About Bertrand Curves and Type-1 Bishop Frame in Three-Dimensional Weyl Space $W_3$

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Abstract

In this paper, we have defined Bertrand curves in three-dimensional Weyl space. Then we have given the relations between the Bishop vector fields of Bertrand curve pair. Finally, while $(C, \overline{C})$ is Bertrand curve pair, we have obtained the equalities depending on $K_1$ and $K_2$.

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1 Introduction

A manifold with a conformal metric $g_{ij}$ and a symmetric connection $\nabla_k$ satisfying the compatibility condition

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0$$

is called a Weyl space which will be denoted by $W(g_{ij}, T_k)$. The vector field $T_k$ is named the complementary vector field. Under a normalization of the metric tensor $g_{ij}$ in the form

$$\tilde{g}_{ij} = \lambda^2 g_{ij}$$

the complementary vector field $T_k$ is transformed by the law
\[ \bar{T}_k = T_k + \partial_k \ln \lambda \]  

(3)

where \( \lambda \) is a scalar function \([9]\). If under the transformation (2), the quantity \( A \) is called a satellite of \( g_{ij} \) with weight \([p]\).

The prolonged derivative and prolonged covariant derivative of \( A \) are defined as

\[ \dot{\partial}_k = \partial_k A - pT_k A \]  

(4)

and

\[ \dot{\nabla}_k A = \nabla_k A - pT_k A \]  

(5)

respectively \([4], [10]\). The \( v^i \) \((i, r = 1, 2, 3)\) be the contravariant components of the vector field \( v \) in \( W_3(g_{ij}, T_k) \). Suppose that the vector field \( v \) are normalized by the conditions \( g^i_j v^i v^j = 1 \( j, r = 1, 2, 3 \). The prolonged covariant derivative of the vector field \( v \) is given by \([14]\)

\[ \dot{\nabla}_k v^i = \frac{s}{r} \frac{T_k v^i}{s} \( s = 1, 2, 3 \). \]  

(6)

The quantities

\[ \frac{q}{rs} = \frac{q}{r} \frac{v^k}{s} \( q = 1, 2, 3 ; r \neq s \) \]  

(7)

and

\[ \frac{r}{s} = \frac{T_k v^k}{s} \]  

(8)

are called the Chebyshev curvature of the first kind and geodesic curvature of the net \((v_1, v_2, v_3)\), respectively \([14]\). The vector fields

\[ a^i_{rs} = \frac{q}{rs} \frac{v^i}{q} , \ c^i_{rs} = \frac{r}{s} \frac{v^i}{r} \( i, q, r, s = 1, 2, 3 \) \]  

(9)

are called the Chebyshev vector fields of the first kind and geodesic vector fields of the net \((v_1, v_2, v_3)\), respectively \([14]\). Since the net \((v_1, v_2, v_3)\) is an orthogonal net, we have \([14]\)

\[ \frac{T_k}{r} = 0 , \ \frac{T_k}{r} + \frac{T_k}{p} = 0 \( r \neq p \). \]  

(10)
Bertrand curves are named after J. Bertrand who determined them in 1850. Later, these curves have been studied by L.R. Pears in 1935, J.K. Wittemore in 1940, and J.F. Burke in 1960. Besides the properties provided in various studies, they have been handled in different spaces such as Riemann-Otsuki space, Galilean space, three-dimensional sphere, three-dimensional space forms, Euclidean 3-space, and Minkowski space-time.

2 Preliminaries

Let $C : \mathbf{x}^i = \mathbf{x}^i(s)$ be a curve in three-dimensional Weyl space $W_3$ ($s$ is the arc length parameter of $C$). Let $\{v, \mathbf{v}, \mathbf{v}\}$ be Frenet frame and $\{v, n, n\}$ be Bishop frame of the curve $C$ such that $K_1, K_2$ are the first and second curvatures and $k_1, k_2$ are the Bishop curvatures of $C$. The Frenet and Bishop formulas of $C$ are

\[
\begin{align*}
\nabla_1 v^i_1 & = K_1 v^i_1, \\
\nabla_1 v^i_2 & = - K_1 v^i_1 + K_2 v^i_3, \\
\nabla_1 v^i_3 & = - K_2 v^i_2, \\
\end{align*}
\]

and

\[
\begin{align*}
\nabla_1 n^i_1 & = k_1 v^i_1 + k_2 v^i_2, \\
\nabla_1 n^i_2 & = - k_1 v^i_1, \\
\nabla_1 n^i_3 & = - k_2 v^i_2.
\end{align*}
\]

Since $v^i_2$ is orthogonal to $v$, $v^i_2$ can be written as $v^i_2 = a n^i_1 + b n^i_2$ [6] where $a = g_{ij} v^j_1 n^j_2 = \cos \theta$, $b = g_{ij} v^j_2 n^j_2 = \cos(\frac{\pi}{2} - \theta) = \sin \theta$ and $\theta = \angle(v, n)$. In addition, since $v^i_3 = \varepsilon_{ijk} v^j_1 v^k_2$ ($k = 1, 2, 3$), the following equality is satisfied:

\[
\begin{align*}
\varepsilon_{ijk} v^j_1 v^k_2 & = \varepsilon_{ijk} (\cos \theta n^k_1 + \sin \theta n^k_2), \\
\varepsilon_{ijk} v^j_1 v^k_2 & = n^i_2 \cos \theta - n^i_1 \sin \theta.
\end{align*}
\]

Therefore

\[
\begin{pmatrix}
    v^i_1 \\
    v^i_2 \\
    v^i_3
\end{pmatrix}
= \begin{pmatrix}
    1 & 0 & 0 \\
    0 & \cos \theta & \sin \theta \\
    0 & -\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
    n^i_1 \\
    n^i_2 \\
    n^i_3
\end{pmatrix}.
\]
is valid.

From (11), $v^k \nabla_k v^i = K_1 v^i_2$. Multiplying this equality by $g_{ij} v^j_2$ and taking summation on $i$ and $j$, we get [6]

$$K_1 = v^k_1 (\nabla_k v^i_2) g_{ij} v^j_2,$$

$$K_1 = T_k v^k_1 g_{ij} v^j_2,$$

$$K_1 = g_{ij} c^j v^i_2,$$

or

$$K_1 = \frac{2}{1}$$

(15)

where $g_{ij} v^i_2 v^j_2 = 1$, $g_{ij} v^i_3 v^j_2 = 0$ and $T_k = 0$.

Theorem 2.1 If $K_1 = 0$, then the geodesic vector field $c^i$ of the net $(v_1, v_2, v_3)$ is orthogonal to $v^i_2$.

From (11), $v^k \nabla_k v^i = -K_2 v^i_2$. Multiplying this relation by $g_{ij} v^j_2$ and summing on $i$ and $j$, we obtain [6]

$$-K_2 = v^k_1 (\nabla_k v^i_2) g_{ij} v^j_2,$$

$$-K_2 = T_k v^k_1 g_{ij} v^j_2,$$

$$K_2 = -g_{ij} a^i v^j_2,$$

or

$$K_2 = -\frac{2}{31}$$

(17)

where $g_{ij} v^i_2 v^j_2 = 0$, $g_{ij} v^i_3 v^j_2 = 1$ and $T_k = 0$.

Theorem 2.2 If $K_2 = 0$, then the Chebyshev vector field of the first kind $a^i$ of the net $(v_1, v_2, v_3)$ is orthogonal to $v^i_2$.

Using $v^k \nabla_k v^i = -K_2 v^i_2$ and [13] and then multiplying obtained equation by $g_{ij} v^j_2$, we have $K_2 = v^k_1 \nabla_k \theta (\theta = \theta(s))$ where $g_{ij} v^i_2 v^j_2 = \sin \theta$, $g_{ij} n^i v^j_2 = \cos \theta$ and $g_{ij} v^i_2 v^j_2 = 0$. Multiplying $v^k \nabla_k v^i = k_1 n^i + k_2 n^i_2 = K_1 v^i_2$ by $g_{ij} n^j$ we get
\[(v^k_1 \nabla_k v^i_1)g_{ij} n^j_1 = k_1 = K_1 \cos \theta\]  
(19)

and multiplying \(v^k_1 \nabla_k v^i_1 = k_1 n^i_1 + k_2 n^i_2 = K_1 v^i_2\) by \(g_{ij} n^j_2\) we have

\[(v^k_1 \nabla_k v^i_1)g_{ij} n^j_2 = k_2 = K_1 \sin \theta.\]  
(20)

From (19) and (20), we obtain

\[k_1^2 + k_2^2 = K_1^2.\]

**Theorem 2.3** If \(k_1 = 0\), then the geodesic vector field \(c^i_1\) of the net \((v_1, v_2, v_3)\) is orthogonal to \(n_1\).

**Theorem 2.4** If \(k_2 = 0\), then the geodesic vector field \(c^i_1\) of the net \((v_1, v_2, v_3)\) is orthogonal to \(n_2\).

### 3 Bertrand Curves in \(W_3\)

Let \(\overline{C} : \bar{x}^i = \bar{x}^i(\bar{s})\) be other curve in \(W_3\) (\(\bar{s}\) is the arc length parameter of \(\overline{C}\)). Let us denote Frenet and Bishop components of \(\overline{C}\) by \(\{\overline{v}_1, \overline{v}_2, \overline{v}_3, \overline{K}_1, \overline{K}_2\}\) and \(\{v_1, \overline{v}_1, \overline{v}_2, k_1, k_2\}\), respectively.

**Definition 3.1** If the principal normal vector fields of the curves \(C\) and \(\overline{C}\) are linear dependent, the curve pair \((C, \overline{C})\) is called Bertrand curve pair.

If the curve pair \((C, \overline{C})\) is Bertrand curve pair the following equality is satisfied:

\[C(s) = \overline{C}(\bar{s}) + \lambda(\bar{s}) \overline{v}^i(\bar{s}).\]  
(21)

Taking prolonged covariant derivative of (21) in the direction of \(v^i_1\) we have

\[v^k_1 \nabla_k C = (v^k_1 \nabla_k C) f(s) = v^i_1 f(s) = \]

\[= v^i_1 + (v^k_1 \nabla_k \lambda) v^i_2 + \lambda(-K_1 v^i_1 + K_2 v^i_2)\]

(22)

and multiplying (22) by \(g_{ij} v^j_2\), we find

\[f(s) g_{ij} v^i_1 v^j_2 = f(s) g_{ij} v^i_1 v^j_2 = v^k_1 \nabla_k \lambda\]

(23)

or

\[0 = v^k_1 \nabla_k \lambda\]

(24)

where \(g_{ij} v^i_1 v^j_2 = 0\), \(g_{ij} v^i_1 v^j_2 = 1\), \(g_{ij} v^i_1 v^j_2 = 0\) and \(g_{ij} v^i_1 v^j_3 = 0\).
From (24), we get $\lambda$ is prolonged covariant constant [12]. On the other hand

$$f(s) = \pm \sqrt{(1 - \lambda K_1)^2 + \lambda^2 K_2^2}.$$ 

Let the angle $\alpha$ be between the tangent vector fields $\mathbf{v}_1$ and $\mathbf{v}$ of Bertrand curve pair $(C, \overline{C})$. Since $(C, \overline{C})$ is Bertrand curve pair and $\mathbf{v}_1 \perp \mathbf{v}_2$, $\mathbf{v}_1 \perp \mathbf{v}_2$ is obtained. Then $\mathbf{v}_1$ can be written $\mathbf{v}_1^i = a \mathbf{v}_1^i + b \mathbf{v}_3^i$ where $a = g_{ij} \mathbf{v}_1^i \mathbf{v}_1^j = \cos \alpha$ and $b = g_{ij} \mathbf{v}_1^i \mathbf{v}_3^j = \sin \alpha$.

$$\mathbf{v}_3^i = \varepsilon_{ijk} \mathbf{v}_1^j \mathbf{v}_2^k = \varepsilon_{ijk} (\mathbf{v}_1^j \cos \alpha + \mathbf{v}_3^j \sin \alpha) \mathbf{v}_2^k$$

$$\mathbf{v}_3^i = \mathbf{v}_3^i \cos \alpha - \mathbf{v}_1^i \sin \alpha.\tag{26}$$

From here, the following equality can be written as:

$$\begin{pmatrix}
\mathbf{v}_1^i \\
\mathbf{v}_2^i \\
\mathbf{v}_3^i
\end{pmatrix}
= 
\begin{pmatrix}
\cos \alpha & 0 & \sin \alpha \\
0 & 1 & 0 \\
-sin \alpha & 0 & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\mathbf{v}_1^i \\
\mathbf{v}_2^i \\
\mathbf{v}_3^i
\end{pmatrix}.	ag{27}$$

**Theorem 3.2** If $(C, \overline{C})$ is a Bertrand curve pair, then there are the following relations between Bishop vector fields of $C$ and $\overline{C}$:

$$\overline{v}_1^i = v_1^i \cos \alpha + n_1^i \sin \alpha \sin \theta - n_2^i \sin \alpha \cos \theta \tag{28}$$

$$\overline{\pi}_1^i = (-\sin \overline{\theta} \sin \alpha) v_1^i + (\cos \overline{\theta} \cos \theta + \sin \overline{\theta} \sin \theta \cos \alpha) n_1^i +$$

$$+(\sin \theta \cos \overline{\theta} - \sin \overline{\theta} \cos \theta \cos \alpha) n_2^i \tag{29}$$

$$\overline{\pi}_2^i = (\cos \overline{\theta} \sin \alpha) v_1^i + (\sin \overline{\theta} \cos \theta - \cos \overline{\theta} \sin \theta \cos \alpha) n_1^i +$$

$$+(\sin \overline{\theta} \sin \theta + \cos \overline{\theta} \cos \theta \cos \alpha) n_2^i.\tag{30}$$

**Proof.** If $(C, \overline{C})$ is Bertrand curve pair, from (14) and (27), we have

$$\begin{pmatrix}
\overline{v}_1^i \\
\overline{v}_2^i \\
\overline{v}_3^i
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \overline{\theta} \sin \overline{\theta} & 0 \\
0 & \sin \overline{\theta} \cos \overline{\theta} & \sin \alpha \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\cos \alpha & 0 & -\sin \alpha \\
0 & 1 & 0 \\
0 & \cos \theta \sin \theta & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{v}_1^i \\
\mathbf{v}_2^i \\
\mathbf{v}_3^i
\end{pmatrix}.	ag{31}$$

where $\overline{\theta} = \angle(\overline{\pi}_1^i, \overline{\pi}_2^i)$. 

Theorem 3.3 If \((C, \overline{C})\) is Bertrand curve pair, the equality \(\cos f(s) = 1 - \lambda K_1\) is satisfied.

Proof. If \((C, \overline{C})\) is Bertrand curve pair, we have \(C(s) = \overline{C}(s) + \lambda \overline{v}^i(s)\) where \(\lambda\) is prolonged covariant constant. Taking prolonged covariant derivative of this equality in the direction of \(\overline{v}^k\), we get

\[
\overline{v}^k \overline{\nabla}_k C = \overline{v}^k \overline{\nabla}_k \overline{C} + \lambda \overline{v}^k \overline{\nabla}_k \overline{v}^i
\]

\[
\overline{v}^i f(s) = \overline{v}^i + \lambda \left( -K_1 \overline{v}^i + K_2 \overline{v}^j \right)
\]

\[
= (1 - \lambda K_1) \overline{v}^i + \lambda K_2 \left( -\sin \theta \overline{n}_1^i + \cos \theta \overline{n}_2^i \right). \tag{32}
\]

Multiplying (32) by \(g_{ij} \overline{v}^j\), we have \(\cos f(s) = 1 - \lambda K_1\) where \(g_{ij} \overline{v}^i\) is obtained where \(g_{ij} \overline{v}^i \overline{v}^j = 1\) and \(g_{ij} \overline{v}^i \overline{v}^j = 0\).

Proof 2.

\[
\overline{v}^i f(s) = \overline{v}^i + \lambda \left( T_{\overline{v}^k 1 1} \overline{v}^i \right)
\]

\[
= \overline{v}^i + \lambda \left( T_{\overline{v}^k 1 1} \overline{v}^i + \frac{3}{2} T_{\overline{v}^k 1 3} \overline{v}^i \right)
\]

\[
= \overline{v}^i + \lambda \left( -T_{\overline{v}^k 1 1} \overline{v}^i + \frac{2}{3} T_{\overline{v}^k 1 3} \overline{v}^i \right)
\]

\[
= \overline{v}^i + \lambda \left( -\overline{v}^i \overline{v}^j - \overline{v}^i \overline{v}^j \right). \tag{33}
\]

Multiplying (33) by \(g_{ij} \overline{v}^j\) and using (27), we get

\[
g_{ij} \overline{v}^j f(s) = g_{ij} \overline{v}^i \overline{v}^j - \lambda \overline{v}^2_{\overline{C}} \overline{v}^i \overline{v}^j
\]

\[
\cos f(s) = 1 - \lambda \overline{v}^2_{\overline{C}} = 1 - K_1. \tag{34}
\]

is obtained where \(g_{ij} \overline{v}^i \overline{v}^j = 1\) and \(g_{ij} \overline{v}^i \overline{v}^j = 0\).

Theorem 3.4 If \((C, \overline{C})\) is Bertrand curve pair, \(\sin f(s) = \lambda K_2\) is valid.

Proof. Multiplying (32) by \(g_{ij} \overline{v}^j\) and using Theorem 3.2, we obtain

\[
g_{ij} \overline{v}^j f(s) = -\lambda K_2 \sin \theta
\]

\[
-\sin \theta \sin \alpha f(s) = -\lambda K_2 \sin \theta
\]

\[
\sin f(s) = \lambda K_2. \tag{35}
\]

Proof 2. Multiplying (33) by \(g_{ij} \overline{v}^j\) and using (27), we get
\[ g_{ij} v^i_1 v^j_3 f(s) = \lambda \left( -\frac{2}{3^i} \right) \lambda K_2 \] (36)

and

\[
\begin{pmatrix}
\cos \alpha & 0 & -\sin \alpha \\
0 & 1 & 0 \\
\sin \alpha & 0 & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
v^i_1 \\
v^i_2 \\
v^i_3
\end{pmatrix}
= \begin{pmatrix}
v^i_1' \\
v^i_2' \\
v^i_3'
\end{pmatrix}
\]
(37)

and

\[ g_{ij} v^i_1 (\sin \alpha v^j_1 + \cos \alpha v^j_3) f(s) = \lambda K_2 \\
\sin \alpha f(s) = \lambda K_2 
\]
(38)

where \( g_{ij} v^i_1 v^j_1 = 0 \), \( g_{ij} v^i_3 v^j_3 = 1 \), \( g_{ij} v^i_1 v^j_1 = 1 \) and \( g_{ij} v^i_1 v^j_3 = 0 \).

References


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