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About Bertrand Curves and Type-1 Bishop Frame in Three-Dimensional Weyl Space W_3

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Abstract

In this paper, we have defined Bertrand curves in three-dimensional Weyl space. Then we have given the relations between the Bishop vector fields of Bertrand curve pair. Finally, while $(C, \overline{C})$ is Bertrand curve pair, we have obtained the equalities depending on $\overline{K}_1$ and $\overline{K}_2$.

Mathematics Subject Classification: 53B25, 53A25

Keywords: Weyl space, type-1, Bishop frame, Bertrand curve

1 Introduction

A manifold with a conformal metric g_{ij} and a symmetric connection ∇_k satisfying the compatibility condition

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \tag{1}$$

is called a Weyl space which will be denoted by $W(g_{ij}, T_k)$. The vector field T_k is named the complementary vector field. Under a normalization of the metric tensor g_{ij} in the form

$$\tilde{g}_{ij} = \lambda^2 g_{ij} \tag{2}$$

the complementary vector field T_k is transformed by the law

$$\tilde{T}_k = T_k + \partial_k ln\lambda \tag{3}$$

where λ is a scalar function [9]. If under the transformation (2), the quantity A is called a satellite of g_{ij} with weight $\{p\}$.
The prolonged derivative and prolonged covariant derivative of A are defined as

$$\dot{\partial}_k = \partial_k A - pT_k A \tag{4}$$

and

$$\dot{\nabla}_k A = \nabla_k A - pT_k A \tag{5}$$

respectively [4], [10]. The $\underset{r}{v}^i$ $(i, r = 1, 2, 3)$ be the contravariant components of the vector field $\underset{r}{v}$ in $W_3(g_{ij}, T_k)$. Suppose that the vector field $\underset{r}{v}$ are normalized by the conditions $g_{ij}\underset{r}{v}^i\underset{r}{v}^j = 1$ $(j, r = 1, 2, 3)$.
The prolonged covariant derivative of the vector field $\underset{r}{v}$ is given by [14]

$$\dot{\nabla}_k \underset{r}{v}^i = \overset{s}{\underset{r}{T_k}} \underset{s}{v}^i \ (s = 1, 2, 3). \tag{6}$$

The quantities

$$\overset{q}{\underset{rs}{\tau}} = \overset{q}{\underset{r}{T_k}} \underset{s}{v}^k \ (q = 1, 2, 3; r \neq s) \tag{7}$$

and

$$\overset{r}{\underset{s}{\text{Ƨ}}} = \overset{r}{\underset{s}{T_k}} \underset{s}{v}^k \tag{8}$$

are called the Chebyshev curvature of the first kind and geodesic curvature of the net (v_1, v_2, v_3), respectively [14].
The vector fields

$$\underset{rs}{a}^i = \overset{q}{\underset{rs}{\tau}} \underset{q}{v}^i \ , \ \underset{s}{c}^i = \overset{r}{\underset{s}{\text{Ƨ}}} \underset{r}{v}^i \ (i, q, r, s = 1, 2, 3) \tag{9}$$

are called the Chebyshev vector fields of the first kind and geodesic vector fields of the net (v_1, v_2, v_3), respectively [14].
Since the net (v_1, v_2, v_3) is an orthogonal net, we have [14]

$$\overset{r}{\underset{r}{T_k}} = 0 \ , \ \overset{p}{\underset{r}{T_k}} + \overset{r}{\underset{p}{T_k}} = 0 \ (r \neq p). \tag{10}$$

Bertrand curves have been determined by J. Bertrand [1] in 1850. Later, Bertrand curves have been studied by L.R. Pears [13] in 1935, by J.K. Wittemore [16] in 1940 and J.F. Burke [2] in 1960, respectively. Besides the properties of these curves provided in various studies [5], they have been handled in different spaces such as Riemann-Otsuki space [17], Galilean space [11], three-dimensional sphere [7], three-dimensional space forms [3], Euclidean 3-space [8] and Minkowski space-time[15].

2 Preliminaries

Let $C\ :\ x^i = x^i(s)$ be a curve in three-dimensional Weyl space W_3 (s is the arc length parameter of C). Let $\{\underset{1}{v}, \underset{2}{v}, \underset{3}{v}\}$ be Frenet frame and $\{\underset{1}{v}, \underset{1}{n}, \underset{2}{n}\}$ be Bishop frame of the curve C such that K_1, K_2 are the first and second curvatures and k_1, k_2 are the Bishop curvatures of C. The Frenet and Bishop formulas [6] of C are

$$\begin{aligned} \underset{1}{v}^k\ \dot{\nabla}_k\ \underset{1}{v}^i =& K_1\ \underset{2}{v}^i, \\ \underset{1}{v}^k\ \dot{\nabla}_k\ \underset{2}{v}^i =& -K_1\ \underset{1}{v}^i + K_2\ \underset{3}{v}^i \\ \underset{1}{v}^k\ \dot{\nabla}_k\ \underset{3}{v}^i =& -K_2\ \underset{2}{v}^i, \end{aligned} \tag{11}$$

and

$$\begin{aligned} \underset{1}{v}^k\ \dot{\nabla}_k\ \underset{1}{v}^i =& k_1\ \underset{1}{v}^i + k_2\ \underset{2}{v}^i, \\ \underset{1}{v}^k\ \dot{\nabla}_k\ \underset{1}{n}^i =& -k_1\ \underset{1}{v}^i, \\ \underset{1}{v}^k\ \dot{\nabla}_k\ \underset{2}{n}^i =& -k_2\ \underset{1}{v}^i. \end{aligned} \tag{12}$$

Since $\underset{2}{v}^i$ is orthogonal to $\underset{1}{v}$, $\underset{2}{v}^i$ can be written as $\underset{2}{v}^i = a\underset{1}{n}^i + b\underset{2}{n}^i$ [6] where $a = g_{ij}\ \underset{2}{v}^i\ \underset{1}{n}^j = cos\theta,\ b = g_{ij}\ \underset{2}{v}^i\ \underset{2}{n}^j = cos(\frac{\pi}{2} - \theta) = sin\theta$ and $\theta = \measuredangle\big(\underset{2}{v}, \underset{1}{n}\big)$. In addition, since $\underset{3}{v}^i = \varepsilon_{ijk}\ \underset{1}{v}^j\ \underset{2}{v}^k$ ($k = 1, 2, 3$), the following equality is satisfied:

$$\begin{aligned} \underset{3}{v}^i =& \varepsilon_{ijk}\ \underset{1}{v}^j (cos\theta\underset{1}{n}^k + sin\theta\underset{2}{n}^k), \\ \underset{3}{v}^i =& \underset{2}{n}^i\ cos\theta - \underset{1}{n}^i\ sin\theta. \end{aligned} \tag{13}$$

Therefore

$$\begin{pmatrix} \underset{1}{v}^i \\ \underset{2}{v}^i \\ \underset{3}{v}^i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} \underset{1}{v}^i \\ \underset{1}{n}^i \\ \underset{2}{n}^i \end{pmatrix} \tag{14}$$

is valid.
From (11), $\underset{1}{v}^k\,\dot{\nabla}_k\,\underset{1}{v}^i = K_1\,\underset{2}{v}^i$. Multiplying this equality by $g_{ij}\,\underset{2}{v}^j$ and taking summation on i and j, we get [6]

$$
\begin{aligned}
K_1 =& \underset{1}{v}^k\,\big(\dot{\nabla}_k\,\underset{1}{v}^i\big) g_{ij}\,\underset{2}{v}^j,\\
K_1 =& \underset{1}{\overset{p}{T}}_k\,\underset{1}{v}^k\,\underset{p}{v}^i\,g_{ij}\,\underset{2}{v}^j,\\
K_1 =& g_{ij}\,\underset{1}{c}^i\,\underset{2}{v}^j,
\end{aligned}
\tag{15}
$$

or

$$
K_1 = \overset{2}{\underset{1}{\varsigma}} \tag{16}
$$

where $g_{ij}\,\underset{2}{v}^i\,\underset{2}{v}^j = 1$, $g_{ij}\,\underset{3}{v}^i\,\underset{2}{v}^j = 0$ and $\underset{1}{\overset{1}{T}}_k = 0$.

Theorem 2.1 *If $K_1 = 0$, then the geodesic vector field $\underset{1}{c}^i$ of the net (v_1, v_2, v_3) is orthogonal to $\underset{2}{v}$.*

From (11), $\underset{1}{v}^k\,\dot{\nabla}_k\,\underset{3}{v}^i = -K_2\,\underset{2}{v}^i$. Multiplying this relation by $g_{ij}\,\underset{2}{v}^j$ and summing on i and j, we obtain [6]

$$
\begin{aligned}
-K_2 =& \underset{1}{v}^k\,\big(\dot{\nabla}_k\,\underset{3}{v}^i\big) g_{ij}\,\underset{2}{v}^j,\\
-K_2 =& \underset{3}{\overset{p}{T}}_k\,\underset{1}{v}^k\,\underset{p}{v}^i\,g_{ij}\,\underset{2}{v}^j,\\
K_2 =& -g_{ij}\,\underset{31}{a}^i\,\underset{2}{v}^j,
\end{aligned}
\tag{17}
$$

or

$$
K_2 = -\overset{2}{\underset{31}{\tau}} \tag{18}
$$

where $g_{ij}\,\underset{1}{v}^i\,\underset{2}{v}^j = 0$, $g_{ij}\,\underset{2}{v}^i\,\underset{2}{v}^j = 1$ and $\underset{3}{\overset{3}{T}}_k = 0$.

Theorem 2.2 *If $K_2 = 0$, then the Chebyshev vector field of the first kind $\underset{31}{a}^i$ of the net (v_1, v_2, v_3) is orthogonal to $\underset{2}{v}$.*

Using $\underset{1}{v}^k\,\dot{\nabla}_k\,\underset{3}{v}^i = -K_2\,\underset{2}{v}^i$ and [13] and then multiplying obtained equation by $g_{ij}\,\underset{2}{v}^j$, we have $K_2 = \underset{1}{v}^k\,\dot{\nabla}_k\,\theta\ (\theta = \theta(s))$ where $g_{ij}\,\underset{2}{n}^i\,\underset{2}{v}^j = sin\theta$, $g_{ij}\,\underset{1}{n}^i\,\underset{2}{v}^j = cos\theta$ and $g_{ij}\,\underset{1}{v}^i\,\underset{2}{v}^j = 0$. Multiplying $\underset{1}{v}^k\,\dot{\nabla}_k\,\underset{1}{v}^i = k_1\,\underset{1}{n}^i + k_2\,\underset{2}{n}^i = K_1\,\underset{2}{v}^i$ by $g_{ij}\,\underset{1}{n}^j$ we get

$$\big(\underset{1}{v}^k\ \dot{\nabla}_k\ \underset{1}{v}^i\big) g_{ij}\ \underset{1}{n}^j = k_1 = K_1\ cos\theta \tag{19}$$

and multiplying $\underset{1}{v}^k\ \dot{\nabla}_k\ \underset{1}{v}^i = k_1\ \underset{1}{n}^i + k_2\ \underset{2}{n}^i = K_1\ \underset{2}{v}^i$ *by* $g_{ij}\ \underset{2}{n}^j$ *we have*

$$\big(\underset{1}{v}^k\ \dot{\nabla}_k\ \underset{1}{v}^i\big) g_{ij}\ \underset{2}{n}^j = k_2 = K_1\ sin\theta. \tag{20}$$

From (19) and (20), we obtain $k_1^2 + k_2^2 = K_1^2$.

Theorem 2.3 *If* $k_1 = 0$*, then the geodesic vector field* $\underset{1}{c}^i$ *of the net* (v_1, v_2, v_3) *is orthogonal to* n_1*.*

Theorem 2.4 *If* $k_2 = 0$*, then the geodesic vector field* $\underset{1}{c}^i$ *of the net* (v_1, v_2, v_3) *is orthogonal to* n_2*.*

3 Bertrand Curves in W_3

Let $\overline{C} : \overline{x}^i = \overline{x}^i(\overline{s})$ be other curve in W_3 ($\overline{s}$ is the arc length parameter of $\overline{C}$). Let us denote Frenet and Bishop components of $\overline{C}$ by $\{\underset{1}{\overline{v}}, \underset{2}{\overline{v}}, \underset{3}{\overline{v}}, \overline{K}_1, \overline{K}_2\}$ and $\{\underset{1}{\overline{v}}, \underset{1}{\overline{n}}, \underset{2}{\overline{n}}, \overline{k}_1, \overline{k}_2\}$, respectively.

Definition 3.1 *If the principal normal vector fields of the curves* C *and* $\overline{C}$ *are linear dependent, the curve pair* $(C, \overline{C})$ *is called Bertrand curve pair.*

If the curve pair $(C, \overline{C})$ is Bertrand curve pair the following equality is satisfied:

$$C(s) = \overline{C}(\overline{s}) + \lambda(\overline{s})\ \underset{2}{\overline{v}}^i(\overline{s}). \tag{21}$$

Taking prolonged covariant derivative of (21) in the direction of $\underset{1}{\overline{v}}$ we have

$$\begin{aligned} \underset{1}{\overline{v}}^k\ \dot{\overline{\nabla}}_k\ C =& \big(\underset{1}{v}^k\ \dot{\nabla}_k\ C\big)\ f(s) = \underset{1}{v}^i\ f(s) = \\ =& \underset{1}{\overline{v}}^i + \big(\underset{1}{\overline{v}}^k\ \dot{\overline{\nabla}}_k\ \lambda\big)\underset{2}{\overline{v}}^i + \lambda\big(-\overline{K}_1\underset{1}{\overline{v}}^i + \overline{K}_2\underset{3}{\overline{v}}^i\big) \end{aligned} \tag{22}$$

and multiplying (22) by $g_{ij}\ \underset{2}{\overline{v}}^j$, we find

$$f(s)\ g_{ij}\ \underset{1}{v}^i\ \underset{2}{\overline{v}}^j = f(s)\ g_{ij}\ \underset{1}{v}^i\ \underset{2}{v}^j = \underset{1}{\overline{v}}^k\ \dot{\overline{\nabla}}_k \lambda \tag{23}$$

or

$$0 = \underset{1}{\overline{v}}^k\ \dot{\overline{\nabla}}_k\ \lambda \tag{24}$$

where $g_{ij}\ \underset{1}{v}^i\ \underset{2}{v}^j = 0\,,\ \ g_{ij}\ \underset{2}{\overline{v}}^i\ \underset{2}{\overline{v}}^j = 1\ \ , g_{ij}\ \underset{1}{\overline{v}}^i\ \underset{2}{\overline{v}}^j = 0$ and $g_{ij}\ \underset{3}{\overline{v}}^i\ \underset{2}{\overline{v}}^j = 0$.

From (24), we get λ is prolonged covariant constant [12]. On the other hand $f(s) = \pm\sqrt{(1-\lambda\overline{K}_1)^2 + \lambda^2\overline{K}_2^2}$.
Let the angle α be between the tangent vector fields $\underset{1}{v}$ and $\underset{1}{\overline{v}}$ of Bertrand curve pair $(C,\overline{C})$. Since $(C,\overline{C})$ is Bertrand curve pair and $\underset{1}{v} \perp \underset{2}{v}$, $\underset{1}{v} \perp \underset{2}{\overline{v}}$ is obtained. Then $\underset{1}{v}$ can be written $\underset{1}{v^i} = a\,\underset{1}{\overline{v}^i} + b\,\underset{3}{\overline{v}^i}$ where $a = g_{ij}\,\underset{1}{v^i}\,\underset{1}{\overline{v}^j} = cos\alpha$ and $b = g_{ij}\,\underset{1}{v^i}\,\underset{3}{\overline{v}^j} = sin\alpha$.

$$\underset{3}{v^i} = \varepsilon_{ijk}\,\underset{1}{v^j}\,\underset{2}{v^k} = \varepsilon_{ijk}\,(\underset{1}{\overline{v}^j}\,cos\alpha + \underset{3}{\overline{v}^j}\,sin\alpha)\underset{2}{\overline{v}^k} \tag{25}$$

$$\underset{3}{v^i} = \underset{3}{\overline{v}^i}\,cos\alpha - \underset{1}{\overline{v}^i}\,sin\alpha. \tag{26}$$

From here, the following equality can be written as:

$$\begin{pmatrix} \underset{1}{v^i} \\ \underset{2}{v^i} \\ \underset{3}{v^i} \end{pmatrix} = \begin{pmatrix} cos\alpha & 0 & sin\alpha \\ 0 & 1 & 0 \\ -sin\alpha & 0 & cos\alpha \end{pmatrix} \begin{pmatrix} \underset{1}{\overline{v}^i} \\ \underset{2}{\overline{v}^i} \\ \underset{3}{\overline{v}^i} \end{pmatrix}. \tag{27}$$

Theorem 3.2 *If $(C,\overline{C})$ is a Bertrand curve pair, then there are the following relations between Bishop vector fields of C and $\overline{C}$:*

$$\underset{1}{\overline{v}^i} = \underset{1}{v^i}\,cos\alpha + \underset{1}{n^i}\,sin\alpha\,sin\theta - \underset{2}{n^i}\,sin\alpha\,cos\theta \tag{28}$$

$$\begin{aligned} \underset{1}{\overline{n}^i} =& (-sin\overline{\theta}\,sin\alpha)\underset{1}{v^i} + (cos\overline{\theta}\,cos\theta + sin\overline{\theta}\,sin\theta\,cos\alpha)\underset{1}{n^i} + \\ &+ (sin\theta\,cos\overline{\theta} - sin\overline{\theta}\,cos\theta\,cos\alpha)\underset{2}{n^i} \end{aligned} \tag{29}$$

$$\begin{aligned} \underset{2}{\overline{n}^i} =& (cos\overline{\theta}\,sin\alpha)\underset{1}{v^i} + (sin\overline{\theta}\,cos\theta - cos\overline{\theta}\,sin\theta\,cos\alpha)\underset{1}{n^i} + \\ &+ (sin\overline{\theta}\,sin\theta + cos\overline{\theta}\,cos\theta\,cos\alpha)\underset{2}{n^i}. \end{aligned} \tag{30}$$

Proof. *If $(C,\overline{C})$ is Bertrand curve pair, from (14) and (27), we have*

$$\begin{pmatrix} \underset{1}{\overline{v}^i} \\ \underset{1}{\overline{n}^i} \\ \underset{2}{\overline{n}^i} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\overline{\theta} & sin\overline{\theta} \\ 0 & sin\overline{\theta} & cos\overline{\theta} \end{pmatrix} \begin{pmatrix} cos\alpha & 0 & -sin\alpha \\ 0 & 1 & 0 \\ sin\alpha & 0 & cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} \underset{1}{\overline{v}^i} \\ \underset{1}{\overline{n}^i} \\ \underset{2}{\overline{n}^i} \end{pmatrix} \tag{31}$$

where $\overline{\theta} = \measuredangle(\underset{2}{\overline{v}^i}, \underset{1}{\overline{n}^i})$.

Theorem 3.3 *If* $(C, \overline{C})$ *is Bertrand curve pair, the equality* $cos\alpha f(s) = 1 - \lambda \overline{K}_1$ *is satisfied.*
Proof. *If* $(C, \overline{C})$ *is Bertrand curve pair, we have* $C(s) = \overline{C}(\overline{s}) + \lambda \underset{2}{\overline{v}}^i(\overline{s})$ *where* λ *is prolonged covariant constant. Taking prolonged covariant derivative of this equality in the direction of* $\underset{1}{\overline{v}}^k$*, we get*

$$
\begin{aligned}
\underset{1}{\overline{v}}^k \, \dot{\overline{\nabla}}_k C =& \big(\underset{1}{v}^k \, \dot{\nabla}_k \, C\big) f(s) = \underset{1}{\overline{v}}^k \, \dot{\overline{\nabla}}_k \, \overline{C} + \lambda \, \underset{1}{\overline{v}}^k \, \dot{\overline{\nabla}}_k \, \underset{2}{\overline{v}}^i \\
\underset{1}{v}^i \, f(s) =& \underset{1}{\overline{v}}^i + \lambda\big(- \overline{K}_1 \, \underset{1}{\overline{v}}^i + \overline{K}_2 \, \underset{3}{\overline{v}}^i\big) \\
=& \big(1 - \lambda \overline{K}_1\big) \underset{1}{\overline{v}}^i + \lambda \overline{K}_2 \big(- sin\overline{\theta} \, \underset{1}{\overline{n}}^i + cos\overline{\theta} \, \underset{2}{\overline{n}}^i\big).
\end{aligned}
\tag{32}
$$

Multiplying (32) by $g_{ij} \, \underset{1}{\overline{v}}^j$*, we have* $cos\alpha f(s) = 1 - \lambda \overline{K}_1$ *where* $g_{ij} \, \underset{1}{v}^i \, \underset{1}{\overline{v}}^j = cos\alpha\big(\alpha = \measuredangle\big(\underset{1}{v}^i, \underset{1}{\overline{v}}^i\big)\big)$, $g_{ij} \, \underset{1}{\overline{n}}^i \, \underset{1}{\overline{v}}^j = g_{ij} \, \underset{2}{\overline{n}}^i \, \underset{1}{\overline{v}}^j = 0$.
Proof 2.

$$
\begin{aligned}
\underset{1}{v}^i \, f(s) =& \underset{1}{\overline{v}}^i + \lambda \, \overset{p}{\underset{2}{\overline{T}}}_k \, \underset{1}{\overline{v}}^k \, \underset{p}{\overline{v}}^i \\
=& \underset{1}{\overline{v}}^i + \lambda\big(\overset{1}{\underset{2}{\overline{T}}}_k \, \underset{1}{\overline{v}}^k \, \underset{1}{\overline{v}}^i + \overset{3}{\underset{2}{\overline{T}}}_k \, \underset{1}{\overline{v}}^k \, \underset{3}{\overline{v}}^i\big) \\
=& \underset{1}{\overline{v}}^i + \lambda\big(- \overset{2}{\underset{1}{\overline{T}}}_k \, \underset{1}{\overline{v}}^k \, \underset{1}{\overline{v}}^i + \overset{2}{\underset{3}{\overline{T}}}_k \, \underset{1}{\overline{v}}^k \, \underset{3}{\overline{v}}^i\big) \\
=& \underset{1}{\overline{v}}^i + \lambda\big(- \overset{2}{\underset{1}{\overline{\text{Ƨ}}}} \, \underset{1}{\overline{v}}^i - \overset{2}{\underset{31}{\overline{\tau}}} \, \underset{3}{\overline{v}}^i\big).
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
g_{ij} \, \underset{1}{v}^i \, \underset{1}{\overline{v}}^j \, f(s) =& g_{ij} \, \underset{1}{\overline{v}}^i \, \underset{1}{\overline{v}}^j - \lambda \, \overset{2}{\underset{1}{\overline{\text{Ƨ}}}} \, g_{ij} \, \underset{1}{\overline{v}}^i \, \underset{1}{\overline{v}}^j \\
cos\alpha \, f(s) =& 1 - \lambda \, \overset{2}{\underset{1}{\overline{\text{Ƨ}}}} = 1 - \overline{K}_1.
\end{aligned}
\tag{34}
$$

is obtained where $g_{ij} \, \underset{1}{\overline{v}}^i \, \underset{1}{\overline{v}}^j = 1$ *and* $g_{ij} \, \underset{3}{\overline{v}}^i \, \underset{1}{\overline{v}}^j = 0$

Theorem 3.4 *If* $(C, \overline{C})$ *is Bertrand curve pair,* $sin\alpha f(s) = \lambda \overline{K}_2$ *is valid.*
Proof. Multiplying (32) by $g_{ij} \, \underset{1}{\overline{n}}^j$ *and using Theorem 3.2, we obtain*

$$
\begin{aligned}
g_{ij} \, \underset{1}{v}^i \, \underset{1}{\overline{n}}^j \, f(s) =& - \lambda \overline{K}_2 sin\overline{\theta} \\
-sin\overline{\theta} \, sin\alpha \, f(s) =& - \lambda \overline{K}_2 sin\overline{\theta} \\
sin\alpha \, f(s) =& \lambda \overline{K}_2.
\end{aligned}
\tag{35}
$$

Proof 2. Multiplying (33) by $g_{ij} \, \underset{3}{\overline{v}}^j$ *and using (27), we get*

$$g_{ij}\underset{1}{v^i}\,\underset{3}{\overline{v}^j}\,f(s)=\lambda\left(-\frac{2}{\underset{31}{T}}\right)=\lambda\overline{K}_2 \tag{36}$$

and

$$\begin{pmatrix} cos\alpha & 0 & -sin\alpha \\ 0 & 1 & 0 \\ sin\alpha & 0 & cos\alpha \end{pmatrix}\begin{pmatrix} \underset{1}{v^i} \\ \underset{2}{v^i} \\ \underset{3}{v^i} \end{pmatrix}=\begin{pmatrix} \underset{1}{\overline{v}^i} \\ \underset{2}{\overline{v}^i} \\ \underset{3}{\overline{v}^i} \end{pmatrix} \tag{37}$$

and

$$\begin{aligned} g_{ij}\underset{1}{v^i}\left(sin\alpha\,\underset{1}{v^j}+cos\alpha\underset{3}{v^j}\right)f(s)&=\lambda\overline{K}_2 \\ sin\alpha\,f(s)&=\lambda\overline{K}_2 \end{aligned} \tag{38}$$

where $g_{ij}\,\underset{1}{\overline{v}^i}\,\underset{3}{\overline{v}^j}=0$, $g_{ij}\,\underset{3}{\overline{v}^i}\,\underset{3}{\overline{v}^j}=1$, $g_{ij}\,\underset{1}{v^i}\,\underset{1}{v^j}=1$ *and* $g_{ij}\,\underset{1}{v^i}\,\underset{3}{v^j}=0.$

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