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The Split Feasibility Problem and Fixed Point Problem of Bregman Totally Quasi-Asymptotically Non-expansive Mapping in Banach Spaces

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Abstract

In this article, an iterative algorithm is proposed for solving the split feasibility problem and fixed point problem of Bregman totally quasi-asymptotically nonexpansive mapping in p-uniformly convex and uniformly smooth real Banach spaces. We obtained and proved the strong convergence theorem of the iterative scheme presented. Then, our main result is used to solve split feasibility problem and equilibrium problem.

Mathematics Subject Classification: 47H09, 47J25

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1 Introduction

Censor and Elfving [1] initially put forward the split feasibility problem (shortly, SFP) in finite-Hilbert spaces for modeling inverse problems which originate from medical image reconstruction and phase retrievals [2]. It is formulated as below: finding a point d^*, such that:

$$d^* \in C \quad and \quad Gd^* \in Q, \tag{1.1}$$

where C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. $G : H_1 \to H_2$ is a bounded linear operator. The solution set of SFP (1.1) is represented by $\Gamma = \{d^* \in C : Gd^* \in Q\}$. SFP has attracted a great deal of attention from authors owing to it is an incredibly effective tool in various disciplines such as radiation therapy treatment planning, signal processing, computer tomograph and image restoration, for details see [3-4].

If C and Q are the sets of fixed points of two nonlinear mappings S and T, respectively, C and Q are nonempty closed convex subsets, then SFP (1.1) is generalized as the split common fixed point problem (shortly, $SCFP$) for S and T. That is, finding a point $d^* \in H_1$ such that:

$$d^* \in F(S) \quad and \quad Gd^* \in F(T), \tag{1.2}$$

where $S : H_1 \to H_1$, $T : H_2 \to H_2$ be two nonlinear operators, $F(S)$ and $F(T)$ represent the fixed point sets of S and T, respectively.

Many researchers have been working on constructing iterative algorithms for finding solution of SFP and got numerous weak or strong convergence theorems in Hilbert space. See, for example [5-11] and the references therein. Recently, attempt to solve SFP in Banach spaces have been concerned by a number of authors. However, there are some difficulties to surmount, for instance, the projection operators are no longer expansive, and dual mappings are nonlinear. In 2008, Schöpfer et al. [12] first established weak convergence theorem for SFP in p-uniformly convex and uniformly smooth real Banach spaces. Until 2015 and 2016, Tang et al.[13], Tian et al.[14] got strong convergence theorems for $SCFP$ under the assumption of semi-compactness on mappings. Afterwards, for solving split feasibility problem and fixed point problem in Banach space, Ma et al.[15] studied the problem of finding a point $d^* \in E_1$ with the property:

$$d^* \in F(S) \quad and \quad Gd^* \in Q, \tag{1.3}$$

where E_1 be a Banach space, Q be nonempty closed and convex subsets of Banach space E_2, $S : E_1 \to E_1$ be a closed quasi-ϕ-nonexpansive mapping. They proposed an iterative algorithm to approximate a solution of (1.3) and the strong convergence theorem is obtained without the assumption of semi-compactness on mapping in 2-uniformly convex and 2-uniformly smooth Banach spaces. These works inspire us to consider the following question.

Question. Can we propose an iterative algorithm converges strongly to a solution of (1.3) for a more general than quasi-nonexpansive mapping in p-uniformly convex and uniformly smooth real Banach spaces which $p > 1$.

So, in this article, we keep on study the problem (1.3) for Bregman totally quasi-asymptotically nonexpansive mapping in p-uniformly convex and uniformly smooth real Banach spaces. A new iterative algorithm was established and strong convergence theorem of proposed algorithm was obtained and proved in the absence of the assumption of semi-compactness on mapping. Our result complement and extend the corresponding results on the topic in the literature.

2 Preliminaries

Let E be a real Banach space and let $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. The modulus of convexity $\delta_E : [0, 2] \to [0, 1]$ is defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{\|d + e\|}{2} : \|d\| = \|e\| = 1, \|d - e\| \geq \epsilon\}.$$

E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$; p-uniformly convex if there is a constant $c_p > 0$ satisfies $\delta_E(\epsilon) \geq c_p \epsilon^p$ for any $\epsilon \in (0, 2]$. The modulus of smoothness of E: $\rho_E(\tau) : [0, \infty) \to [0, \infty)$ is defined by

$$\rho_E(\tau) = sup\{\frac{\|d + \tau e\| + \|d - \tau e\|}{2} - 1 : \|d\| = \|e\| = 1\}.$$

E is said to be uniformly smooth if $\lim_{n\to\infty} \frac{\rho_E(\tau)}{\tau} = 0$; q-uniformly smooth if there is a constant $C_q > 0$ satisfies $\rho_E(\tau) \leq C_q \tau^q$ for any $\tau > 0$. We know that if E is p-uniformly convex and uniformly smooth, then its dual space E^* is q-uniformly smooth and uniformly convex. In this condition, the duality mapping $\mathcal{J}_E^p$ is one-to-one, single-valued and contents $\mathcal{J}_E^p = (\mathcal{J}_{E^*}^q)^{-1}$, where $\mathcal{J}_{E^*}^q$ is the duality mapping of E^*.

Definition 2.1. ([16]) The duality mapping $\mathcal{J}_E^p : E \to 2^{E^*}$ is defined by

$$\mathcal{J}_E^p(d) = \{d^* \in E^* : \langle d, d^* \rangle = \|d\|^p, \|d^*\| = \|d\|^{p-1}\}, \forall d \in E.$$

The duality mapping $\mathcal{J}_E^p$ is called weak-to-weak continuous if

$$d_n \rightharpoonup d \Rightarrow \langle \mathcal{J}_E^p d_n, e \rangle \to \langle \mathcal{J}_E^p d, e \rangle, \forall e \in E.$$

Lemma 2.2. ([17]) Let $d, e \in E$. If E is q-uniformly smooth, then there is a constant $C_q > 0$ with

$$\|d-e\|^q \leq \|d\|^q - q\langle e, \mathcal{J}_E^q(d)\rangle + C_q\|e\|^q.$$

Definition 2.3. There is a Gâteaux differentiable convex function $f : E \to R$. The Bregman distance related to f is defined by:

$$\Delta_f(d,e) := f(e) - f(d) - \langle f'(d), e-d\rangle, e, d \in E.$$

As is known to all, the duality mapping $\mathcal{J}_E^p$ is the derivative of the function $f_p(d) = \frac{1}{p}\|d\|^p$. For convenience, the $\Delta_{f_p}(d,e)$ is denoted by $\Delta_p(d,e)$, then the Bregman distance related to f_p can be written as follows

$$\begin{aligned}
\Delta_p(d,e) &= \frac{1}{q}\|d\|^p - \langle \mathcal{J}_E^p d, e\rangle + \frac{1}{p}\|e\|^p \\
&= \frac{1}{p}(\|e\|^p - \|d\|^p) + \langle \mathcal{J}_E^p d, d-e\rangle \\
&= \frac{1}{q}(\|d\|^p - \|e\|^p) - \langle \mathcal{J}_E^p d - \mathcal{J}_E^p e, e\rangle.
\end{aligned}$$

By the definition of $\Delta_p(\cdot,\cdot)$, we have

$$\Delta_p(d,e) = \Delta_p(d,w) + \Delta_p(w,e) + \langle w-e, \mathcal{J}_E^p d - \mathcal{J}_E^p w\rangle, \tag{2.1}$$

and

$$\Delta_p(d,e) + \Delta_p(e,d) = \langle d-e, \mathcal{J}_E^p d - \mathcal{J}_E^p e\rangle. \tag{2.2}$$

for any $d, e, w \in E$.

Lemma 2.4. ([12]) Let E be a p-uniformly convex space, the following inequality relationship hold:

$$\tau\|d-e\|^p \leq \Delta_p(d,e) \leq \langle d-e, \mathcal{J}_E^p d - \mathcal{J}_E^p e\rangle, d, e \in E.$$

Where $\tau > 0$ is some fixed number.

Evidently, if $\{d_n\}$ and $\{e_n\}$ are both bounded sequences of a p-uniformly convex and uniformly smooth space E, then $d_n - e_n \to 0$ $(n \to \infty)$ means that $\Delta_p(d_n, e_n) \to 0$ as $n \to \infty$.

The metric projection

$$P_C d = argmin_{e\in C}\|d-e\|, \quad d \in E,$$

is the unique minimizer of the norm distance, which can be described by a variational inequality:

$$\langle \mathcal{J}_E^p(d - P_C d), w - P_C d\rangle \leq 0, \quad \forall w \in C. \tag{2.3}$$

Where C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E.

Likewise the definition of metric projection, the Bregman projection is defined as follows:

$$\Pi_C d = argmin_{e \in C} \Delta_p(d, e), d \in E,$$

it is the unique minimizer of the Bregman distance. The Bregman projection can also be described by a variational inequality:

$$\langle \mathcal{J}_E^p d - \mathcal{J}_E^p(\Pi_C d), w - \Pi_C d\rangle \leq 0, \forall w \in C. \tag{2.4}$$

from (2.4), we have

$$\Delta_p(\Pi_C d, w) \leq \Delta_p(d, w) - \Delta_p(d, \Pi_C d), \forall w \in C. \tag{2.5}$$

Following [18,19], the function $V_p : E^* \times E \to [0, +\propto)$ related to f_p, which is defined by

$$V_p(\bar{d}, d) = \frac{1}{q}\|\bar{d}\|^q - \langle \bar{d}, d\rangle + \frac{1}{p}\|d\|^p, \forall d \in E, \bar{d} \in E^*.$$

Obviousely, V_p is nonnegative and

$$V_p(\bar{d}, d) = \Delta_p(\mathcal{J}_{E^*}^p(\bar{d}), d), \forall d \in E, \bar{d} \in E^*. \tag{2.6}$$

Furthermore, by the subdifferential inequality, we have

$$V_p(\bar{d}, d) + \langle \bar{e}, \mathcal{J}_{E^*}^p(\bar{d}) - d\rangle \leq V_p(\bar{d} + \bar{e}, d). \tag{2.7}$$

for all $d \in E$ and $\bar{d}, \bar{e} \in E^*$ (see [20]). In addition, V_p is convex in the first variable. Thus,

$$\Delta_p(\mathcal{J}_{E^*}^q(\sum_{i=1}^{N} t_i \mathcal{J}_E^p d_i), w) \leq \sum_{i=1}^{N} t_i \Delta_p(d_i, w), \forall w \in E, \tag{2.8}$$

where $\{d_i\}$ and $\{t_i\}$ satisfy $\{d_i\}_{i=1}^N \subset E$, $\{t_i\}_{i=1}^N \subset (0,1)$ and $\sum_{i=1}^N t_i = 1$.

Let C be a subset of E and T be a self-mapping of C. A point $p \in C$ is called an $asymptotic\ fixed\ point$ of T if C includes a sequence $\{d_n\}_{n=1}^\infty$ which converges weakly to p and $\lim_{n\to\infty} \|d_n - Td_n\| = 0$. $\widehat{F}(T)$ is used to denote the set of $asymptotic\ fixed\ points$ of T.

Definition 2.5. Let C be a subset of E. The set of fixed points of mapping T denoted by $F(T) = \{d \in C : Td = d\}$, A mapping $T : C \to C$ is said to be:
(i) Bregman quasi-nonexpansive mapping [21], if $F(T) \neq \emptyset$ and

$$\Delta_p(Td, \bar{d}) \leq \Delta_p(d, \bar{d}),\ \forall d \in C, \bar{d} \in F(T);$$

(ii) Bregman relatively nonexpansive [21], if $\widehat{F}(T) = F(T)$ and

$$\Delta_p(Td, \bar{d}) \leq \Delta_p(d, \bar{d}), \ \forall d \in C, \bar{d} \in F(T);$$

(iii)Bregman firmly nonexpansive [22], if

$$\langle \mathcal{J}_E^p(Td) - \mathcal{J}_E^p(Te), Td - Te \rangle \leq \langle \mathcal{J}_E^p(Td) - \mathcal{J}_E^p(Te), d - e \rangle;$$

for any $d, e \in C$, or equivalently,

$$\Delta_p(Td, Te) + \Delta_p(Te, Td) + \Delta_p(d, Td) + \Delta_p(e, Te) \leq \Delta_p(d, Te) + \Delta_p(e, Td);$$

(iv) Bregman totally quasi-asymptotically nonexpansive mapping [23], if $F(T) \neq \emptyset$ and there are nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$(as $n \to \infty$) and a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$ such that

$$\Delta_p(T^n d, \bar{d}) \leq \Delta_p(d, \bar{d}) + \nu_n \zeta(\Delta_p(d, \bar{d})) + \mu_n, \ \forall d \in C, \bar{d} \in F(T).$$

Lemma 2.6.([23]) Let C be a nonempty, closed and convex subset of real reflexive Banach space E and $f : X \to (-\infty, +\infty]$ be a Legendre function which is total convex on bounded subsets of E. Let $T : C \to C$ be a closed and Bregman totally quasi-asymptotically nonexpansive mapping with nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$, then the fixed point set $F(T)$ of T is a closed and convex subset of C.

Definition 2.7. A mapping $T : C \to C$ is said to be closed if for any sequence $d_n \subset C$ with $d_n \to d \in C$ and $Td_n \to e$ as $n \to \infty$, then $Td = e$.

Definition 2.8. A mapping $T : C \to C$ is said to be uniformly L-Lipschitz continuous, if there exist a constant $L > 0$ such that$\|T^n d - T^n e\| \leq L\|d - e\|, \ \forall d, e \in C, \ \forall n \geq 1$.

3 Main Results

Theorem 3.1. Let E_1, E_2 be two p-uniformly convex and uniformly smooth real Banach spaces. Let Q be nonempty closed and convex subset of E_2. Let $G : E_1 \to E_2$ be a bounded linear operators and $G^* : E_2^* \to E_1^*$ be the adjoint operator of G. Let $T : E_1 \to E_1$ with $C := F(T) \neq \emptyset$ be a *closed Bregman totally quasi-asymptotically nonexpansive mapping* with nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ and a strictly increasing continuous function $\zeta : R^+ \to R^+$ such that $\nu_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$), $\zeta(0) = 0$.

Assume that T is uniformly L-Lipschitz continuous. Let $d_1 \in E_1$ and $C_1 = E_1$, and $\{d_n\}$ be a sequence generated by

$$\begin{cases} w_n = \mathcal{J}^q_{E_1^*}(\mathcal{J}^p_{E_1} d_n - \gamma_n G^* \mathcal{J}^p_{E_2}(Gd_n - P_Q Gd_n)) \quad n \geq 1 \\ e_n = \mathcal{J}^q_{E_1^*}[\alpha_n \mathcal{J}^p_{E_1} w_n + (1-\alpha_n)\mathcal{J}^p_{E_1} T^n d_n] \\ C_{n+1} = \{u \in C_n : \Delta_p(e_n, u) \leq \Delta_p(d_n, u) + \delta_n; \Delta_p(w_n, u) \leq \Delta_p(d_n, u)\} \\ d_{n+1} = \Pi_{C_{n+1}} d_1, \end{cases} \tag{3.1}$$

where $\delta_n = \nu_n sup_{\eta \in \Gamma} \zeta(\Delta_p(d_n, \eta)) + \mu_n$, P_Q is the metric projection of E_2 onto Q, $\{\alpha_n\} \subset [k, l] \subset (0, 1)$ and $\{\gamma_n\} \subset [a, b] \subset (0, (\frac{q}{C_q \|G\|^q})^{\frac{1}{q-1}})$. If $\Gamma = \{y \in F(T) : Gy \in Q\} \neq \emptyset$, then the sequence $\{d_n\}$ converges strongly to a point $\Pi_\Gamma d_1$.

Proof. We shall divided the proof into four steps.

Step 1. We first prove that C_n is closed and convex for any $n \geq 1$.

We know that C_1 is closed and convex by $C_1 = E_1$. Presuming C_n is closed and convex. For any $u \in C_n$, we obtain

$$\Delta_p(e_n, u) \leq \Delta_p(d_n, u) + \delta_n \Leftrightarrow \langle \mathcal{J}^p_{E_1} d_n - \mathcal{J}^p_{E_1} e_n, u\rangle \leq \frac{1}{q}(\|d_n\|^p - \|e_n\|^p) + \delta_n, \tag{3.2}$$

$$\Delta_p(w_n, u) \leq \Delta_p(d_n, u) \Leftrightarrow \langle \mathcal{J}^p_{E_1} d_n - \mathcal{J}^p_{E_1} w_n, u\rangle \leq \frac{1}{q}(\|d_n\|^p - \|w_n\|^p). \tag{3.3}$$

These indicate that C_{n+1} is closed. It is evident that C_{n+1} is convex. Therefore, $\{d_{n+1}\}$ is well defined.

Step 2. We prove that $\Gamma \subseteq C_n$ for any $n \geq 1$.

Let $y \in \Gamma$, by (3.1) and Lemma 2.2, we have

$$\begin{aligned} \Delta_p(w_n, y) &= \Delta_p(\mathcal{J}^q_{E_1^*}(\mathcal{J}^p_{E_1} d_n - \gamma_n G^* \mathcal{J}^p_{E_2}(Gd_n - P_Q Gd_n)), y) \\ &= \frac{1}{q}\|\mathcal{J}^p_{E_1} d_n - \gamma_n G^* \mathcal{J}^p_{E_2}(Gd_n - P_Q Gd_n)\|^q - \langle \mathcal{J}^p_{E_1} d_n, y\rangle \\ &\quad + \gamma_n \langle \mathcal{J}^p_{E_2}(Gd_n - P_Q Gd_n), Gy\rangle + \frac{1}{p}\|y\|^p \\ &\leq \frac{1}{q}\|\mathcal{J}^p_{E_1} d_n\|^q - \gamma_n \langle \mathcal{J}^p_{E_2}(Gd_n - P_Q Gd_n), Gd_n\rangle \\ &\quad + \frac{C_q(\gamma_n\|G\|)^q}{q}\|\mathcal{J}^p_{E_2}(Gd_n - P_Q Gd_n)\|^q - \langle \mathcal{J}^p_{E_1} d_n, y\rangle \\ &\quad + \gamma_n \langle \mathcal{J}^p_{E_2}(Gd_n - P_Q Gd_n), Gy\rangle + \frac{1}{p}\|y\|^p \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{q}\|d_n\|^p - \langle \mathcal{J}_{E_1}^p d_n, y\rangle + \frac{1}{p}\|y\|^p \\
&\quad + \gamma_n \langle \mathcal{J}_{E_2}^p (Gd_n - P_Q Gd_n), Gy - Gd_n\rangle \\
&\quad + \frac{C_q(\gamma_n\|G\|)^q}{q}\|\mathcal{J}_{E_2}^p (Gd_n - P_Q Gd_n)\|^q \qquad (3.4)\\
&= \Delta_p(d_n, y) + \gamma_n \langle \mathcal{J}_{E_2}^p (Gd_n - P_Q Gd_n), Gy - Gd_n\rangle \\
&\quad + \frac{C_q(\gamma_n\|G\|)^q}{q}\|\mathcal{J}_{E_2}^p (Gd_n - P_Q Gd_n)\|^q
\end{aligned}
$$

It follows from (2.3) that

$$
\begin{aligned}
\langle \mathcal{J}_{E_2}^p (Gd_n - P_Q Gd_n), Gy - Gd_n\rangle &= -\langle \mathcal{J}_{E_2}^p (Gd_n - P_Q Gd_n), Gd_n - Gy\rangle \\
&= -\langle \mathcal{J}_{E_2}^p (Gd_n - P_Q Gd_n), Gd_n - P_Q Gd_n\rangle \\
&\quad - \langle \mathcal{J}_{E_2}^p (Gd_n - P_Q Gd_n), P_Q Gd_n - Gy\rangle \\
&= -\|(Gd_n - P_Q Gd_n)\|^p \\
&\quad - \langle \mathcal{J}_{E_2}^p (Gd_n - P_Q Gd_n), P_Q Gd_n - Gy\rangle \\
&\leq -\|(Gd_n - P_Q Gd_n)\|^p.
\end{aligned} \tag{3.5}
$$

Substituting (3.5) into (3.4), we get

$$
\begin{aligned}
\Delta_p(w_n, y) &\leq \Delta_p(d_n, y) - \gamma_n\|(Gd_n - P_Q Gd_n)\|^p + \frac{C_q(\gamma_n\|G\|)^q}{q}\|Gd_n - P_Q Gd_n\|^p \\
&= \Delta_p(d_n, y) - (\gamma_n - \frac{C_q(\gamma_n\|G\|)^q}{q})\|Gd_n - P_Q Gd_n\|^p.
\end{aligned} \tag{3.6}
$$

Since $\{\gamma_n\} \subset [a, b] \subset (0, (\frac{q}{C_q\|G\|^q})^{\frac{1}{q-1}})$, we get

$$
\Delta_p(w_n, y) \leq \Delta_p(d_n, y). \tag{3.7}
$$

Furthermore, from (3.1), (2.8) and (3.7), we obtain

$$
\begin{aligned}
\Delta_p(e_n, y) &= \Delta_p(\mathcal{J}_{E_1^*}^q [\alpha_n \mathcal{J}_{E_1}^p w_n + (1-\alpha_n)\mathcal{J}_{E_1}^p T^n d_n], y) \\
&\leq \alpha_n \Delta_p(w_n, y) + (1-\alpha_n)\Delta_p(T^n d_n, y) \\
&\leq \alpha_n \Delta_p(w_n, y) + (1-\alpha_n)\{\Delta_p(d_n, y) + \nu_n \zeta(\Delta_p(d_n, y)) + \mu_n\} \\
&\leq \alpha_n \Delta_p(d_n, y) + (1-\alpha_n)\Delta_p(d_n, y) + \nu_n sup_{\eta\in\Gamma}\zeta(\Delta_p(d_n, \eta)) + \mu_n \\
&= \Delta_p(d_n, y) + \delta_n,
\end{aligned} \tag{3.8}
$$

where $\delta_n = \nu_n sup_{\eta\in\Gamma}\zeta(\Delta_p(d_n, \eta)) + \mu_n$. (3.7) and (3.8) show that $y \in C_{n+1}$, which implies $\Gamma \subseteq C_{n+1}$.

step 3. We show that $\{d_n\}$ is a Cauchy sequence.

It follows from the definition of C_n, we have $d_n = \Pi_{C_n} d_1$, for all $n \geq 1$. From (2.5) and for all $y \in \Gamma$, we have

$$\begin{aligned}\Delta_p(d_1, d_n) &= \Delta_p(d_1, \Pi_{C_n} d_1) \\ &\leq \Delta_p(d_1, y) - \Delta_p(\Pi_{C_n} d_1, y) \\ &\leq \Delta_p(d_1, y).\end{aligned} \tag{3.9}$$

This shows that $\Delta_p(d_1, d_n)$ is bounded. Hence $\{d_n\}$ is bounded.
For some positive integers m, n with $m \geq n$, we have $d_m = \Pi_{C_m} d_1 \in C_m \subset C_n$ and

$$\begin{aligned}\Delta_p(d_n, d_m) &= \Delta_p(\Pi_{C_n} d_1, d_m) \\ &\leq \Delta_p(d_1, d_m) - \Delta_p(d_1, \Pi_{C_n} d_1) \\ &= \Delta_p(d_1, d_m) - \Delta_p(d_1, d_n).\end{aligned} \tag{3.10}$$

which implies that $\Delta_p(d_1, d_m) \geq \Delta_p(d_1, d_n)$ $\ for\ all\ m \geq n$. Therefore, $\{\Delta_p(d_1, d_n)\}$ is nondecreasing and bounded and hence the limit $\lim_{n\to\infty} \Delta_p(d_1, d_n)$ exists. It follows from (3.10) that $\Delta_p(d_n, d_m) \to 0$ as $m, n \to \infty$. From Lemma 2.4 we have $\|d_n - d_m\| \to 0$ as $m, n \to \infty$. Hence $\{d_n\}$ is a Cauchy sequence. So, there exists $d^* \in E_1$ such that $d_n \to d^*$ as $n \to \infty$.

Step 4. We prove that $d^* \in \Gamma$.

Since $d_{n+1} = \Pi_{C_{n+1}} d_1 \in C_{n+1} \subset C_n$, we have

$$\begin{aligned}\Delta_p(d_n, d_{n+1}) &= \Delta_p(\Pi_{C_n} d_1, d_{n+1}) \\ &\leq \Delta_p(d_1, d_{n+1}) - \Delta_p(d_1, \Pi_{C_n} d_1) \\ &= \Delta_p(d_1, d_{n+1}) - \Delta_p(d_1, d_n).\end{aligned} \tag{3.11}$$

In addition, $\lim_{n\to\infty} \Delta_p(d_1, d_n)$ exist, hence,

$$\lim_{n\to\infty} \Delta_p(d_n, d_{n+1}) = 0. \tag{3.12}$$

So, by Lemma 2.4, we get

$$\lim_{n\to\infty} \|d_n - d_{n+1}\| = 0. \tag{3.13}$$

From definition of C_{n+1}, we have

$$\Delta_p(e_n, d_{n+1}) \leq \Delta_p(d_n, d_{n+1}) + \delta_n; \quad \Delta_p(w_n, d_{n+1}) \leq \Delta_p(d_n, d_{n+1}),$$

where $\delta_n = \nu_n sup_{\eta\in\Gamma} \zeta(\Delta_p(d_n, \eta)) + \mu_n$. By $\Delta_p(d_n, d_{n+1}) \to 0,\ \nu_n \to 0,\ \mu_n \to 0$ as $n \to \infty$, we obtain $\lim_{n\to\infty} \Delta_p(e_n, d_{n+1}) = 0$ and $\lim_{n\to\infty} \Delta_p(w_n, d_{n+1}) = 0$. From Lemma 2.4, we conclude that

$$\lim_{n\to\infty} \|e_n - d_{n+1}\| = 0 \ \ and \ \ \lim_{n\to\infty} \|w_n - d_{n+1}\| = 0, \tag{3.14}$$

and so

$$\lim_{n\to\infty}\|e_n-w_n\|=0,\quad \lim_{n\to\infty}\|w_n-d_n\|=0 \;\; and \;\; \lim_{n\to\infty}\|e_n-d_n\|=0. \quad (3.15)$$

From (3.1), we have

$$\begin{aligned}\|\mathcal{J}^p_{E_1}e_n-\mathcal{J}^p_{E_1}d_n\| &= \|\alpha_n(\mathcal{J}^p_{E_1}w_n-\mathcal{J}^p_{E_1}d_n)+(1-\alpha_n)(\mathcal{J}^p_{E_1}T^nd_n-\mathcal{J}^p_{E_1}d_n)\|\\ &\geq (1-\alpha_n)\|\mathcal{J}^p_{E_1}T^nd_n-\mathcal{J}^p_{E_1}d_n\|-\alpha_n\|\mathcal{J}^p_{E_1}w_n-\mathcal{J}^p_{E_1}d_n\|.\end{aligned}$$

This implies that

$$(1-\alpha_n)\|\mathcal{J}^p_{E_1}T^nd_n-\mathcal{J}^p_{E_1}d_n\|\leq \alpha_n\|\mathcal{J}^p_{E_1}w_n-\mathcal{J}^p_{E_1}d_n\|+\|\mathcal{J}^p_{E_1}e_n-\mathcal{J}^p_{E_1}d_n\|. \quad (3.16)$$

Since $\mathcal{J}^p_{E_1}$ is norm-to-norm uniformly continuous, so, from (3.15), (3.16) and $\{\alpha_n\}\subset[k,l]\subset(0,1)$, we have

$$\lim_{n\to\infty}\|T^nd_n-d_n\|=0. \quad (3.17)$$

Note that

$$\|T^nd_n-d^*\|\leq\|T^nd_n-d_n\|+\|d_n-d^*\|,$$

hence, we have

$$\lim_{n\to\infty}\|T^nd_n-d^*\|=0. \quad (3.18)$$

On the other hand, by the supposition that T is uniformly L-Lipschitz continuous, thus we get

$$\begin{aligned}\|T^{n+1}d_n-T^nd_n\| &\leq \|T^{n+1}d_n-T^{n+1}d_{n+1}\|+\|T^{n+1}d_{n+1}-d_{n+1}\|\\ &\quad+\|d_{n+1}-d_n\|+\|d_n-T^nd_n\|\\ &\leq (L+1)\|d_{n+1}-d_n\|+\|T^{n+1}d_{n+1}-d_{n+1}\|+\|d_n-T^nd_n\|.\end{aligned} \quad (3.19)$$

From (3.13) and (3.17), we obtain

$$\lim_{n\to\infty}\|T^{n+1}d_n-T^nd_n\|=0. \quad (3.20)$$

Further, we have

$$\|T^{n+1}d_n-d^*\|\leq\|T^{n+1}d_n-T^nd_n\|+\|T^nd_n-d^*\|.$$

By (3.18) and (2.20), we get

$$\lim_{n\to\infty}\|T^{n+1}d_n-d^*\|=0, \quad (3.21)$$

Owing to the closedness of T, it yields that $Td^*=d^*$, i.e., $d^*\in F(T)$. From (3.6), we obtain

$$(\gamma_n-\frac{C_q(\gamma_n\|G\|)^q}{q})\|Gd_n-P_QGd_n\|^p\leq\Delta_p(d_n,y)-\Delta_p(w_n,y).$$

By (3.15), we have

$$\lim_{n\to\infty} \|Gd_n - P_Q Gd_n\| = 0. \tag{3.22}$$

In addition, from (2.3), we get

$$\begin{aligned}
\|Gd^* - P_Q Gd^*\|^p &= \langle \mathcal{J}^p_{E_2}(Gd^* - P_Q Gd^*), Gd^* - P_Q Gd^* \rangle \\
&= \langle \mathcal{J}^p_{E_2}(Gd^* - P_Q Gd^*), Gd^* - Gd_n \rangle \\
&\quad + \langle \mathcal{J}^p_{E_2}(Gd^* - P_Q Gd^*), Gd_n - P_Q Gd_n \rangle \\
&\quad + \langle \mathcal{J}^p_{E_2}(Gd^* - P_Q Gd^*), P_Q Gd_n - P_Q Gd^* \rangle \\
&\le \langle \mathcal{J}^p_{E_2}(Gd^* - P_Q Gd^*), Gd^* - Gd_n \rangle \\
&\quad + \langle \mathcal{J}^p_{E_2}(Gd^* - P_Q Gd^*), Gd_n - P_Q Gd_n \rangle.
\end{aligned}$$

By (3.22) and $Gd_n \to Gd^*$ as $n \to \infty$, we have $\|Gd^* - P_Q Gd^*\|^p \to 0(as\ n \to \infty)$, this implies that $Gd^* \in Q$. Thus, we conclude that $d_n \to d^* \in \Gamma$.

Since $d_n = \Pi_{C_n} d_1$ and $\Gamma \subset C_n$, so, by (2.4), we have

$$\langle \mathcal{J}^p_{E_1} d_1 - \mathcal{J}^p_{E_1} d_n, y - d_n \rangle \le 0 \quad y \in \Gamma. \tag{3.23}$$

By setting $n \to \infty$ in (3.23), we have

$$\langle \mathcal{J}^p_{E_1} d_1 - \mathcal{J}^p_{E_1} d^*, y - d^* \rangle \le 0 \quad y \in \Gamma. \tag{3.24}$$

which implies that $d^* = \Pi_\Gamma d_1$. This completes the proof.

4 Application to Split feasibility problem and equilibrium problem

Here, we use Theorem3.1 to solve the following problem,

$$find \quad d^* \in E_1 \quad such \quad that \quad F(d^*, w) \ge 0, \quad and \quad Gd^* \in Q, \quad \forall w \in E_1, \tag{4.1}$$

where Q is a nonempty closed and convex subset of E_2, E_1 and E_2 are p-uniformly convex and p-uniformly smooth real Banach spaces, $G : E_1 \to E_2$ is a bounded linear operator, $F : E_1 \times E_1 \to R$ is a bi-function satisfying the following conditions (A1)-(A4).

(A1) $F(d, d) = 0, \forall d \in E_1$;

(A2) F is monotone, that is, $F(d, e) + F(e, d) \le 0, \forall d, e \in E_1$;

(A3) For all $d, e, w \in E_1$, $\lim_{t\downarrow 0} F(tw + (1-t)d, e) \le F(d, e)$;

(A4) For each $d \in E_1$, the function $e \longmapsto F(d, e)$ is convex and lower semicontinuous.

The resolvent mapping of F is defined as

$$Res_F(d) = \{w \in E_1 : F(w, e) + \langle e - w, \mathcal{J}^p_{E_1} w - \mathcal{J}^p_{E_1} d\rangle \geq 0, \forall e \in E_1\}.$$

$EP(F)$ is the solution set of the equilibrium problem of $F(d^*, w) \geq 0$. Then the following assertions hold [24]:

(a) Res_F is single-valued ;

(b) Res_F is a Bregman firmly nonexpansive mapping;

(c) $F(Res_F) = EP(F)$;

(d) $EP(F)$ is closed and convex.

Furthermore, we get $F(Res_F) = \widehat{F}(Res_F)$ from [22] and hence $Res_F F$ is a relatively nonexpansive mapping. So, as a consequence of Theorem 3.1, we have the following result.

Theorem 4.1 Let E_1, E_2 be two p-uniformly convex and uniformly smooth real Banach spaces. Let Q be nonempty closed and convex subsets of E_2. Let $G : E_1 \to E_2$, be a bounded linear operators and $G^* : E_2^* \to E_1^*$ be the adjoint operator of G. Let $F : E_1 \times E_1 \to R$ be bi-function satisfying the condition (A1)-(A4) and Res_F be the resolvent mapping of F. Let $d_1 \in E_1$ and $C_1 = E_1$, and $\{d_n\}$ be a sequence generated by

$$\begin{cases} w_n = \mathcal{J}^q_{E_1^*}(\mathcal{J}^p_{E_1} d_n - \gamma_n G^* \mathcal{J}^p_{E_2}(Gd_n - P_Q G d_n)) \quad n \geq 1 \\ e_n = \mathcal{J}^q_{E_1^*}[\alpha_n \mathcal{J}^p_{E_1} w_n + (1 - \alpha_n)\mathcal{J}^p_{E_1} Res_F d_n] \\ C_{n+1} = \{u \in C_n : \Delta_p(e_n, u) \leq \Delta_p(d_n, u); \Delta_p(w_n, u) \leq \Delta_p(d_n, u)\} \\ d_{n+1} = \Pi_{C_{n+1}} d_1, \end{cases} \tag{4.2}$$

where P_Q is the metric projection of E_2 onto Q, $\{\alpha_n\} \subset [k, l] \subset (0, 1)$ and $\{\gamma_n\} \subset [a, b] \subset (0, (\frac{q}{C_q\|G\|^q})^{\frac{1}{q-1}})$. If $\Gamma = \{y \in EP(F) : Gy \in Q\} \neq \emptyset$, then the sequence $\{d_n\}$ converges strongly to a point $\Pi_\Gamma d_1$.

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