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On Unbounded Convergence Properties in $\mathbf{C_p(X,[0,1])}$

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Abstract

I want to introduce the notion of unbounded order convergence in $C_p(\mathrm{X},[0,1])$ which is the space of all continuous $[0,1]$-valued functions on a Tychonoff space X with the topology of pointwise convergence. I will give the fact that the unbounded order convergence on a Baire space X agrees the pointwise convergence on a co-meagre set.

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1 Introduction

The notion of unbounded order convergence, shortly uo-convergence, is firstly studied in the article [5] by Hidegoro Nakano. The aim of Nakano was to define almost everywhere convergence in terms of lattice operations without direct use of measure theory and he defined "individual convergence". Then, it was named as "unboundedly order convergence" in the paper [3] by Ralph DeMaar. After couple of years, some mathematicians began studying this topic such as Anthony Wickstead, Jan Harm van der Walt -who is still studying- ([6], [7], [8]) and Samuel Kaplan ([4]). In this article, Kaplan characterized the usage of weak order units to

define unbounded order convergence. Walt studied this convergence type on the family of continuous functions and Wickstead stated the relationship between weak convergence and uo-convergence in [9]. In 2014, the mathematicians Nishuan Gao, Vlademir Troitsky and Foivos Xanthos published several papers which include important generalizations to different families, spaces and algebras. I refer the reader to [2] for a survey of some convergence types on vector lattices.
In this work, my aim is to give an approach to the characterization of unbounded order convergence in the space of $C_p(\mathrm{X}, [0,1])$ which includes all $[0,1]$-valued continuous functions on X, specifically Tychonoff space during this paper.
In section 2, I give some fundamental definitions, theorems and unbounded order convergence in Riesz spaces, also some topological concepts dealed with my work.
In section 3, I characterize uo-convergence of nets of [0,1]-valued continuous functions on a Tychonoff space, inspired by [1]. Moreover, I give an important relation between ou-convergence and pointwise convergence in a co-meagre of Baire Spaces at the end.

2 Preliminaries

X is said to be an ordered set whenever the following conditions are satisfied:
i. $x \leq x$ for every $x \in X$,
ii. $x \leq y$ and $y \leq x$ implies that $x = y$,
iii. $x \leq y$ and $y \leq z$ implies that $x \leq z$.
An ordered real vector space X with the property that for every x, $y \in X$ the supremum and infimum of $\{x, y\}$ exist in X is called a Riesz space or a vector lattice. We denote the following notations for supremum and infimum: $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$. Let X be a Riesz space. The positive cone X^+ consists of all $x \in X$ such that $x \geq 0$. Furthermore, for every $x \in X$ let $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, and $|x| = x \vee (-x)$ be the positive part, the negative part and the absolute value of x, respectively.
A vector lattice X is called Archimedean if $\frac{x}{n} \downarrow 0$ holds in X for every $x \in X^+$. In this work, I will assume that all vector lattices are Archimedean. The set $[x, y] = \{ z \in \mathrm{E} : x \leq z \leq y \}$ is said to be an order interval. A subset A of E is called an order bounded set if there exist $x, y \in E$ such that $A \in [x, y]$. A function from a directed set A to an arbitrary set X is said to be a net in X indexed by A. It is denoted by $(x_\alpha)_{\alpha\in A}$. A net $(x_\alpha)_{\alpha\in A}$ is said to be increasing whenever $x_\alpha \leq x_\beta$ for all $\alpha,\beta \in A$ such that $\alpha \leq \beta$. If $(x_\alpha)_{\alpha\in A}$ is an increasing net and $x = sup\{x_\alpha : \alpha \in A\}$, then we write $x_\alpha \uparrow x$ as $\alpha \in A$. A net $(x_\alpha)_{\alpha\in A}$ is said to be decreasing whenever $x_\alpha \geq x_\beta$ for all $\alpha,\beta \in A$ such that $\alpha \leq \beta$. If $(x_\alpha)_{\alpha\in A}$ is a decreasing net and $x = inf\{x_\alpha : \alpha \in A \}$, then we write $x_\alpha \downarrow x$ as $\alpha \in A$. A vector $e \in X^+$ is said to be an order unit (or strong unit) if $X_e = X$ satisfies. If $B_e = X$, then e is said to be a weak unit, that is, if $(x \wedge ne) \uparrow x$ holds for each $x \in X^+$.
If X is Archimedean, then a vector $e > 0$ is a weak unit if and only if $|x| \wedge e = 0$ whenever $x = 0$. It can be easily seen that every order unit is a weak unit. Let (x_α) be a net indexed by a directed set A. For $\alpha_0 \in A$ fixed, let $A_0 = \{ \alpha \in \ : \alpha \geq \alpha_0\}$

which is again a directed set under the pre-order induced from A. The restriction of the function x to A_0 is called a tail of (x_α), and it is denoted by $(x_\alpha)_{\alpha \geq \alpha_0}$.

Unbounded Order Convergence

Definition 2.1 Let $(x_\alpha)_{\alpha\in A}$ be a net in X. (x_α) is said to be order convergent to x if there exists a net $(y_\alpha)_{\alpha\in A}$ such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all $\alpha \in A$. It is denoted as $x_\alpha \overset{o}{\to} x$.

Lemma 2.2 [1] For a net (x_α) in a vector lattice X, $x_\alpha \overset{o}{\to} x$ if and only if there exists a set $G \subseteq X^+$ such that $inf G = 0$ and every element of G dominates a tail of (x_α), that is, for every $g \in G$ there exists α_0 such that $|x_\alpha| \leq g$ for all $\alpha \geq \alpha_0$.

Definition 2.3 [1] A net (x_α) is said to be unbounded order converges to x if $|x_\alpha - x| \wedge y \overset{o}{\to} 0$ for every $y \geq 0$. We will use the notation (x_α) uo-converges to x, for short, and we will denote it as $x_\alpha \overset{uo}{\to} x$.
Order and uo-convergences agree for order bounded nets. If $w \geq 0$ is a weak unit then $x_\alpha \overset{uo}{\to} x$ if and only if $|x_\alpha - x| \wedge w \overset{o}{\to} 0$.

Definition 2.4 i. A subspace Y of X is called a sublattice of Y if $x \vee y \in Y$ and $x \wedge y \in Y$ for all $x, y \in Y$.
ii. A sublattice Y of X is called order dense if $0 < x \in X$ implies that there exists $y \in Y$ with $0 < y \leq x$.
iii. A sublattice is regular if the inclusion map is order continuous, that is, it preserves order convergence of nets.
Every order dense sublattice is regular. For a net (x_α) in a regular sublattice Y of X, $x_\alpha \overset{uo}{\to} 0$ in X if and only if $x_\alpha \overset{uo}{\to} 0$ in Y.

Definition 2.5 A net $(x_\alpha)_{\alpha\in A}$ is called order Cauchy (or simply o-Cauchy) if the double net $\left(x_\alpha - x_\beta\right)_{A^2}$ is order null. Unbounded order Cauchy (simply uo-Cauchy) is defined in the same way.

Some Topological Concepts

Let A be a subset of a Hausdorff topological space. Then, A is said to be nowhere dense if $\mathrm{Int}(\bar{A}) = \emptyset$.
Let A be a subset of X, which is defined as in the previous definition. Then,
i) It is called meagre or of the first category if it can be represented as a union of a sequence of nowhere dense sets.
ii) it is called co-meagre or residual if its complement is of the first category.
X is said to be a Baire space if it satisfies the Baire condition, that is, if every intersection of countably many dense open sets is dense.

Equivalently, X is Baire if every co-meagre set is dense. Every locally compact Hausdorff space is Baire. Every nonempty open subspace of a Baire space is a Baire space. A space is Baire if and only if every point has a neighborhood which is also Baire.
Topological vector space X is said to be normal if it satisfies Axiom T_4: every disjoint closed sets of X have disjoint open neighborhoods, that is, for disjoint closed sets A, $B \subset X$ there exists neighborhoods $A \subset U$ and $A \subset V$ such that $U \cap V = \emptyset$.

Theorem 2.6 (Uryson's Lemma) A topological space X is normal if and only if for any two disjoint nonempty closed subsets Y, $Z \subseteq X$ there is a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ for all $x \in$ Y and $f(x) = 1$ for all $x \in Z$.
A topological space X is completely regular if the points can be seperated from closed sets via continuous real-valued functions. That is, for any closed set $A \subseteq X$ and any point $x \in X \setminus A$, there exists a real-valued continuous function $f : X \rightarrow \mathbb{R}$ such that $f(x) = 1$ and $f|_A = 0$.
As a consequence of Uryson's Lemma, we can say that every locally compact Hausdorff space or every normal space is completely regular.
Also, X is said to be Tychonoff or completely T_3 space if it is a completely regular Hausdorff space.

3 Unbounded Order Convergence in $\mathbf{C_p(X, [0, 1])}$

Throughout this section, X stands for a completely regular Hausdorff topological space (Tychonoff space), which is exactly the class of Hausdorff spaces where the conclusion of Uryson's lemma holds. Recall that every locally compact Hausdorff space or every normal space is completely regular.
$C_p(\mathrm{X}, [0,1])$ denotes the space of all continuous [0,1]-valued functions on a Tychonoff space X with the topology of pointwise convergence and $\mathbb{1}$ is the constant one function.

Lemma 3.1 [1] Suppose that X is a completely regular Hausdorff topological space and $G \subset C_p(\mathrm{X}, [0,1])$. The following are equivalent:
i. inf$G$ = 0;
ii. for every non-empty U and every $\epsilon > 0$ there exists $t \in$ U and $g \in G$ with $g(t) < \epsilon$;
iii. for every non-empty U and every $\epsilon > 0$ there exists a non-empty open set V $\subseteq$ U and $g \in G$ such that $g(t) < \epsilon$ for all $t \in$ V.

Proof. (i) ⇒ (ii) Suppose that inf$G$ = 0, but (ii) fails, i.e., there is a non-empty U and $\epsilon > 0$ such that every $g \in G$ is greater than or equal to ϵ on U. Since X is completely regular, we find a non-zero $f \in C(X)_+$ such that $f \leq \epsilon\mathbb{1}$ and f vanishes outside of U. Then $f \leq G$, which contradicts infG = 0.

(ii) ⇒ (i) Assume that infG ≠ 0. Then there is $f \in C_p(\mathrm{X},[0,1])$ with $0 < f \leq G$. We can find an open non-empty set U and ϵ > 0 such that f is greater than ϵ on U. Followingly, every $g \in G$ is greater than ϵ on U, which contradicts (ii).

Now, I will define uo-convergence and characterize it on $C_p(\mathrm{X},[0,1])$. Since $f_\alpha \xrightarrow{uo} f$ if and only if $|f_\alpha - f| \xrightarrow{uo} 0$, it sufficies to characterize order convergence of positive nets to zero.

Definition 3.2 Let (f_α) be a net in $C_p(\mathrm{X},[0,1])$. $f_\alpha \xrightarrow{uo} f$ if and only if $|\, f_\alpha - f \,| \wedge g \xrightarrow{o} 0$ for all $g \geq 0$.

Theorem 3.3 [1] Let X be Tychonoff space and (f_α) a net in $C_p(\mathrm{X},[0,1])$. Then $f_\alpha \xrightarrow{uo} 0$ if and only if for every non-empty open set U and every ϵ > 0 there exists an open non-empty $V \subseteq$ U and an index α_0 such that f_α is less than ϵ on V whenever $\alpha \geq \alpha_0$.

Proof. Suppose that $f_\alpha \xrightarrow{uo} 0$. Then $f_\alpha \wedge \mathbb{1} \xrightarrow{o} 0$. By Lemma 2.2, there exists a set $\mathrm{G} \subset C_p(\mathrm{X},[0,1])$ such that infG = 0 and every member of G dominates a tail of $(f_\alpha \wedge \mathbb{1})$. Fix a nonempty open set U and ϵ ∈ (0, 1). Let V and g be as in Lemma 3.1 (iii). Since g dominates a tail of $(f_\alpha \wedge \mathbb{1})$, there is an α_0 such that $f_\alpha \wedge \mathbb{1} \leq g$ for every $\alpha \geq \alpha_0$. In particular, $f_\alpha(\mathrm{s}) \wedge \mathbb{1}(\mathrm{s}) \leq g(\mathrm{s}) < \epsilon$, hence $f_\alpha(\mathrm{s}) < \epsilon$ for all s ∈ V. This proves the forward implication.
To prove the converse, consider that the condition in the theorem is satisfied. Since $\mathbb{1}$ is a weak unit, it suffices to prove that $f_\alpha \wedge \mathbb{1} \xrightarrow{o} 0$. We will use Lemma 3.1 again. Fix an open non-empty set U and ϵ > 0. Let V and α_0 be as in the assumption. Choose any $t \in$ V. Since X is completely regular, there is an $\mathrm{h} \in C(X)_+$ such that $\mathrm{h}(t) = 0$ and h equals 1 outside of V. Put $g = \mathrm{h} \vee \epsilon\, 1$. Then $g(t) = \epsilon$. We claim that $f_\alpha \wedge \mathbb{1} \leq g$ for every $\alpha \geq \alpha_0$. Indeed, if s ∈ V then $f_\alpha(\mathrm{s}) < \epsilon \leq g(\mathrm{s})$, and if s is not in V then $(f_\alpha \wedge \mathbb{1})(\mathrm{s}) \leq 1 = h(\mathrm{s}) \leq g(\mathrm{s})$.
Repeat this process for every pair (U, ϵ), where U is open and non-empty and ϵ > 0; let G be the set of the resulting functions g. Each such g dominates a tail of $(f_\alpha \wedge \mathbb{1})$. Lemma 2.2 yields infG = 0. This completes the proof.

Corollary 3.4 Let X be completely regular Hausdorff and (f_α) a net in $C_p(\mathrm{X},[0,1])$. Then (f_α) is uo-Cauchy if and only if for every non-empty open set U and every ϵ > 0 there exists an open non-empty V ⊆ U and an index α_0 such that
$|\, f_\alpha(t) - f_\beta(t)| < \epsilon$ for all t ∈ V and $\alpha, \beta \geq \alpha_0$.

Lemma 3.5 [1] Let X be a completely regular Hausdorff Baire space. For $G \subseteq C_p(\mathrm{X},[0,1])$, the following are equivalent:
i) inf G = 0;

ii) There exists a dense set D such that $\inf_{g\in G} g(t) = 0$ for every $t \in D$;
iii) There exists a co-meagre set D such that $\inf_{g\in G} g(t) = 0$ for every $t \in D$.

Proof. (ii) ⇒ (i) is a consequence of Lemma 3.1.

(iii) ⇒ (ii) satisfies because every co-meagre set in a Baire space is dense.

(i) ⇒ (iii), consider that inf $G = 0$. For $n \in N$, put $W_n = \bigcup_{g\in G}\left\{g < \frac{1}{n}\right\}$. Then W_n is open. For every non-empty open set U, Lemma 3.1 yields a point $t \in$ U and $g \in G$ such that $g(t) < \frac{1}{n}$. Hence, $t \in W_n$. That means W_n is dense. Take $D :=\bigcap_{n=1}^{\infty} W_n$, then D is co-meagre. Let $t \in D$, then for all $n \in N$ we get $t \in W_n$, hence $\inf_{g\in G} g(t) = 0$.

Theorem 3.6 [1] Let X be a completely regular Hausdorff Baire space and (f_α) a net in C_p(X, [0,1]). If $f_\alpha \xrightarrow{uo} f$ then f_α converges to f pointwise on a co-meagre set. The converse is true for countable nets.

Proof. Without loss of generality, let choose $f = 0$. Consider that $f_\alpha \xrightarrow{uo} 0$. Then $|f_\alpha| \wedge \mathbb{1} \xrightarrow{o} 0$. There exists a net (g_λ) satisfying $g_\lambda \downarrow 0$ and for all λ there is α_0 such that $|f_\alpha| \wedge \mathbb{1} \le g_\lambda$ whenever $\alpha \ge \alpha_0$. Fix $G = \{g_\lambda\}$.
Then inf G = inf $g_\lambda = 0$. By the previous lemma, there exists a co-meagre set D such that for all $t \in D$, we get $0 = \inf_{g\in G} g(t) = \inf_\lambda g_\lambda(t)$. Therefore, $\lim_\alpha f_\alpha(t) = 0$.

For simplicity, we use sequences instead of nets. Assume that a sequence (f_n) is convergent to zero on a co-meagre set D. We will use Theorem 3.3 to show that $f_n \xrightarrow{uo} 0$. Without loss of generality, $f_n \ge 0$ for all n. Take an open non-empty set U and $\epsilon > 0$. For each m, put $W_m = \bigcup_{n\ge m}\{f_n > \epsilon\}$. So, W_m is open. If $t \in \bigcap_m W_m$ then for all m there exists $n \ge m$ such that $f_n(t) > \epsilon$, and therefore $t \notin D$. This provides that $\bigcap_m W_m$ is contained in $X \setminus D$, hence is meagre. Since W_m is open, ∂W_m is nowhere dense for every m. We conclude that $\bigcup_m \partial W_m$ is meagre. It follows from $\bigcap_m \overline{W_m} \subset (\bigcap_m W_m) \subset (\bigcup_m W_m)$ that $\bigcap_m \overline{W_m}$ is meagre, so that its complement $\bigcup_m (X \setminus \overline{W_m})$ is co-meagre, hence dense. Therefore, it meets U. Consequently, U $\cap$ $(X \setminus W_m) \neq \emptyset$ for some m. Therefore, the condition in Theorem 3.3 is satisfied.

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