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Strongly Pure Ideals

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Abstract

Let R be a ring then the ideal I known as right (left) strongly pure ideal if , for every $x \in I$ there exist a prime element $p \in I$ such that $x = x.p(x = p.x)$. And several properties of this class of ideals are discussed . Finally we presented another definition that is , Let R be a ring then I is called right (left) strongly pure ideal of prime elements if , for every prime element $p_i \in I$ there exist a prime element $p \in I$ such that $p_i = p_i.p(p_i = p.p_i)$, with some important results.

Keywords: right (left) strongly pure ideal, right (left) strongly pure ideal of prime elements

1. Introduction

In this article , R be a ring with identity. We write $\cap$, $\cup$, $\oplus$ for the Intersection , Union and Direct sum, respectively . A ring R is cycle ring iff R can be written as $R = (a) = \{a^n : n \in Z\}$ **[5]** .

Generalization of pure ideal have been discussed in many papers (see **[1]** , **[2] , [3]** , **[4]**) , . In (**[1]** defined right (left) strongly pure ideals (for short SP-Ideal).The nice structure of SP-Ideal draws our attention to study some properties of this kind of ideals and to define right (left) SP-Ideal of prime elements .

As usual , $I \in R$ is called idempotent if $I^2 = I$ **[6]** . A non-empty set $I \in R$ is ideal if , for every $x \in I$, $y \in R$ implies that $x.y \in I \ \wedge \ y.x \in I)$.

2. SP-Ideals

Definition 2.1: An ideal $I \in R$ is called right (left) pure ideal if , for every $x \in I$ there exist $y \in I$ such that $x = x.y(x = y.x)$. **[1]** , **[4]**

Definition 2.2: Let R be a ring then I is called right (left) SP-Ideal if, for every $x \in I$ there exist a prime element $p \in I$ such that $x = x.p(x = p.x)$.

Note: Every right (left) SP-Ideal is right (left) pure ideal, but the converse is not true .

Example: In the rings $\mathcal{Z}_6$ an ideal $I = \{0 , 2 , 4\}$ is right (left) pure ideal but it is not right (left) SP-Ideal because there is no a prime element $p \in I$ such that $x = x.p(x = p.x)$.

Note: Not every right SP-Ideal is left SP-Ideal, and never the converse.

Example: In the rings $R = \left\{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} , a, b, d \in z_2\right\}$ an ideals $I = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right\}$
and $J = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right\}$

So I is left SP-Ideal but it is not right SP-Ideal because there is no a prime element $p \in I$ such that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot P$.

And J is right SP-Ideal but it is not left SP-Ideal because there is no a prime element $p \in J$ such that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = P \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Lemma 2.3: Let I be a SP-Ideal of R . Then $J.I = J \cap I$, for every ideal J of R. **[1]**

Lemma 2.4: every idempotent ideal generated by prime element is SP-Ideal. And every SP-Ideal is idempotent ideal . **[1]**

Theorem 2.5: If $R = Z_n$, $I = (p), J = (k)$, I and J are two deferent SP-Ideals in R then either $I = R$ or $J = R$. where p , k are two prime elements.

Proof: Let $I \neq R$, since I and J are two deferent SP-Ideals in R , then $p.k \in I \wedge p.k \in J$, so we have three cases either $k = 1$ implies that , $J = (k) = R$. Or $k < p \Longrightarrow k \cdot k < n \Longrightarrow k \cdot k \neq k$ (contradiction with

J is SP-Ideal) . Or $p < k \Longrightarrow p \cdot p < n \Longrightarrow p \cdot p \neq p$ (contradiction with I is SP-Ideal) . Therefore $J = R$. By the same method we can proof that $I = R$ if $J \neq R$.

Note: (This suggest in general is not true) let I , J ideals in $R = Z_n$ and I SP-Ideal in R then $I \cap J$ SP-Ideal in R . See the next example

Example: Let $I = (3) = \{0,3\}$, $J = \{0\}$ such that I , J ideals in Z_6, and I SP-Ideal in Z_6 . But $I \cap J = \{0\}$ is not SP-Ideal in Z_6 .

Theorem 2.6: If I and J are SP-Ideals in $R = Z_n$ then $I \cap J$, $I \cup J$ and $I \oplus J$ are SP-Ideals in $R = Z_n$.

Proof: by **Theorem 2.5** then either $I = R$ or $J = R$

1 - $I \cap J$ equal to either I or J , so $I \cap J$ is SP-Ideal .

2 - $I \cup J$ equal to R , so $I \cup J$ is SP-Ideal.

3 - $I \oplus J$ equal to R , so $I \oplus J$ is SP-Ideal.

but the converse is not true see the next examples:

Examples:

(1) $I = (3) = \{0,3\}$, $J = \{0,2,3\} \Longrightarrow I \cap J = \{0,3\}$. Then $I \cap J$ SP-Ideal in Z_6 . But J is not ideal in Jin Z_6 .

(2) $I = (5) = \{0,5,10,15\}$, $J = (10) = \{0,10\}$in Z_{20} . Then $I \cup J = \{0,5,10,15\}$ is SP-Ideal in Z_{20} . But J is not SP-Ideal in Z_{20} , because there is no prime element $P \in J$such that $10 = 10 \cdot p$.

(3) $I = (2) = \{0,2,4\}$, $J = (3) = \{0,3\}$ in Z_6 .Then $I \oplus J = Z_6$ is SP-Ideal in Z_6. But I is not SP-Ideal in Z_6 .

Note: (This suggest in general is not true) let I , J ideals in $R = Z_n$ and $I \oplus J$ SP-Ideal in $R = Z_n$ either I or J SP-Ideal in R : See the next example

Example: Let $I = (3) = \{0,3,6,9\}$, $J = (4) = \{0,4,8\}$ in Z_{12} , then $I \oplus J = Z_{12}$is SP-Ideal in Z_{12}. But I is not SP-Ideal in Z_{12}. And J is not SP-Ideal in Z_{12}.

Corollary 2.7: let I , J ideals in $R = Z_n$ and $I \oplus J$ SP-Ideal in $R = Z_n$ either I or J SP-Ideal in R if $I \subseteq J$ or $J \subseteq I$.

Proof: Let $j \subseteq J$ implies that $j \in I \oplus J$, since $I \oplus J$ SP-Ideal in $R = Z_n$, so there exist a prime element $p \in I \oplus J$, so $j = j.p$, since $I \subseteq J$ so $p \in I \oplus J = J$ so J is SP-Ideal in R . by the same way can we proof I is SP-Ideal in R .

3 – SP-Ideals of prime element

Definition 3.1: Let I any ideal of R we said I is right (left) SP-Ideal of prime element on a ring R if for every $P_i \in I$ such that P_i is prime element , there exist a prime element $p \in I$ such that $P_i = P_i \cdot P$ $(P_i = P \cdot P_i)$

Proposition 3.2: Every SP-Ideal is SP-Ideal of prime element .

Proof: let $I = (p)$ SP-Ideal in $R = z_n$. So either $I = R$ implies that for every prime element $P_i \in I$ there exist $1 \in I$ such that $P_i = P_i \cdot 1$, therefore I SP-Ideal of prime element in R . Or $I \subseteq R \Longrightarrow I = (p) = \{0, p, 2p, ...\}$ Implies that $p = p \cdot p$ (because I SP-Ideal in R) , so I is SP-Ideal of prime element in R .

But the converse is not true (See the next Example and the next Proposition)

Example: Let $J = (4) = \{0,4,8\}$ in Z_{12} , then J is SP-Ideal of prime element in Z_{12}. But J is not SP-Ideal in Z_{12}.

Proposition 3.3: The SP-Ideal of prime element I in $R = z_n$ is SP-Ideal If there exist a prime element $p \in I$.

Proof: let $P_i \in I$ then there exist $p \in I$, such that $P_i = P_i \cdot p$ Either $I = z_n$, so I SP-Ideal in z_n Or $I \subseteq R \Longrightarrow I = (p) = \{0, p, 2p, \cdots\}$ $\Longrightarrow P = P \cdot P \Longrightarrow 2P = 2P \cdot P$. If we continuo in this way we get I is SP-Ideal .

Proposition 3.4: Let I SP-Ideal of prime element in $R = z_n$. Then $J \cdot I = J \cap I$, for all J ideal in $R = z_n$

Proof : by Implies that ISP-Ideal in R . By Implies that $J \cdot I = J \cap I$

Proposition 3.5: Idempotent ideal generated by prime element is SP-Ideal of prime element. .

Proof: By **Lemma 2.4** every idempotent ideal generated by prime element is SP-Ideal . By **Proposition 3.2** implies every idempotent ideal generated by prime element is SP-Ideal of prime element

Proposition 3.6: Every SP-Ideal of prime element is idempotent ideal If there exist a prime element $p \in I$.

Proof: By **Proposition 3.3** the SP-Ideal of prime element I in $R = z_n$ is SP-Ideal If there exist a prime element $p \in I$. By **Lemma 2.4** every SP-Ideal of prime element is idempotent ideal.

Note: Not every left SP-Ideal of prime element in R , is right SP-Ideal of prime element in R . and the converse is true : see example

Example: Let $R = \left\{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, a, b, d \in z_2\right\}$

$$I = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right\}$$

$$J = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right\}$$

Then I is left SP-Ideal of prime element in R , but it is not right SP-Ideal of prime element in R , because the prime element $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in I$ but there is no prime element $p \in I$, such that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot P$. And J is right SP-Ideal of prime element in R but it is not left SP-Ideal of prime element in R , because the prime element $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in J$ but there is no prime element $p \in J$ such that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = P \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Theorem 3.7: Let I , J are SP-Ideals of prime element in $R = Z_n$ such that I , J contains prime elements

(1) $I \cap J$ is SP-Ideal of prime element in $R = Z_n$
(2) $I \cup J$ is SP-Ideal of prime element in $R = Z_n$
(3) $I \oplus J$ is SP-Ideal of prime element in $R = Z_n$

Proof: By **Proposition 3.2 ,** implies that I , J SP-Ideal in $R = Z_n$. So we can proof (1) , (2) , (3) by **Theorem 2.6 (1) , (2) , (3)** respectively.

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