International Mathematical Forum, Vol. 19, 2024, no. 2, 57 - 63
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2024.914442

A Note on Mobility Measures in Mixed Markov Processes: An Application to Firm-Level Productivity

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Abstract

Mixed Markov transition probabilities can be used to implement estimation and testing procedures for between-group homogeneity and within-group mobility measures. The aim of this note is to examine the procedure for obtaining a Markov matrix and mobility measures, and to assess their statistical properties. The methodology is then applied to study TFP dynamics in a sample of European firms with the aim of detecting persistence and differences by firm size.

Keywords: transition probability, quintile matrix, persistence

1 Introduction

The purpose of this note is to shed light on the statistical inference procedures for comparing Mixed Markov transition probabilities across heterogeneous groups and for testing mobility measures using transition matrices. It represents the first stage of a research project on the dynamics of firm-level productivity in European countries. The Markov chain model is a modelling approach widely used in the literature to analyse dynamic stochastic processes within a given population, where future states depend with some probability on past states. A transition matrix documents the movement of individuals between different classes. Mixed Markov chain models have been useful for analysing dynamic Markov processes in heterogeneous populations. Using quantile transition matrices, I describe the largesample properties of matrix estimates for heterogeneous subgroups and test procedures for between-group homogeneity and within-group mobility measures. The procedure described in this note is applied to productivity mobility at the level of European firms.

2 Statistical inference procedures in Mixed Finite Markov Processes

Consider a Markov process with a finite state space $\mathcal{S} = \{s_1, s_2, \dots, s_k\}$ where the set of states $\mathcal{K}$ has k states and n is the total number of agents in the population. Assuming a discrete-time process, x_{it} is the state of agent i $(i = 1, 2, \dots, n)$ at time t, $(t = 0, 1, \dots, T)$. The variable $x_{it} = j$, $(j = 1, 2, \dots, k)$ indicates agent i is in state j at time t. The column vector $\boldsymbol{x}_t = (x_{1t} \quad x_{2t} \quad \cdots \quad x_{nt})'$ is a distribution where the evolution over time may be described by the following law of motion:

$$\boldsymbol{x}_{t+1} = M \cdot \boldsymbol{x}_t \tag{1}$$

Each state $\mathcal{K}$ maps to a finite probability distribution of the next states transitioned to from a given state. Transition probabilities, defined on $\mathcal{P}: \mathcal{K} \times \mathcal{S} \rightarrow [0, 1]$, are represented as elements of the $k \times k$ matrix M. If the process is constant over time the Markov chain is completely determined by the Markov transition matrix

$$M = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1k} \\ p_{21} & p_{22} & \cdots & p_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ p_{k1} & p_{k2} & \cdots & p_{kk} \end{bmatrix} \tag{2}$$

where $p_{rj} = p(x_{it+1} = r | x_{it} = j) \geq 0$ and $\sum_{j=1}^{k} p_{rj} = 1$. The initial distribution $H_0 = (h_{10} \quad h_{20} \quad \dots \quad h_{k0})$, $\sum_{j=1}^{k} h_{j0} = 1$, describes the starting probabilities of the k states.

Assumption 1: The probability distribution satisfies the condition $p(x_{it+1} = r | x_{it} = j) = p(x_{it} = r | x_{it-1} = j)$ for all t (stationary Markov process).

Assumption 2: A Markov process is first-order if for any t the cumulative distribution function of $\boldsymbol{x}_{t+1}$ depends only on $\boldsymbol{x}_t$ (Markov property).

The statistical inference of the transition probabilities in (2) is outlined in Statement 1, as shown in [3].

Statement 1: The maximum likelihood (ML) estimator, $\hat{p}_{rj} = n_{rj}/n_r$, is consistent and asymptotically normally distributed, where n_{rj} and n_r are the observed frequencies for all $r, j = 1, 2, \dots, k$.

Likelihood ratio (LR) and Pearson χ^2 tests can be used to compare the transition probabilities estimated by the ML approach under specific null and alternative hypotheses.

Definition 1: The Pearson χ^2 and the LR test statistics to evaluate the null assumption, H0: $p_{rj} = p_r p_j$ are defined as

$$Q = k \sum_{r=1}^{k} \sum_{j=1}^{k} n_{gr} \frac{[\hat{p}_{rj} - \hat{p}_r \hat{p}_j]^2}{\hat{p}_r \hat{p}_j} \quad (3)$$

$$LR = 2k \sum_{r=1}^{k} \sum_{j=1}^{k} \hat{p}_{rj} ln \left(\frac{\hat{p}_{rj}}{\hat{p}_r \hat{p}_j} \right) \quad (4)$$

where $\hat{p}_{rj}$ are estimated transition probabilities from state r to state j, $\hat{p}_r$ and $\hat{p}_j$ are estimated marginal probabilities of states r and j, respectively.

Statement 2: The Pearson χ^2 and the LR test statistics are asymptotically distributed as χ^2 with second-order correction (see [5]). The null hypothesis H0 is accepted if Q (or LR) is less than a pre-specified critical value (p-value > 0.05).

Assumption 3: The observed sample of agents is divided into G mutually exclusive and exhaustive homogeneous groups $(g = 1, 2, \dots, G)$ where agents of the same group are characterized by a similar Markovian process.

Definition 2: A transition model is a Mixed Finite Markov process if transition probabilities are heterogeneous, i.e. $p_{grj} = p(x_{it+1} = r | x_{it+1} = j, g_i = g)$, $g = 1, 2, \dots, G$ and $r, j = 1, 2, \dots, k$.

Statement 3: If individual or specific groups of firms follow a Mixed Finite Markov model, the transition matrix and limit distribution estimated using a homogeneous Markov process is misleading (see [2]).

Two statistics to test the validity of the assumption of homogeneity across subgroups against the alternative assumption that transition probabilities vary across groups are reported in the next definition.

Definition 3: The Pearson χ^2 and the LR test statistics to evaluate the null assumption, H0: $\hat{p}_{grj} = \hat{p}_{rj}$ are defined as

$$Q_g = \sum_{g=1}^{G} \sum_{r=1}^{k} \sum_{j=1}^{k} n_{gr} \frac{[\hat{p}_{grj} - \hat{p}_{rj}]^2}{\hat{p}_{rj}} \quad (5)$$

$$LR_g = 2 \sum_{g=1}^{G} \sum_{r=1}^{k} \sum_{j=1}^{k} n_{grj} ln \left(\frac{\hat{p}_{grj}}{\hat{p}_{rj}} \right) \quad (6)$$

where $\hat{p}_{grj}$ and $\hat{p}_{rj}$ are estimated transition probabilities from state r to state j for subsample g and the full sample, respectively; n_{gr} and n_{grj} are observed frequencies within subsample g, $g = 1, 2, \dots, G$.

Statement 4: The Pearson χ^2 and the LR test statistics are asymptotically distributed as follows

$$Q_g \sim \text{asy } \chi^2\left(\sum_{r=1}^{k}(c_r-1)(b_r-1)\right)$$

$$LR_g \sim \text{asy } \chi^2\left(\sum_{r=1}^{k}(c_r-1)(b_r-1)\right)$$

where b_r is the number of elements of the set of nonzero transition probabilities in the r-th row of the transition matrix estimated from the full sample; c_r is the number of groups with a positive number of transition probabilities in the r-th row of the transition matrices estimated from the subsamples (see [3]).

Estimating the transition matrix in the presence of a (strictly) positive number of observations for each matrix element gives $\sum_{r=1}^{k}(c_r-1)(b_r-1)=k(k-1)(G-1)$.

3 Matrix-based mobility measures

The Markov process methodology is useful for analyzing intra-distribution mobility; [6] and [7] proposed mobility measures based on transition matrices. In this note, I will concentrate on measures that evaluate the diagonal elements separately from the off-diagonal elements.

Definition 4: A mobility measure can be defined as a function $F(M)$ that maps M into a scalar.

Statement 5: $F(M)$ satisfies the property $F(I)\leq F(M)\leq F(Q)$, where I is a $k\times k$ identity matrix and Q is a $k\times k$ matrix with identical rows [6].

The identity matrix is assigned the lowest mobility level and the mobility matrix with identical rows is assigned the highest mobility level.

Definition 5: The trace based mobility index is the normalised distance of M away from the identity matrix I, i.e. $T=\frac{k-tr(M)}{k-1}$.

Statement 6: The trace-based mobility measure $T=\frac{k-tr(M)}{k-1}$ is a function of the diagonal estimated elements of M. Then, the corresponding standard error can be computed to implement testing procedures, as shown by [4].

Definition 6: Upward mobility and downward mobility are the probability of attaining an upward status ($x_{it}<x_{it+1}$) and the probability of attaining a downward

status ($x_{it} > x_{it+1}$), i.e. $\hat{p}_{rj}$, where $r < j$ and $r > j$, respectively.

Statement 7: The statistic to evaluate the null assumption, H0: $p_{rj} = p_{jr}$, with $r \neq j$, $r, j = 1, 2, .., k$ is distributed as Student's t with $n_{rj} + n_{jr} - 2$ degrees of freedom.

4 Application to total factor productivity dynamics

Let us consider an application to total factor productivity (TFP) mobility at the firm level over time. The empirical analysis is based on data from EFIGE. The TFP data in 2008 and 2014 refer to a representative sample of firms for the manufacturing sector in four European economies (France, Italy, Spain and the United Kingdom). Firms are classified according to the number of employees: size 1 (10-19) size 2 (20-49), size 3 (50-249) and size 4 (more than 249). See [1] for a detailed description of the methods and contents of the survey.
The full range of TFP values is divided into four classes corresponding to the 25th, 50th and 75th percentiles of TFP log levels, $(j = 1, 2, 3, 4)$. A Markov transition probability is then defined as the probability p_{rj} that a firm is in TFP class j at time $t + 1$, given that it was in class r at time t. Three main facts are assessed from the estimated transition matrix for the full sample ($n =$ 1435) and the size sub-samples reported in Table 1. First, there is evidence of persistence in TFP classes since all diagonal elements are larger than the off-diagonal elements. Second, extreme classes tend to be more absorbing than intermediate classes; the probability of remaining in classes 1 and 4 is greater than 50%. Third, transitions between extreme classes are very unlikely. LR and Pearson χ^2 tests confirm that there is a significant difference between the four TFP classes for the full sample matrix, such that $p_{rj} \neq p_r p_j$. This result also holds for all sub-groups by firm size. The null hypothesis of homogeneous size groups cannot be rejected for all of them at the 1% significance level. Finally, I focus on the elements of the transition matrix to examine TFP mobility for some TFP classes of firms. On the one hand, I consider within-class differences by analysing the off-diagonal elements. I find that the probability of attaining upward status is less than the probability of attaining downward status. The null hypothesis cannot be accepted in the full sample so that $p_{12} < p_{21}$ and $p_{34} < p_{43}$ (at 5% level). By size, $p_{12} < p_{21}$ (at 1% level) is reported for size 2, while $p_{34} < p_{43}$ is reported for sizes 1 and 2 (at 1% level and 10% levels, respectively).

On the other hand, using a trace-based mobility measure, I find that the average probability of a firm leaving its initial class in the following period is greater than 50% in the full sample and appears to decrease with size (Table 2). In fact, the null assumption of an identical mobility index between any two size subgroups cannot be accepted at the 1% level for all pairs.

Table 1: Quantile transition matrix, full sample and by size

TFP class			Transition probability				
2008	2014	N	All	Size 1	Size 2	Size 3	Size 4
1	1	246	79.10	69.09	82.61	76.92	88.89
1	2	54	17.36	25.45	13.04	23.08	3.7
1	3	7	2.25	3.64	2.9	0	3.7
1	4	4	1.29	1.82	1.45	0	3.7
2	1	86	24.16	27.18	26.99	15.28	16.67
2	2	165	46.35	44.66	50.31	40.28	44.44
2	3	87	24.44	24.27	19.63	31.94	38.89
2	4	18	5.06	3.88	3.07	12.5	0
3	1	28	7.20	2.94	13.46	3.95	0
3	2	97	24.94	27.94	21.79	25	28.57
3	3	190	48.84	51.47	46.79	48.68	47.62
3	4	74	19.02	17.65	17.95	22.37	23.81
4	1	7	1.85	2.38	2.65	0	0
4	2	24	6.33	6.35	7.28	6.76	0
4	3	101	26.65	37.3	25.83	17.57	7.14
4	4	247	65.17	53.97	64.24	75.68	92.86

Note: Size classes from 1 to 4 refer to 10-19, 20-49, 50-250 and over 250 employees.

Table 2: Trace-based mobility index

	T	St. error
Full sample	0.6374	0.00046
Size 1	0.6027	0.00164
Size 2	0.5202	0.00102
Size 3	0.5281	0.00199
Size 4	0.4206	0.00615

Note: Size classes from 1 to 4 refer to 10-19, 20-49, 50-250 and over 250 employees.

5 Concluding remarks

In this note I described how quantile transition matrices can be used to implement estimation and testing procedures for between-group homogeneity and within-group mobility measures. I then proposed an application to TFP mobility, which is driven by a Markov process that is very persistent, where upward mobility is lower than downward mobility for small sized firms and intra-group mobility decreases as firm size increases. There are a number of open questions that deserve to be explored in the future. First, distribution-independent measures of mobility should be added to take into account whether total TFP is rising or falling. Second, the testing procedures should include log-linear models. Finally, the analysis of TFP in terms of volatility and mobility risk requires richer datasets.

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Received: April 11, 2024; Published: May 10, 2024