International Mathematical Forum, Vol. 19, 2024, no. 2, 101 - 106
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2024.914480

On Collectively Multiplicative Order Convergence on the Lattice Ordered Algebras

Omer Gok

Yildiz Technical University
Faculty of Arts and Sciences
Mathematics Department
Davutpasa, Istanbul, Turkey

Abstract

By using the order convergence and multiplicative order convergence, we investigate the collectively multiplicative order convergence on the lattice ordered algebras.

Mathematics Subject Classification: 46A40, 46B42, 47B65, 47B60, 47B07

Keywords: vector lattice, order convergence, lattice ordered algebra, collectively multiplicative order convergence

1 Introduction

The theory of order convergence, unbounded order convergence in the vector lattices, unbounded norm convergence, multiplicative norm convergence in the normed lattice ordered algebras and multiplicative order convergence in the f-algebras was studied by the some authors, for example, in [1], [4], [5], [6], [7], [8], [9], [10].

The aim of the present paper is to give collective versions of the multiplicative order convergence.

We refer to the books [3], [11], [13] for any unexplained notion and terminology on vector lattices.

2 Preliminary Notes

Suppose that E is a real vector space.E is called an ordered vector space if it is equipped with an order relation $\leq$ (that is, transitive,reflexive and antisymmetric relation $\leq$) compatible with algebraic structure of E in the sense that it satisfies the following properties:

(1) If $x, y \in E$ satisfy $x \leq y$, then $x + z \leq y + z$ for all $z \in E$.

(2) If $x, y \in E$ satisfy $x \leq y$,then $\alpha x \leq \alpha y$ for every $\alpha \geq 0$.

We say that an ordered vector space E is a vector lattice (Riesz space) if the infimum and supremum of $\{x, y\}$ exist for every $x, y \in E$. We give the notations called lattice operations $\vee, \wedge$ for $x \vee y = sup\{x, y\}$ and $x \wedge y = inf\{x, y\}$.The positive part of a vector $x \in E$ is defined by $x^+ = x \vee 0$, the negative part of a vector $x \in E$ is defined by $x^- = -x \vee 0$ and the absolute value of a vector $x \in E$ is defined by $|x| = x^+ + x^-$. The set $E^+ = \{x \in E : x \geq 0\}$ is called the cone of a Riesz space E. A vector lattice E is called Archimedean whenever $\frac{1}{n}x \downarrow 0$ holds in E for every $x \in E^+$, where $\downarrow 0$ means that decreasing and infimum is 0. A vector lattice E under an associative multiplication is called a Riesz algebra or lattice ordered algebra whenever the multiplication makes E an algebra with the usual properties, and in addition, it satisfies the property: the multiplication of positive elements is positive. A lattice ordered algebra E is said to be commutative if $xy = yx$ for every $x, y \in E$. There are a lot of examples of lattice ordered alebras. For example, some lattice ordered algebras are f-algebras, almost f-algebras, d-algebras.

Definition 2.1 [12] *A lattice ordered algebra E is said to be an f-algebra if E has the property that $x \wedge y = 0$ implies $(xz) \wedge y = (zx) \wedge y = 0$ for all $z \in E^+$.*

Every Archimedean f-algebra is commutative.

Definition 2.2 *A lattice ordered algebra E is said to be an almost f-algebra if E has the property that $x \wedge y = 0$ implies $xy = 0$ for all $x, y \in E$.*

Every f-algebra is an almost f-algebra.

Definition 2.3 *A lattice ordered algebra E is said to be a d-algebra if E has the property that $x \wedge y = 0$ implies $(xz) \wedge (yz) = (zx) \wedge (zy) = 0$ for all $z \in E^+$.*

Every f-algebra is a d-algebra.

A norm $\|.\|$ on a vector lattice E is called a lattice norm whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for every $x, y \in E$. A normed vector lattice E is called a normed lattice ordered algebra if it is a lattice ordered algebra and

$$\|x.y\| \leq \|x\|\|y\|$$

holds for every $x, y \in E$.

In this paper, we assume that all vector lattices are real and Archimedean and all operators are linear and bounded.

3 Main Results

Definition 3.1 [1] *A net* (x_α) *in a vector lattice* E *is called order convergent to* $x \in E$ *if there is a net* (y_β) *in* E *such that* $y_\beta \downarrow 0$ *and,for each beta there exists an* α_β *with* $|x_\alpha - x| \leq y_\beta$ *for all* $\alpha \geq \alpha_\beta$. *We denote it by o-convergence* $x_\alpha \to x$.

Definition 3.2 [6] *A net* $\{x_\alpha\}$ *in a vector lattice* E *is called order convergent to an element* $u \in E$ *whenever there exists another net* $\{y_\alpha\}$ *of* E *satisfying* $|x_\alpha - u| \leq y_\alpha \downarrow 0$. *The element* u *is called the order limit of the net* $\{x_\alpha\}$. *The order limits are uniquely determined.*

Definition 3.3 [4] *A net* $\{x_\alpha\}$ *in a lattice ordered algebra* A *is called multiplicative order convergent to* $x \in A$ *if* $(|x_\alpha - x|.u)$ *order converges to* 0 *for every* $u \in A^+$.

Definition 3.4 [6, 9, 10] *A sequence* (x_n) *in a vector lattice* E *is called unbounded order convergent to* $x \in E^+$ *if for every* $u \in E^+$ *the sequence* $(\mid x_n - x \mid \wedge u)$ *converges to zero in order. A sequence* (x_n) *in a normed vector lattice* E *is called unbounded norm convergent to* $x \in E^+$ *if* $(\|\mid x_n - x \mid \wedge u \|)$ *converges to zero for every* $u \in E^+$.

Definition 3.5 [7] *Let* $B = \{(x^b_\alpha)_{\alpha \in A}\}_{b \in B}$ *be a set of nets in a lattice ordered algebra* U *indexed by a directed set* A*.Then,* B *is is called collective o-convergent to* 0 *if there exists a net* $(y_\beta) \downarrow 0$ *such that , for each* β *, there is an* α_β *with* $|x^b_\alpha| \leq y_\beta$ *for all* $\alpha \geq \alpha_\beta$ *and* $b \in B$.

Definition 3.6 *Let* $B = \{(x^b_\alpha)_{\alpha \in A}\}_{b \in B}$ *be a set of nets in a lattice ordered algebra* U *indexed by a directed set* A*.Then,* B *is called collective multiplicative order convergent to* 0 *if* $\{(|x^b_\alpha|.u)_{\alpha \in A}\}_{b \in B}$ *order converges to* 0 *for every* $u \in U^+$.

Definition 3.7 [5]*Let* E *be a normed lattice ordered algebra* E. *A net* (x_α) *in* E *is called multiplicative norm convergent to* $x \in E$ *if* $\||x_\alpha - x|.u\|$ *converges to* 0 *for all* $u \in E^+$.

Definition 3.8 *Let* $B = \{(x^b_\alpha)_{\alpha \in A}\}_{b \in B}$ *be a set of nets in a normed lattice ordered algebra* E *indexed by a directed set* A*.Then,* B *is is called collective multiplicative norm convergent to* 0 *if* $\||x^b_\alpha|.u\|$ *converges to* 0 *for all* $u \in E^+$ *and* $b \in B$.

Theorem 3.9 [7] *Let $B = \{(x_\alpha^b)\}_{b\in B}$ and $C = \{(x_\alpha^c)\}_{c\in C}$ be non-empty set of nets in an lattice ordered algebra E indexed by the same directed set A and let $r \in R$. Then,*

If B and C are collective multiplicative order convergent to 0, then rB, $sol(B)$, $B\cup C$, $B+C$ and $co(B)$ are collective multiplicative order convergent to 0, where

$$sol(B) = \{(x_\alpha) : (\exists b \in B)(\forall \alpha \in A)|x_\alpha| \leq |x_\alpha^b|\},$$

$$B + C = \{(x_\alpha^b + x_\alpha^c)\}_{b\in B, c\in C},$$

$$co(B) = \{(x_\alpha) : (\exists b_1, b_2, ..., b_n \in B, r_1, r_2, ..., r_n \in R^+,),$$
$$\sum_{k=1}^{n} r_k = 1, (x_\alpha) = (\sum_{k=1}^{n} r_k x_\alpha^{b_k})\}.$$

Proof. *Let $B \to 0$ and $C \to 0$ are collective multiplicative order convergent. Hence, we take nets $y_\beta \downarrow 0$ and $z_\gamma \downarrow 0$ in E such that , for every β and γ, there are indices α_β and α_γ satisfying $|x_\alpha^b|.u \leq y_\beta$ for all $\alpha \geq \alpha_\beta$ and $b \in B$ and $u \in E^+$ and $|x_\alpha^c|.u \leq z_\gamma$ for all $\alpha \geq \alpha_\gamma$ and $c \in C$ and $u \in E^+$.*

That $sol(B) \to 0$ and $rB \to 0$ are collective multiplicative order convergence is easy. Now, we show that $B \cup C \to 0$ is collective multiplicative order convergence. We consider $(y_\beta + z_\gamma) \downarrow 0$. $B\cup C = \{(x_\alpha^d)\}_{\alpha\in A, d\in B\cup C}$. This implies that $|x_\alpha^d|.u \leq y_\beta + z_\gamma$ for all $\alpha \geq \alpha_\beta, \alpha_\gamma$ and $d \in B\cup C$ and $u \in E^+$. In order to show $B + C \to 0$ collective multiplicative order convergence, as $(y_\beta + z_\gamma) \downarrow 0$, pick an $\alpha_{(\beta,\gamma)} \geq \alpha_\beta, \alpha_\gamma$. From here, we get $|x_\alpha^b + x_\alpha^c|.u \leq (y_\beta + z_\gamma)$ for all $\alpha \geq \alpha_{(\beta,\gamma)}, b \in B, c \in C$ and $u \in E^+$.

Now, we prove that $co(B) \to 0$ is collective multiplicative order convergent. For this, let $b_1, b_2, ..., b_n \in B$ and $r_1, r_2, ..., r_n \in R^+$ with $\sum_{k=1}^{n} r_k = 1$. Since $\alpha \geq \alpha_\beta$,

and $b_1, b_2, ..., b_n \in B$ and $r_1, r_2, ..., r_n \in R^+$ and $u \in E^+$ are arbitrary and

$$|\sum_{k=1}^{n} r_k x_\alpha^{b_k}|.u \leq \sum_{k=1}^{n} r_k |x_\alpha^{b_k}|.u \leq \sum_{k=1}^{n} r_k y_\beta = y_\beta$$

So, $co(B) \to 0$ is collective multiplicative order convergent.

Let E, F be real vector lattices. By $L(E, F)$, we denote the set of all linear bounded operators, by $L_{ob}(E, F)$ we denote the set of all order bounded operators, and $L_{oc}(E, F)$ represents the set of all order continuous operators.

Definition 3.10 *Let $\tau \subseteq L(E, F)$, where E, F are lattice ordered algebras. We say that τ is collectively multiplicative order continuous if $\tau(x_\alpha)$ is* *collectively multiplicative order convergent to* 0 *whenever* (x_α) *is multiplicative order convergent to* 0. *We denote it by* $\tau \in L_{moc}(E, F)$.

Theorem 3.11 *Let* $B = \{(x_\alpha^b)\}_{b\in B}$ *and* $C = \{(x_\alpha^c)\}_{c\in C}$ *be non-empty set of nets in a normed lattice ordered algebra* E *indexed by the same directed set* A *and let* $r \in R$. *Then,*

If B *and* C *are collective multiplicative norm convergent to* 0, *then* rB, $sol(B)$, $B\cup C$, $B+C$ *and* $co(B)$ *are collective multiplicative normed convergent to* 0, *where*

$$sol(B) = \{(x_\alpha) : (\exists b \in B)(\forall \alpha \in A)|x_\alpha| \leq |x_\alpha^b|\},$$

$$B + C = \{(x_\alpha^b + x_\alpha^c)\}_{b\in B, c\in C},$$

$$co(B) = \{(x_\alpha) : (\exists b_1, b_2, ..., b_n \in B, r_1, r_2, ..., r_n \in R^+,),$$
$$\sum_{k=1}^{n} r_k = 1, (x_\alpha) = (\sum_{k=1}^{n} r_k x_\alpha^{b_k})\}.$$

Let r_1, r_2 be real numbers and $\tau_1, \tau_2 \subseteq L_{moc}(E, F)$, $r_1\tau_1 + r_2\tau_2 = \{r_1T_1 + r_2T_2 : T_1 \in \tau_1, T_2 \in \tau_2\}$.

Lemma 3.12 [7] *If* τ_1, τ_2 *are nonempty subsets of* $L_{moc}(E, F)$, *then* $\tau_1 \cup \tau_2$, $r_1\tau_1 + r_2\tau_2$ *are subsets of* $L_{moc}(E, F)$, *where* $r_1, r_2 \in R$.

The collectively compact sets of operators on normed spaces were first studied by Anselone et al in [2]. We give the following result.

Theorem 3.13 [7] *Let* E, F *be lattice ordered algebras with* F *Archimedean. Then, the following statements are true.*

(i) Let $\tau \subseteq L(E, F)$. *If* $\tau(x_\alpha) \to 0$ *collectively order converges whenever* $x_\alpha \downarrow 0$, *then* $\tau \subseteq L_{ob}(E, F)$. *So,* $L_{oc} \subseteq L_{ob}(E, F)$.

(ii) If $dim(F) < \infty$, *then* $L_{moc}(E, F) \subseteq L_{ob}(E, F)$.

References

[1] Y. A. Abramovich, G. Sirotkin, On order convergence of nets, *Positivity*, **9** (3) (2005), 287-292. https://doi.org/10.1007/s11117-004-7543-x

[2] P. M. Anselone, T. W. Palmer, Collectively compact sets of linear operators, *Pacific J. Math.*, **25** (3) (1968), 417-422. https://doi.org/10.2140/pjm.1968.25.417

[3] C.D. Aliprantis, O. Burkinshaw, *Positive Operators*, Academic Press, London, 1985. https://doi.org/10.1007/978-1-4020-5008-4

[4] A. Aydin, Multiplicative order convergence in f-algebras, *Hacet. J. Math. Stat.*, **49** (3) (2020), 998-1005. https://doi.org/10.15672/hujms.512073

[5] A. Aydin, The multiplicative norm convergence in normed Riesz algebras, *Hacet. J. Math. Stat.*, **50** (1) (2021), 24-32. https://doi.org/10.15672/hujms.638900

[6] Y.A. Dabborasad, E. Y. Emelyanov, M. A. A. Marabeh, $u\tau$ - convergence in locally solid vector lattices, *Positivity*, **22** (2018), 1065-1080. https://doi.org/10.1007/s11117-018-0559-4

[7] E. Emelyanov, Collective order convergence and collectively qualified set of operators, ArXiv, 7 Aug. 2024.

[8] E. Emelyanov, On collectively L-weakly compact sets of operators, ArXiv, 2024.

[9] N. Gao, V.G. Troitsky, and F. Xanthos, Uo-convergence and its applications to Cesaro means in Banach lattices, *Israel J. Math.*, **220** (2017), 649-689. https://doi.org/10.1007/s11856-017-1530-y

[10] N. Gao and F. Xanthos, Unbounded order convergence and application to martingales without probability, *J. Math. Anal. Appl.*, **415** (2014), 931-947. https://doi.org/10.1016/j.jmaa.2014.01.078

[11] P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, New York, 1991. https://doi.org/10.1007/978-3-642-76724-1

[12] B. de Pagter, f-algebras and orthomorphisms, PhD Thesis, Leiden, 1981.

[13] B.Z. Vulikh, *Introduction to the Theory of Partially Ordered Spaces,* Wolters-Noordhoff Scientific Publications,Groningen, 1967.

Received: December 5, 2024; Published: December 23, 2024