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Study of Errors in Integration from APOS Theory

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Abstract

The research aims to detect and analyze the errors made by students of first university semesters, during the process of understanding the methods of integration for the calculation of antiderivatives and to determine their possible causes. This fact is studied by means of the development of the mental constructs called: Action, Process, Object and Scheme in the student's mind when constructing his own mathematical knowledge, contextualized from the APOS theory. The methodology adopts a mixed approach, which seeks through the use of the genetic decomposition of the indefinite integral and adopting a defined error classification framework, to analyze the development of workshops, class activities and interviews, in order to characterize these errors and their possible relationship with the level of development of the indefinite integral scheme. The results indicate that some of the difficulties detected in the realization of indefinite integrals are mainly due to failures classified as slips or misconceptions of the algebraic, trigonometric, functional, or differential processes, and with this possibly being able to infer the current level of development of the scheme of the indefinite integral in the students.

Keywords: APOS Theory, Antiderivatives, Difficulties, Errors, Misconceptions

1 Introduction

Mathematics, mainly Calculus, poses many difficulties for undergraduate students especially in the first semesters of college. On this topic there are multiple researches where the possible causes and consequences of these difficulties are addressed, which can be social, personal or academic, mentioned for example: Analysis of students' difficulties in solving integration problems [4], Handling pupils' misconceptions [5], Critical thinking skills: error identifications on students' with APOS theory [10], Students' understanding of integration [6], The analysis of the meanings of the antiderivative used by students of the first engineering courses [7], among others.

This fact, together with the traditional mathematical difficulties with which students arrive at the Pedagogical and Technological University of Colombia - UPTC, leads us to think that it is necessary to study in depth how students learn the subject of Integral Calculus, what difficulties they have before and during the process of acquiring the mathematical tools necessary to calculate antiderivatives, what mistakes they make, what could be changed or improved, and with this to know how students learn the methods of integration in this subject and if possible, to offer complementary alternatives to the usual ones developed in traditional textbooks.

For this reason, we sought to detect some ways in which the learning of calculus occurs in engineering students, since the acceptable application of mathematical concepts depends on how the student has mentally constructed these concepts. For this study we have used the Action-Process-Object-Scheme (APOS) theory [1] as an approach to analyze the mental constructions made by students during the integration process and to detect the difficulties classified as slips or misconceptions according to the adapted error classification framework and thus propose strategies to help overcome these failures.

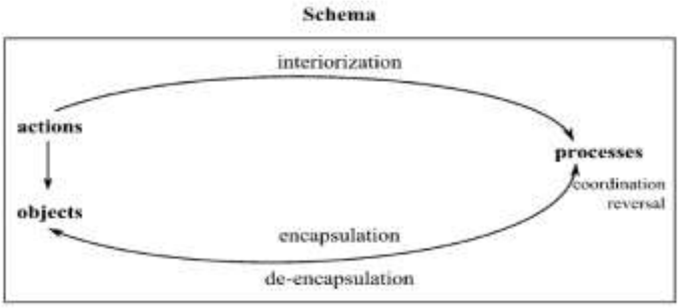
The APOS theory

In the development of this research, we have used the theoretical and analytical tools provided by the APOS theory (Action, Process, Object and Scheme) developed by Dubinsky et al. in [1]. This theory is the result of the interpretation of Piaget's radical constructivism referring to reflective abstraction and applied to the investigation of Advanced Mathematical Thinking trying to study and model the way in which a student learns mathematics, but also, how mathematics can be taught more effectively.

To interpret this theory, it is necessary to mention that the principle of reflective abstraction was considered by Piaget as the main mechanism for any mental construction, through which any mathematical logical structure can develop inside the mind of an individual [1]. On this basis, the APOS theory aims to describe the path and construction of the cognitive, logical, and mathematical structures in the student's mind during the learning process of a specific mathematical concept.

Specifically, the APOS theory states that, to achieve understanding of a particular mathematical concept, a student must go through the mental constructs called Action, Process, Object and Schema, through the mechanisms of internalization, encapsulation, de-encapsulation, reversal, coordination, generalization [1] as illustrated in the following diagram that summarizes the relationship between schemas, structures and the mental mechanisms that generate it:

Figure 1: *Relationship between structures and mechanisms in APOS theory*



Source: Arnon et al. [1]

Therefore, for a student to construct a mathematical concept within himself, he must begin with the manipulation of physical or mental objects, previously constructed, in terms of Actions that are Internalized to form Processes that are Encapsulated to form Objects. In relation to the Processes, these can be generated from the: Coordination, Reversion or Generalization of other Processes previously constructed by the student. Finally, Actions, Processes and Objects can be organized in Schemes [1].

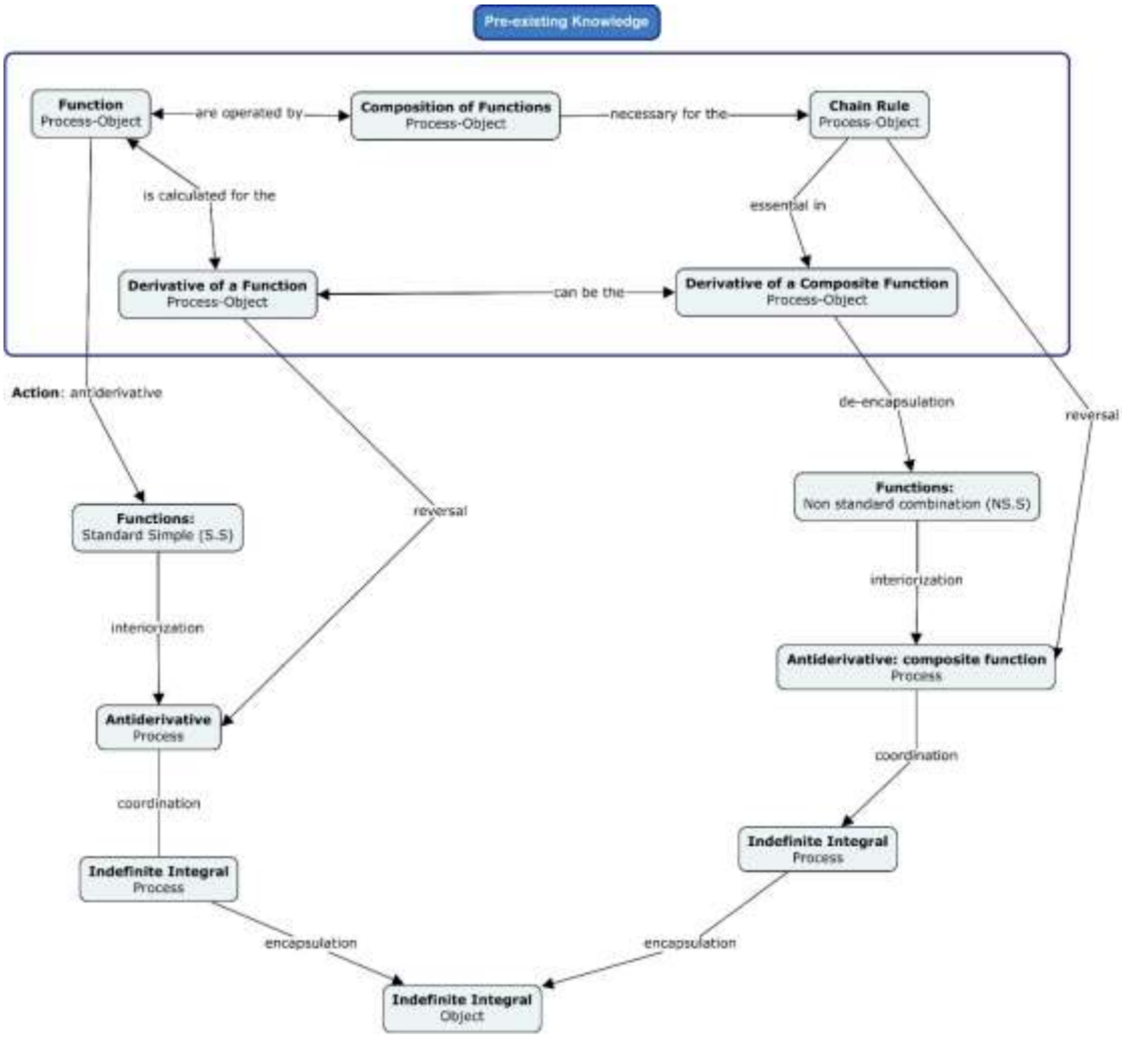
The above steps require the design of an a priori model that describes and predicts the structures and mechanisms necessary to build such concepts or topics in the student's mind, this is the so-called *Genetic Decomposition* (GD) that also guides the design of activities, classes and exercises that are called *ACE cycle* and is specifically the strategy designed to implement the GD. For this study we use the GD proposed for the indefinite integral by Tarr & Maharaj and which was advised by Dubinsky [8]. The ACE cycle can be repeated until the DG is reaffirmed and perfected, this occurs when “students apparently perform the mental constructions proposed by the model and the learning of the concept is satisfactory” Trigueros,[9].

2 Methodology

Research design.
This work adopted a mixed research methodology, because according to Johnson et al. [3] this corresponds to studies where the researcher mixes or combines qualitative and quantitative techniques; additionally they add complexity to the research design by the planning that is done in the integration or combination of these techniques during the implementation of the ACE cycle for each method of integration throughout the research, this because the guidelines of the APOS theory [1] contemplate the advantages of combining each of these approaches.
Genetic Decomposition
In Tarr & Maharaj's DG design [8], two types of functions are established to classify the antiderivatives proposed to the students and thus propose the activities to be

developed:
1. *Simple standard functions (s.s functions):* These are functions whose integral is immediate from a reversal of the table of derivatives, or with a table of antiderivatives, or which are so with a simple numerical adjustment.
For example: $5x^7, sen(x),\ e^x,\ sec^2(x),\ 4x^3,\ \sqrt{x},$
2. *Non-standard combined functions (ns.c functions):* These are functions that require an interpretation of their structure and/or an algebraic or trigonometric manipulation of the integrand to return to a known form.
For example: $sen^2(x)cos(x),\ \frac{4x+2}{x^2+x},\ \ tan(x),$ etc.
The DG used in the research is illustrated in the following figure:

Figure 2: *A Genetic Decomposition for the Indefinite Integral.*



Source: Adapted in CmapTools from Tarr & Maharaj [8]

About errors

In order to study the development of the students' skills and thinking when approaching the calculation of integrals using the integration methods, by means of the questionnaires and workshops implemented during the ACE cycle, it is necessary to analyze the different errors or mistakes that they could make during the development of these activities, since not all of them are the same nor do they reflect the same steps, strategies used or progress in the development of a scheme, and that is a fundamental part of the approach and validation of the GD proposed.

During the development of their research, Tarr & Maharaj [8] propose the application of a mixed error classification framework, combining the proposals arising from Olivier's research [5], Kiat's research [4] and the work developed by Orton [6], since these authors manage to establish links with APOS theory and the levels of the students' schema under the guidelines proposed by Dubinsky [1]. The following figure illustrate the proposed error classification framework used in this research, which are grouped into two types of errors called *misconceptions* and *slips* by affinity in the concepts used in the authors' research Tarr & Maharaj [8]:

Figure 3: *Relationship between structures and mechanisms in APOS theory*

<table>
<tr><th colspan="4">Error Classification Frameworks</th></tr>
<tr><td>Olivier [5]</td><td>Kiat [4]</td><td>Orton [6]</td><th rowspan="2">Anticipated APOS Cognitive Structures</th></tr>
<tr><th colspan="3">Errors</th></tr>
<tr><td rowspan="2">Errors due to misconceptions</td><td rowspan="2">Conceptual Errors
• Arise from failure to grasp the concepts in the problem, or
• Arise from failure to recognize the relationships in the problem</td><td>Arbitrary Errors
• Failure to heed the constraints specified in the given problem and responds arbitrarily.</td><td>Not even at an Action level (“Pre-action” level)</td></tr>
<tr><td>Structural Errors
• Failure to appreciate inherent relationships in the problem, or
• Failure to understand a principle fundamental to the solution thereof</td><td>• Pre-action or Action level may be inferred.
• Interiorization mechanism hindered by misconceptions.
• A constraint or implicit condition from the structure of a symbolic representation is not considered. For example, For $n \neq 1$ $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$</td></tr>
<tr><td rowspan="2">Slips (careless mistakes)</td><td>Procedural Errors
• Arise from failure to correctly perform manipulations or algorithms. Concepts may have been understood.</td><td rowspan="2">Executive Errors
• Arise from incorrect or incomplete manipulations.
• Fundamental concepts or principles may have been understood</td><td rowspan="2">• Process level structures may be present.
• Coordination mechanism and Object level mental constructions hindered by lack of arithmetic/algebraic manipulation skills.
• Links to components of another Schema not well connected.</td></tr>
<tr><td>Technical Errors
• Due to inadequate mathematical content knowledge in other topics
• Due to carelessness</td></tr>
</table>

Source: Adapted from Tarr & Maharaj [8]

Instruments

The data collection instrument was a questionnaire designed by the authors by performing the adaptation of the DG of the antiderivative proposed by Tarr & Maharaj [8] to identify and classify the possible errors made by the students during its development. It should be noted that the questionnaire did not include tasks on applications of the indefinite integral concept outside the analytic-algebraic context. This is because we only focused on the analysis of the errors present in the students during the cognitive construction of the concept of calculus of antiderivatives, prior to its use in the definite integral by means of the fundamental theorem of calculus in various applications and in other contexts. The questionnaire was applied to a group of 30 UPTC students who collaborated in the development of this research and its duration was approximately 90 minutes. The proposed questionnaire, based on the GD is as follows: (Note that each exercise is labeled according to the structure to be studied, **A:** Action, **P**: Process, **O**: Object)

Table 4: *Questionnaire*

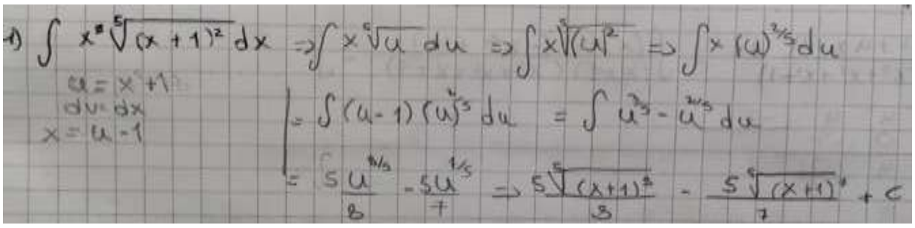
Item	Activity
A1	Identify a composition. Select u and write a $f(u)$, if $f(x) = \sqrt{x^2 + 4sen(3x+2)}$
A2	Calculate the derivative of the function: $h(x) = \frac{4x}{\sqrt{6x^2+1}}$
A3	Calculate the antiderivative and evaluate the derivative of the answer: $\int (2(x+1)^{6/7} + sen(2x))dx$
P1	Calculate $\int \frac{x}{(x+4)^2} dx$
P2	Calculate $\int \frac{2x+1}{x+4} dx$
P3	Calculate $\int sen(ln(x))\, dx$
O1	Calculate $\int \frac{4}{x^2-9} dx$
O2	Calculate $\int x \sqrt[5]{(x+1)^2} dx$

Source: Authors' proposal

3. Results and Discussion

When observing the results of the questionnaires, it is worth mentioning some specific situations detected during the process of data analysis and the adjustment of the students to each respective range. It is observed that most of the students only perform *actions* or *pre-actions*, for example, in **A3**, to calculate an integral such as $sen(2x)$ they still explicitly write $u = 2x$, which evidences that these actions are not yet fully internalized and must be explicit to be performed. It was also observed that some students had an irregular performance, for example, in the item **O2**, student E22 showed a low average performance due to the fact that in the substitution method his performance was affected by some slips that prevented him from reaching a higher level of the scheme as observed in the following illustration:

Figure 5: *Development of the questionnaire by the student E22*

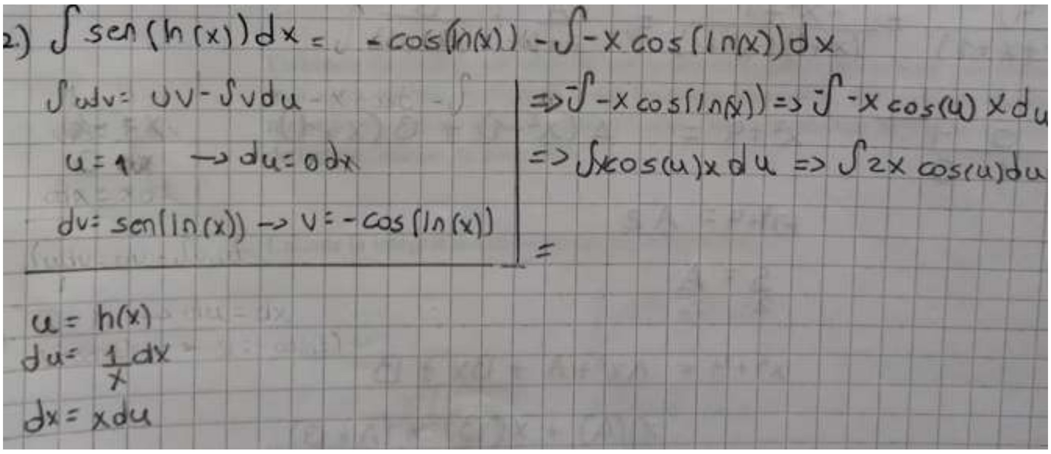


Source: Authors' proposal

Here we can see how an error in the operation of the exponents before integrating with respect to u was enough to not achieve the expected result then this is classified as a *slip*, however omitting that failure, we can see how the rest of the development is coherent and possibly comprehension will be obtained as a *process* in the near future.

The following illustration (figure 6), shows how the student E12 has a *misconception* in the development of the activity **P3** because he tries to perform the method of integration by parts taking u as 1 and therefore obtains du as 0, this leads him to a conflict because when he tries to use the formula, he has the inconvenience that the second integral $\int vdu$ is immediately annulled, then he tries to recompose the situation taking $u = ln\, x$ and trying by substitution.

Figure 6: *Development of the questionnaire by the student E12*



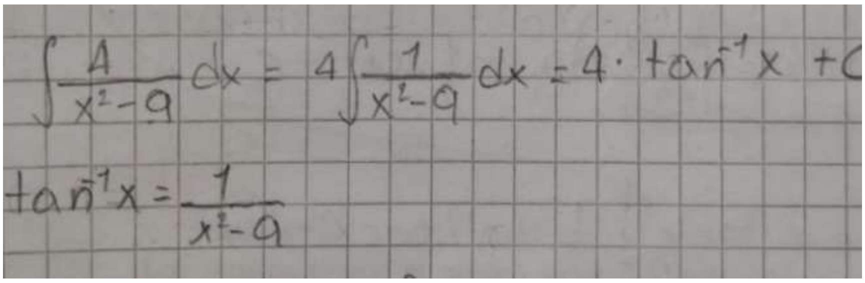
Source: Authors' proposal

However, he reaches a point where the new integral has mixed terms between u and x, which leads him to a new *misconception* because he is not clear on how to eliminate the term x from the new integral, and when operating the product between the x's, he obtains $2x$, which complicates the calculation of the integral.

This is evidence that he still has the algebraic handling of powers and the handling of logarithmic and exponential functions at the *pre-action* in APOS level and still does not recognize them clearly as inverse functions of each other. These failures in the algebraic and functional prerequisites inevitably affect his progress in the development of the scheme of integration. In general, depending on their performance and through activities and exercises that allow them to overcome these errors, they can make it possible to understand the technique of integration by substitution as a *process* structure.

In the following illustration (figure 7), the student E8 makes a *slip* in **O1** because he associates the given integral with the arc tangent and omits the sign $(-)$ in the denominator, and also omits writing $9 = 3^2$ and making that adjustment. It can be observed that he has an idea of the concept of antiderivative, but he still needs to consolidate the *actions*.

Figure 7: *Development of the questionnaire by the student E8*

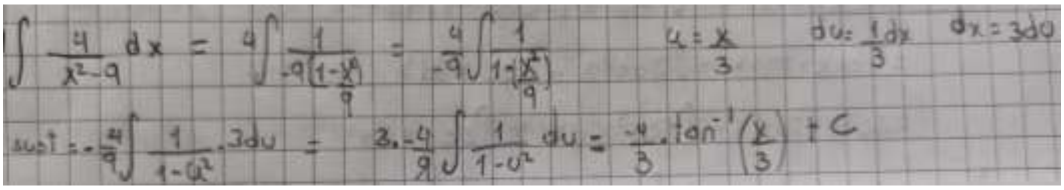


Source: Authors' proposal

Similarly, student E9 has a *slip* similar to the previous one, (in figure 8), however it is seen that he starts to understand the substitution integration technique as a *process* structure, when making the substitution and finding the differential, then he had a greater progress, however a *slip* did not allow him to reach the result correctly because he omits the sign $(-)$.

Therefore, it can be evidenced the attempt to perform the composition of two *processes* to generate a new one, however in that composition mistakes are made, particularly in the *reversion process*. Because it erroneously associates the derivative of the tangent arc with the integrand of the exercise, as follows:

Figure 8: *Development of the questionnaire by the student E9*

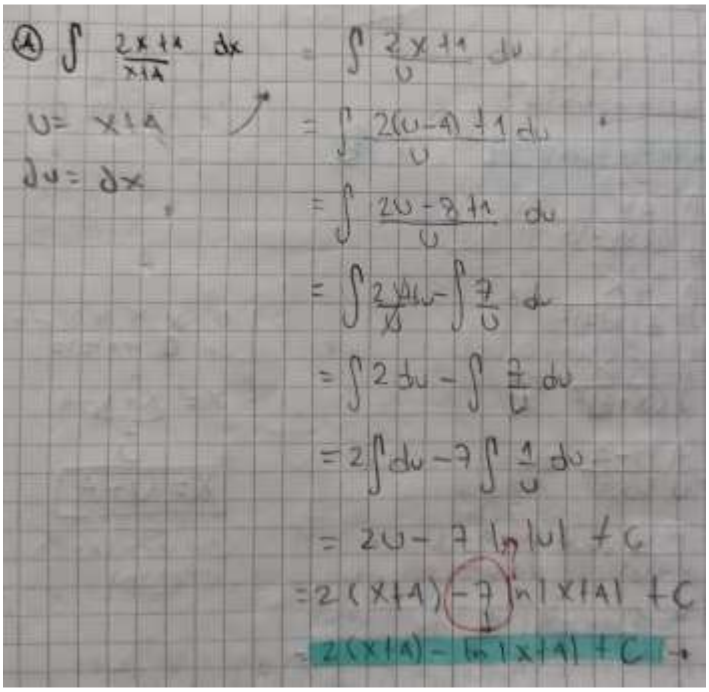


Source: Authors' proposal

In another case, the following illustration (figure 9) shows a *slip* of the student E10 when performing exercise **P2** of the questionnaire, the development was correct and it can be seen that he already performs the substitution as a *process*, however he omitted a 7 when giving the final answer and in case of having a multiple choice such omission could ruin an exercise that was well developed in general.

This is noted in detail below:

Figure 9: *Development of the questionnaire by the student E10*

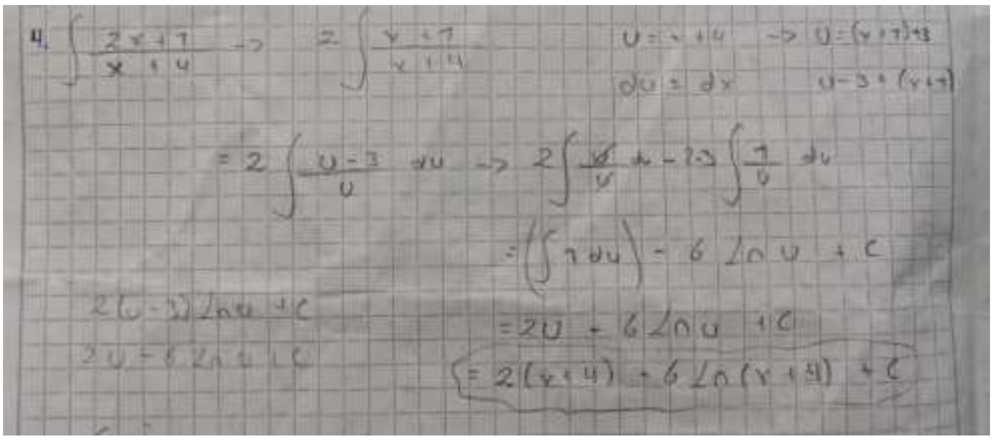


Source: Authors' proposal

In the same exercise **P2**, the student E4 makes a *slip* when factoring the number 2 of the numerator without making the respective adjustment, this slip is an error at the *pre-action* APOS level, this unfortunately affects his result since the *process* of performing the manipulation in the substitution was adequate. It is enough for the student to develop the exercises with more care to obtain satisfactory results.

This can be seen in the following figure:

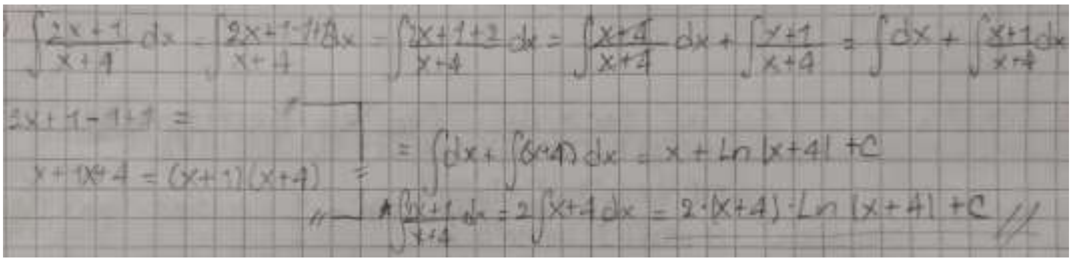
Figure 9: *Development of the questionnaire by the student E4*



Source: Authors' proposal

During the development of the same exercise student E15, (in figure 10), presents *misconceptions* when performing the necessary algebraic operations to carry out the integral, since he manages to perform a factorization in linear factors in a linear term. Hence, the reversion mechanism was hindered, although a *action* level conception may have been constructed for specific functions.

Figure 10: *Development of the questionnaire by the student E15*



Source: Authors' proposal

In this case, it is necessary to strengthen the algebraic operations with some previous arithmetic activities that help the development of these *pre-actions*.

Conclusions

After conducting the research, the following main conclusions were obtained:

Most of the students were able to correctly find derivatives of algebraic and transcendental functions. The errors observed were mostly slips caused by lack of attention or wanting to perform the operations too fast. Therefore, it was observed that the majority of the students had constructed at least one conception as a *process* of calculating derivatives of *s.s functions*.

In many cases it was observed that students tend to adopt a totally algorithmic approach when using integration methods, because they manifest the inherent need to know the "rule" and to find as soon as possible how to put it into practice, so that the errors observed are not due to the incorrect application of the substitution method, but mostly due to errors made in the construction of the mental structure of the previous knowledge (errors in algebraic operations, operations with fractions, arithmetical failures in the handling of numbers, etc.).

In addition, during the research it was observed that in students who mainly perform actions (and very few processes), when evaluating an integral, the use of algorithms was not accompanied by an analytical knowledge of the properties of the *ns.c functions* to be integrated, but by the constant search and application of certain mnemonic rules or abbreviated ways of arriving at the solution.

Particularly, it was observed that in most cases, this absence of analysis by the students did not prevent the presentation of a correct solution of certain exercises, this according to the APOS theory, shows that such tendencies evidence the application of integration methods as an *action*, to a lesser extent they are internalized as a *process* and in very few cases encapsulated as an *object*.

In general, the students knew the rules and how to apply them for certain exercises, but in most of the students who mainly perform *actions*, a clear construction of the mathematical meaning associated with such process was not yet observed, although this is normal, since the progress in the development of the *schemes* is gradual and it is necessary to perform *actions* repeatedly until they are internalized in a *process*.

The above observations should be considered when refining the activities implemented, giving a space during the development of these, to the exercises at the *pre-action* level in order to reduce the *slips* and *misconceptions* presented at the time of calculating the antiderivatives.

As Arnon et al. [1] points out, the APOS Theory is not prescriptive in its pedagogical recommendations, instead it is more used to analyze students' understanding of mathematical concepts, where the teacher studies the mental constructs envisaged in the proposed genetic decomposition and makes the adjustments deemed necessary when implementing the ACE cycle, to facilitate the understanding of the indefinite integral and the methods of integration in this case.

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