b) Iterative process of Algorithms 2 and 3 in Example

Figure 1: Iterative process of process

| Iter. no. | Algorithm1 | Algorithm2 | Algorithm3 |
|---|---|---|---|
| 0 | 5 | 5 | 5 |
| 5 | 1.31058462618559 | 1.32684171792157 | 1.43798265238294 |
| 13 | 1.00005003645558 | 1.00005011782881 | 1.21783907664200 |
| 17 | 1.00000000005475 | 1.00000000005484 | 1.17228751666004 |
| 21 | 1 | 1 | 1.14226302469760 |

Table 1: Computation results of Example

From Table 1 and Figure 1, it is easy to see that Algorithm 1 and 2 converge faster than Algorithm 3. Moreover, we record the calculation time of each algorithm with some different initial points in Table 2.

| Initial point | Algorithm1 | Algorithm2 | Algorithm3 |
|---|---|---|---|
| 5 | 0.021s | 0.021s | 0.137s |
| 10 | 0.026s | 0.026s | 0.141s |
| 100 | 0.026s | 0.029s | 0.142s |
| 1000 | 0.026s | 0.022s | 0.150s |
| 10000 | 0.026s | 0.030s | 0.144s |
| 100000 | 0.028s | 0.030s | 0.156s |

Table 2: The calculation time of each algorithms

Obviously, Algorithm 1 and Algorithm 2 use less calculation time than Algorithm 3.

# 6 Conclusion

This paper focuses on investigating the common solution problem of two pesudomonotone variational inequality problems and fixed point problem of $\lambda$-demicontractive mapping on Hadamard manifold, which is more general problem than that one in [21]. Two convergence theorems are established and the main results presented in this paper are utilized to solve convex minimization problem. Meanwhile, a numerical example is given to show the effectiveness of our algorithms by comparing with the algorithm presented in [17].

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# References

[1] S. Plubtieng and K. Ungchittrakool. Hybrid iterative methods for convex feasibility problems and fixed point problems of relatively nonexpansive mappings in Banach spaces. *Fixed Point Theory and Applications*, 2008:1–19, 2009.

[2] M.A. Rami, U. Helmke, and J.B. Moore. A finite steps algorithm for solving convex feasibility problems. *Journal of Global Optimization*, 38:143–160, 2007.

[3] K.H. Lu, G.S. Jing, and L. Wang. Distributed algorithms for solving the convex feasibility problems. *Science China Information Sciences*, 63:1–3, 2020.

[4] X.P. Zhao and Y.H. Yao. Modified extragradient algorithms for solving monotone variational inequalities and fixed point problems. *Optimization*, 69(9):1987–2002, 2020.

[5] S.S. Chang, H.W.J. Lee, and C.K. Chan. A block hybrid method for solving generalized equilibrium problems and convex feasibility problem. *Advances in Computational Mathematics*, 38:563–580, 2013.

[6] D.V. Thong and D.V. Hieu. Some extragradient-viscosity algorithms for solving variational inequality problems and fixed point problems. *Numerical Algorithms*, 82(3):761–789, 2019.

[7] S.S. Zhang, C.K. Chan, and H.W.J. Lee. Modified block iterative method for solving convex feasibility problem, equilibrium problems and variational inequality problems. *Acta Mathematica Sinica, English Series*, 28(4):741–758, 2012.

[8] D.V. Thong and D.V. Hieu. Modified subgradient extragradient algorithms for variational inequality problems and fixed point problems. *Optimization*, 67(1):83–102, 2018.

[9] X.L. Qin and N.T. An. Smoothing algorithms for computing the projection onto a Minkowski sum of convex sets. *Computational Optimization and Applications*, 74(3):821–850, 2019.

[10] P.L. Combettes. The convex feasibility problem in image recovery. In *Advances in imaging and electron physics*, volume 95, pages 155–270. Elsevier, 1996.

[11] T. Kotzer, N. Cohen, and J. Shamir. Image restoration by a novel method of parallel projection onto constraint sets. *Optics Letters*, 20(10):1172–1174, 1995.

[12] M.L. Sezan and H. Stark. Applications of convex projection theory to image recovery in tomography and related areas. *Image Recovery: Theory and Application*, pages 155–270, 1987.

[13] A. Stachurski. Parallel optimization: Theory, algorithms and applications. *Scalable Computing: Practice and Experience*, 3(4), 2000.

[14] G.C. Bento and J.G. Melo. Subgradient method for convex feasibility on Riemannian manifolds. *Journal of Optimization Theory and Applications*, 152:773–785, 2012.

[15] X.M. Wang, C. Li, and J.C. Yao. Subgradient projection algorithms for convex feasibility on Riemannian manifolds with lower bounded curvatures. *Journal of Optimization Theory and Applications*, 164:202–217, 2015.

[16] X.M. Wang, C. Li, J.H. Wang, and J.C. Yao. Linear convergence of subgradient algorithm for convex feasibility on Riemannian manifolds. *SIAM Journal on Optimization*, 25(4):2334–2358, 2015.

[17] S. Al-Homidan, A.H. Qamrul, and F. Babu. Halpern-and mann-type algorithms for fixed points and inclusion problems on Hadamard manifolds. *Numerical Functional Analysis and Optimization*, 40(6):621–653, 2019.

[18] D. Filali, M. Dilshad, M. Akram, F. Babu, and I. Ahmad. Viscosity method for hierarchical variational inequalities and variational inclusions

on Hadamard manifolds. *Journal of Inequalities and Applications*, 2021:1–20, 2021.

[19] S.S. Chang, J.F. Tang, and C.F. Wen. A new algorithm for monotone inclusion problems and fixed points on Hadamard manifolds with applications. *Acta Mathematica Scientia*, 41(4):1250–1262, 2021.

[20] S.S. Chang, J.C. Yao, M. Liu, L.C. Zhao, and J.H. Zhu. Shrinking projection algorithm for solving a finite family of quasi-variational inclusion problems in Hadamard manifold. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 115(4):166, 2021.

[21] K. Khammahawong, P. Chaipunya, and P. Kumam. Iterative algorithms for monotone variational inequality and fixed point problems on Hadamard manifolds. *Advances in Operator Theory*, 7(4):43, 2022.

[22] S.Z. Németh. Variational inequalities on Hadamard manifolds. *Nonlinear Analysis: Theory, Methods & Applications*, 52(5):1491–1498, 2003.

[23] G.J. Tang and N.J. Huang. Korpelevichs method for variational inequality problems on Hadamard manifolds. *Journal of Global Optimization*, 54:493–509, 2012.

[24] G.J. Tang, L.W. Zhou, and N.J. Huang. The proximal point algorithm for pseudomonotone variational inequalities on Hadamard manifolds. *Optimization Letters*, 7:779–790, 2013.

[25] S. Jana and C. Nahak. A projection-type method for set valued variational inequality problems on Hadamard manifolds. *Mediterranean Journal of Mathematics*, 13:3939–3953, 2016.

[26] J.F. Chen, S.Y. Liu, and X.K. Chang. Modified tseng's extragradient methods for variational inequality on Hadamard manifolds. *Applicable Analysis*, 100(12):2627–2640, 2021.

[27] Z.L. Ma and L. Wang. New convergence theorems for pseudomonotone variational inequality on Hadamard manifolds. *Symmetry*, 15(11):2085, 2023.

[28] T. Sakai. *Riemannian geometry*, volume 149. American Mathematical Soc., 1996.

[29] C. Li, G. López, V. Martín-Márquez, and J.H. Wang. Resolvents of set-valued monotone vector fields in Hadamard manifolds. *Set-Valued and Variational Analysis*, 19:361–383, 2011.

[30] J.H. Wang, G. López, V. Martín-Márquez, and C. Li. Monotone and accretive vector fields on Riemannian manifolds. *Journal of optimization theory and applications*, 146(3):691–708, 2010.

[31] G.J. Tang and N.J. Huang. Korpelevichs method for variational inequality problems on Hadamard manifolds. *Journal of Global Optimization*, 54:493–509, 2012.

[32] M.R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319. Springer Science & Business Media, 2013.

[33] C. Li, G. López, and V. Martín-Márquez. Monotone vector fields and the proximal point algorithm on Hadamard manifolds. *Journal of the London Mathematical Society*, 79(3):663–683, 2009.

[34] S. Saejung and P. Yotkaew. Approximation of zeros of inverse strongly monotone operators in Banach spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 75(2):742–750, 2012.

[35] O.P. Ferreira and P.R. Oliveira. Proximal point algorithm on Riemannian manifolds. *Optimization*, 51(2):257–270, 2002.

[36] G.C. Bento, O.P. Ferreira, and P.R. Oliveira. Local convergence of the proximal point method for a special class of nonconvex functions on Hadamard manifolds. *Nonlinear Analysis: Theory, Methods & Applications*, 73(2):564–572, 2010.

[37] T. Rapcsák. Smooth nonlinear optimization in Rn..(nonconvex optimization and its applications, 19.), 1997.

[38] C. Udriste. *Convex functions and optimization methods on Riemannian manifolds*, volume 297. Springer Science & Business Media, 2013.

[39] L.C. Ceng, A. Petruşel, X. Qin, and J.C. Yao. A modified inertial subgradient extragradient method for solving pseudomonotone variational inequalities and common fixed point problems. *Fixed Point Theory*, 21(1), 2020.

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