

Varieties and Codes from Partial Ruled Sets

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Abstract

We construct linear codes from projective systems in a finite projective space, by considering the points of the lines of partial ruled sets.

In two complementary subspaces we choose known varieties V and V' , $V \cong V'$ via a projectivity, and study both the linear codes and the varieties arising by connecting corresponding points.

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1 Introduction

The close connection between the geometry of the projective varieties, or in general, of suitable subsets of a finite geometry and linear codes through projective systems allows, on the one hand, the elaboration and study of eventually new varieties, and, on the other, the possibility of constructing linear codes with interesting parameters for the various applications in the communication systems.

In [7] we introduced the definition of *ruled set* $\mathcal{X}$ in a finite projective space. By fixing two complementary subspaces $\mathcal{S}, \mathcal{S}'$, an m -set $\mathcal{K} \subset \mathcal{S}$ and an m' -set $\mathcal{K}' \subset \mathcal{S}'$, then $\mathcal{X}$ is defined to be the union of the lines joining the points of $\mathcal{K}$

with those of $\mathcal{K}'$. Such projective systems were used to construct linear codes also showing the technique to calculate their basic parameters and spectra.

In this paper that definition is restricted into that of *partial ruled set* and extended the construction of varieties in $PG(r, q)$ given in [1] starting from a birational correspondence between two varieties lying in two *independent spaces* (cf.[1], pp. 285–293). We fix two complementary subspaces $\mathcal{S}, \mathcal{S}'$ of $PG(r, q)$ and we choose two birationally equivalent varieties $V \subset \mathcal{S}$ and $V' \subset \mathcal{S}'$. Then consider the partial ruled set $\mathcal{V}$ arising by connecting corresponding points of V and V' , respectively.

In Theorem 1 are shown the parameters of $\mathcal{V}$ and of a code $C_{\mathcal{X}}$ related to it in the general case.

In $PG(7, q)$ the Example 1 and Proposition 2 describe the partial ruled set $\mathcal{V}$ arising from two rational normal curves of order 3 and show the parameters of a code related to it. In Proposition 5 is proved that $\mathcal{V}$ is a variety of $PG(7, q)$ of dimension 2 and order 6.

The Example 2 and Proposition 3 describe the partial ruled set $\mathcal{V}$ from two hyperbolic quadrics and calculate the parameters of a code related to it. In Proposition 6 is proved that $\mathcal{V}$ is a variety of $PG(7, q)$ of dimension 3 and order at least 4.

In $PG(9, q)$ the Example 3 and Proposition 4 describe the partial ruled set $\mathcal{V}$ arising from two *celtic varieties* (i.e., the varieties V_2^3) and show the parameters of a code related to it. In Proposition 7 is proved that $\mathcal{V}$ is a variety of $PG(9, q)$ of dimension 2 and order at least 6.

If the result given by Bertini in [1], 7.- ... *La rigata generata dalle rette congiungenti i punti di due curve in corrispondenza birazionale fra di loro ha per ordine la somma degli ordini delle due curve diminuite del numero di punti uniti (se esistono)...* (“...The ruled variety generated by the lines joining the points of two curves in birational correspondence has order the sum of the orders of the two curves reduced by the number of fixed points (if any)...”) can be extended to any pair of varieties in birational correspondence, then the orders of the varieties in Propositions 6 and 7 will be precisely 4 and 6, respectively.

2 Preliminaries

Let $F = GF(q)$ be a finite field, $q = p^s$, p prime. Denote F^n the n -dimensional vector space over F , $P^{n-1} = PrF^n = PG(n-1, q)$ the $(n-1)$ -dimensional projective space contraction of F^n over F .

A linear $[n, k]_q$ -code C of length n is a k -dimensional subspace of the vector space F^n . The dual code of C is the $(n - k)$ -dimensional subspace $C^\perp$ of F^n and it is an $[n, n - k]_q$ -code.

For $t \geq 1$ the t -th *higher weight* of C (see Wei [13]) is defined by

$$d_t = d_t(C) = \min\{\|D\| \text{ for all } D < C, \dim D = t\},$$

where $\|D\|$ is the number of indices i such that there exists $v \in D$ with $v_i \neq 0$. Note that $d_1 = d_1(C)$ is the classical minimum distance of C , the *Hamming distance*.

An $[n, k]_q$ -code C of minimum distance d is also denoted as $[n, k, d]_q$ -code.

An $[n, k]_q$ -*projective system* $\mathcal{X}$ of the projective space P^{k-1} is a collection of n not necessarily distinct points. It is called *non-degenerate* if these n points are not contained in any hyperplane.

Assume that $\mathcal{X}$ consists of n distinct points having rank k .

For each point of $\mathcal{X}$ choose a generating vector. Denote $\mathcal{M}$ the matrix having as rows such n vectors and let $C_{\mathcal{X}}$ be the linear code having $\mathcal{M}^t$ as a generator matrix. The code $C_{\mathcal{X}}$ is the k -dimensional subspace of F^n which is the image of the mapping from the dual k -dimensional space $(F^k)^*$ onto F^n that calculates every linear form over the points of $\mathcal{X}$. Hence the length n of codeword of $C_{\mathcal{X}}$ is the cardinality of $\mathcal{X}$, the dimension of $C_{\mathcal{X}}$ being just k .

The equivalences among $[n, k, d]_q$ -codes are the restrictions of the automorphisms of F^n represented by *monomial matrices*, where a monomial matrix is the product of a permutation matrix and a diagonal matrix (for the basic concepts of coding theory see for example Huffman [5]).

There exists a natural 1-1 correspondence between the equivalence classes of a non-degenerate $[n, k]_q$ -projective system $\mathcal{X}$ and a non-degenerate $[n, k]_q$ -code $C_{\mathcal{X}}$ such that if $\mathcal{X}$ is an $[n, k]_q$ -projective system and $C_{\mathcal{X}}$ is a corresponding code, then the non-zero codewords of $C_{\mathcal{X}}$ correspond to hyperplanes H of the set $\mathfrak{H}$ of P^{k-1} , up to a non-zero factor, the correspondence preserving the parameters n, k, d_t .

Hence the weight of a codeword $\mathbf{c}$ corresponding to the hyperplane $H_{\mathbf{c}}$ is the number of points of $\mathcal{X} \setminus H_{\mathbf{c}}$, thus the minimum weight d of the code $C_{\mathcal{X}}$ is $d = |\mathcal{X}| - \max\{|\mathcal{X} \cap H| \mid H \in \mathfrak{H}\}$.

If d is the minimum weight of a linear code C then C is an s -error-correcting code for all $s \leq \lfloor \frac{d-1}{2} \rfloor$. We call $t = \lfloor \frac{d-1}{2} \rfloor$ the *error-correcting capability* of C .

More generally, sub codes D of C of dimension r correspond to (projective) subspaces of codimension r of P^{k-1} . Consequently, the higher weights of C

are given by $d_t = d_t(C) = n - \max\{|\mathcal{X} \cap S| : S < P^{k-1}, \text{codim } S = t\}$. In particular, $d_1 = d_1(C) = n - \max\{|\mathcal{X} \cap H| : H < P^{k-1}, \text{codim } H = 1\}$.

The *spectrum* of a projective system $\mathcal{X}$ of P^{k-1} is defined by the numbers $A_i^{(s)} = |\{S < P^{k-1} : \text{codim } S = s, |S \cap \mathcal{X}| = n - i\}|$ for all $i = 1, 2, \dots, n$, $s = 1, 2, \dots, k - 2$.

Let $H \in \mathfrak{H}$ be a hyperplane. Then $|\mathcal{X} \cap H|$ is called an *intersection number* of $\mathcal{X}$ (with respect to hyperplanes). The set $M_{\mathcal{P}}$ of all intersection numbers of $\mathcal{X}$ is called the *type of $\mathcal{X}$ with respect to hyperplanes*. For $i \in M_{\mathcal{P}}$ let $t_i := |\{H \in \mathfrak{H} \mid |\mathcal{X} \cap H| = i\}|$ the total number of hyperplanes yielding the intersection number i .

Let $\mathcal{X}$ be a projective system of type $M_{\mathcal{P}}$. Then for $i \in M_{\mathcal{P}}$ there are t_i code words in the related code $C_{\mathcal{X}}$ of weight $|\mathcal{X}| - i$. Analogous definitions can be stated for all subspaces of $\mathfrak{T}$.

Therefore the spectrum of $\mathcal{X}$ induces the *weight distribution of the codewords* of $C_{\mathcal{X}}$.

For the above definitions see Ghorpade, Lachaud [2], Montanucci, Vincenti [9].

In $P^r = PG(r, q)$ with $r \geq 3$, denote $\mathcal{P}$ and $\mathfrak{L}$ the point set and the line set, respectively, $\mathfrak{S}$ the set of all subspaces, $\mathfrak{H}$ the set of the hyperplanes.

The *incidence hull* of a subset $X \subset \mathcal{P}$ is denoted by $\overline{X}$. Thus the joining line of two points $X, Y \in \mathcal{P}$ is $\overline{X, Y} := \overline{\{X, Y\}}$.

Denote $\overline{F}$ the algebraic closure of the field $F = GF(q)$.

The geometry $PG(r, q)$ is considered a sub-geometry of $\overline{PG}(r, q) = \overline{P}^r$, the projective geometry over $\overline{F}$. We refer to the points of P^r as the *rational points* of $\overline{P}^r$.

Choose a coordinate system in P^r so that it is a coordinate system for $\overline{P}^r$ too, denote a point $P \approx (x_0, x_1, \dots, x_r) := \overline{F}^*(x_0, x_1, \dots, x_r)$, $\overline{F}^* = \overline{F} \setminus \{0\}$.

P is a *rational point* if there exists $(x_0, x_1, \dots, x_r) \in F^{r+1}$ such that $P \approx (x_0, x_1, \dots, x_r)$.

A *variety* V_u^v of dimension u and of order v of P^r is the set of the rational points of a projective variety $\overline{V}_u^v$ of $\overline{P}^r$ defined by a finite set of polynomials of $F[x_0, \dots, x_r]$.

Two varieties V_u^v and $V_{u'}^{v'}$ are *birationally equivalent* if there exist rational bijective functions (projectivities) connecting the points of V_u^v with the points of $V_{u'}^{v'}$.

Two rational normal curves C_m and C_{r-m-1} of order m and $r-m-1$ in two complementary subspaces $\mathcal{S}_m$ and $\mathcal{S}_{r-m-1}$ of P^r are projectively equivalent.

In [1], pp. 285–291, is shown the generation of the smooth variety V_2^{r-1} , the ruled rational surface of order $r-1$ of P^r , obtained by connecting corresponding points of C_m and C_{r-m-1} , two directrices of V_2^{r-1} .

Referring to [1], p. 290, comma 7.- we make the following

Conjecture: the same rule holds for any pair of varieties in birational correspondence.

3 Partial ruled sets from varieties and their codes

In the finite projective space $P^r = PG(r, q)$ consider two complementary subspaces $\mathcal{S}, \mathcal{S}' \in \mathfrak{G}$ and two subsets $\mathcal{K} \subset \mathcal{S}$ and $\mathcal{K}' \subset \mathcal{S}'$ with $\overline{\mathcal{K}} = \mathcal{S}$, $\overline{\mathcal{K}'} = \mathcal{S}'$. Denote $m := |\mathcal{K}|$, $m' := |\mathcal{K}'|$.

Set $\mathcal{R} = \{X \in l = \overline{P, P'} \mid P \in \mathcal{K}, P' \in \mathcal{K}'\} \subset \mathfrak{L}$. Then $\mathcal{X} := \bigcup_{X \in \mathcal{R}} X$ is called a *ruled set*.

Any subset of $\mathcal{X}$ is a *partial ruled set*.

Choose as $\mathcal{K}$ and $\mathcal{K}'$ two birationally equivalent varieties V and V' , respectively, with $\overline{V} = \mathcal{S}$ and $\overline{V'} = \mathcal{S}'$. Let $m = |V| = |V'|$ be the number of the points on V (or, V'). Consider the partial ruled set $\mathcal{V}$ consisting of the m lines connecting corresponding points $P_k \in V$, $P'_k \in V'$, that is $\mathcal{V} = \{l_k = \overline{P_k, P'_k} \mid k \in GF(q)\}$. Such m lines are the *generatrices* of $\mathcal{V}$, V and V' are two *directrices*.

Denote $n = |\mathcal{V}|$ the number of the points incident the lines of $\mathcal{V}$.

Set $H_{\mathcal{S}} = H \cap \mathcal{S}$ for $H \in \mathfrak{H}$ and $s = \max\{|V \cap H_{\mathcal{S}}| \mid H \in \mathfrak{H}\}$, $v = \max\{|\mathcal{V} \cap H| \mid H \in \mathfrak{H}\}$. Denote $\mathcal{X}$ the projective system defined by $\mathcal{V}$, $\mathcal{C}_{\mathcal{X}}$ a linear code arising from $\mathcal{X}$.

Theorem 1 1) $n = |\mathcal{V}| = m(q+1)$, $v = m + qs$.

2) $\mathcal{C}_{\mathcal{X}}$ is an $[n, k]_q$ -code with $n = m(q+1)$ and $k = r+1$.

3) The minimum distance of $\mathcal{C}_{\mathcal{X}}$ is $d = (m-s)q$.

Proof. 1) Every two lines l_{k_1} and l_{k_2} are obviously skew (cf. also Kroll, Vincenti [7], Lemma 2, (1)), so that $|\mathcal{V}|$ is the number of the points on m lines, or, $|\mathcal{V}| = 2m + m(q-1) = mq + m = m(q+1)$. The maximum intersection of $\mathcal{V}$ by hyperplanes $H \in \mathfrak{H}$ occurs when H contains one of the two subspaces,

$\mathcal{S}'$, for instance, and $H \cap \mathcal{S} = H_{\mathcal{S}}$ is such that $H_{\mathcal{S}} \cap V$ is maximum. Therefore $v = \max\{|\mathcal{V} \cap H| \mid H \in \mathfrak{H}\} = m + qs$.

2) is trivial by definitions.

3) The minimum distance is $d = n - v = m(q + 1) - m - qs = (m - s)q$. $\square$.

It is clear that the minimum distance of $\mathcal{C}_{\mathcal{X}}$ depends on the *type of $\mathcal{X}$ (or, of $\mathcal{V}$) with respect to the hyperplanes*, while the spectrum of $\mathcal{C}_{\mathcal{X}}$ depends on the *type of $\mathcal{X}$ (or, of $\mathcal{V}$) with respect to the subspaces $\mathfrak{S}$* . Indeed we should know for each case which kind of varieties V and V' are chosen. That is, only by giving examples one can calculate the cardinalities of such intersections.

As far as a ruled set $\mathcal{X}$, Theorem 7 of [7] summarizes in the general case the basic parameters of a related linear code $\mathcal{C}_{\mathcal{X}}$. Note that if $\mathcal{K} = \mathcal{S}$ and $\mathcal{K}' = \mathcal{S}'$ then $\mathcal{X} = \mathcal{P}$, hence $\mathcal{C}_{\mathcal{X}} = \mathcal{C}_{\mathcal{P}}$ is the simplex code of dimension $r + 1$. In this case every code word $\mathbf{c}$ has the same weight since every hyperplane H is contained in $\mathcal{P} = \mathcal{X}$. Thus the minimal weight is $d = |\mathcal{X}| - |H| = q^r$.

Let $PG(7, q)$ be the 7-dimensional geometry over the Galois field $F = GF(q)$. Denote $\mathfrak{H}$ the set of the hyperplanes.

Example 1 - Let $\mathcal{K} = C_3$ and $\mathcal{K}' = C'_3$ be two rational cubic curves chosen and fixed in two complementary 3-dimensional subspaces $\mathcal{S}$ and $\mathcal{S}'$, respectively. Obviously $\overline{C_3} = \mathcal{S}$, $\overline{C'_3} = \mathcal{S}'$. It is $m := |C_3| = q + 1 = m' := |C'_3|$.

Define a bijective rational mapping $\phi : C_3 \rightarrow C'_3$ that maps a point $P_k \in C_3$ onto the point $P'_k = \phi P_k \in C'_3$.

Let $\mathcal{V} = \{l_k = \overline{P_k, P'_k} \mid k \in GF(q)\}$ be the partial ruled set consisting of all lines connecting corresponding points. Such lines are the generatrices, C_3 and C'_3 are two directrices.

Denote $n = |\mathcal{V}|$ the number of the points of $\mathcal{V}$, $H_{\mathcal{S}} = H \cap \mathcal{S}$ for $H \in \mathfrak{H}$ and $s = \max\{|V \cap H_{\mathcal{S}}| \mid H \in \mathfrak{H}\}$, $v = \max\{|\mathcal{V} \cap H| \mid H \in \mathfrak{H}\}$. Denote $\mathcal{X}$ the projective system defined by $\mathcal{V}$, $\mathcal{C}_{\mathcal{X}}$ a linear code arising from $\mathcal{X}$.

Proposition 2 1) $n = |\mathcal{V}| = (q + 1)^2$.

2) $s = 3$, $v = m + qs = 4q + 1$.

3) The linear code $\mathcal{C}_{\mathcal{X}}$ is an $[n, k, d]_q$ -code, when $q \geq 3$, with $n = (q + 1)^2$, $k = 8$, $d = q^2 - 2q$.

Proof. 1) As every two lines l_{k_1} and l_{k_2} are skew, it is enough to count the points on each line l_k times the number of such lines. The points of $C_3 \cup C'_3$ are $2(q + 1)$,

the lines l_k are $(q+1)$, each of them with $q-1$ points, excluded the points of $C_3 \cup C'_3$, that is $(q+1)(q-1)$ points, so that $|\mathcal{V}| = 2(q+1) + (q-1)(q+1) = 2q+2+q^2-1 = q^2+2q+1 = (q+1)^2$.

2) The maximum intersection of $\mathcal{V}$ by hyperplanes $H \in \mathfrak{H}$ occurs when H contains one of the two subspaces, $\mathcal{S}'$, for instance, and $H \cap \mathcal{S} = H_S$ is such that $H_S \cap V$ is maximum. Note that in such a case H_S is a plane. A plane meets the cubic curve C_3 in at most 3 points. Hence $s = 3$ and $v = \max\{|\mathcal{V} \cap H|\} = m + sq = (q+1) + 3q = 4q+1$.

3) The parameters n and k are clear. The minimum distance of $\mathcal{C}_{\mathcal{X}}$ is $d = n - v = (q+1)^2 - (4q+1) = q^2 + 2q + 1 - 4q - 1 = q^2 - 2q$ when $q \geq 3$. $\square$

Example 1' - In $PG(5, q)$ consider a line $\mathcal{S} = \mathcal{K} = r$, a cubic curve $\mathcal{K}' = C_3$ of a 3-dimensional complementary subspace $\mathcal{S}'$ and the variety $\mathcal{V}$ arising from the partial ruled set defined by them. It is easy to get $n = (q+1)^2$. To look for the maximum intersection by hyperplanes, if H is a hyperplane, $\dim H = 4$, contains $\mathcal{S}'$ then $H_S = H \cap r = \{P\}$ is one point, $s = 1$ and $v = m + sq = q+1+q = 2q+1$. If H contains r , then $H'_S = H \cap \mathcal{S}'$ consists of three points, $s = 3$ and $v = m + sq = q+1+3q = 4q+1 > 2q+1$. In such a case the parameters of a code related to $\mathcal{V}$, if $q \geq 3$, are $n = (q+1)^2$, $k = 6$, $d = n - v = (q+1)^2 - 4q - 1 = q^2 - 2q$.

Such a code is comparable with the code of Proposition 2 and we can say that one is better, as for the same n and minimum distance d , the dimension is greater (8 instead of 6).

The variety $\mathcal{V}$ is well known, it is a V_2^4 of $PG(5, q)$ (cf. [1], Cap.13).

Example 2 - Consider two complementary 3-dimensional subspaces $\mathcal{S}, \mathcal{S}'$ and two hyperbolic quadrics $\mathcal{H} \subset \mathcal{S}$ and $\mathcal{H}' \subset \mathcal{S}'$. It is $\overline{\mathcal{H}} = \mathcal{S}$, $\overline{\mathcal{H}'} = \mathcal{S}'$. Denote $\mathcal{R}$ and $\mathcal{R}'$ one of the two reguli ruling $\mathcal{H}$ and $\mathcal{H}'$, respectively. It is $m := |\mathcal{R}| = (q+1)^2 = m' := |\mathcal{R}'|$.

Define a bijective rational mapping $\phi : \mathcal{R} \rightarrow \mathcal{R}'$ that maps a point $P_k \in \mathcal{R}$ onto the point $P'_k = \phi P_k \in \mathcal{R}'$.

Consider the partial ruled set $\mathcal{V}$ consisting of all lines connecting corresponding points $P_k \in \mathcal{R}$, $P'_k \in \mathcal{R}'$, that is $\mathcal{V} = \{l_k = \overline{P_k, P'_k} | k \in GF(q)\}$. Such lines are the generatrices, $\mathcal{R}$ and $\mathcal{R}'$ are two directrices.

Denote $n = |\mathcal{V}|$ the number of the points in $\mathcal{V}$, $H_S = H \cap \mathcal{S}$ for $H \in \mathfrak{H}$ and $s = \max\{|V \cap H_S| \mid H \in \mathfrak{H}\}$, $v = \max\{|\mathcal{V} \cap H| \mid H \in \mathfrak{H}\}$. Denote $\mathcal{X}$ the projective system defined by $\mathcal{V}$, $\mathcal{C}_{\mathcal{X}}$ a linear code arising from $\mathcal{X}$.

Proposition 3 1) $n = |\mathcal{V}| = (q+1)^3$.

- 2) $s = q + 1$, $v = m + qs = 2q^2 + 3q + 1$.
- 3) The linear code $\mathcal{C}_{\mathcal{X}}$ is an $[n, k, d]_q$ -code with $n = (q + 1)^3$, $k = 8$, $d = q^3 + q^2$.

Proof. 1) As every two lines l_{k_1, λ_1} and l_{k_2, λ_2} are skew, it is enough to count the points on each line $l_{k, \lambda}$ times the number of the lines $l_{k, \lambda}$. The points of $\mathcal{R} \cup \mathcal{R}'$ are $2(q + 1)^2$ and there are $(q + 1)^2$ lines $l_{k, \lambda}$ each of them with $q - 1$ points, excluded the points of $\mathcal{R} \cup \mathcal{R}'$, that is $(q + 1)^2(q - 1)$ points, so that $|\mathcal{V}| = 2(q^2 + 2q + 1) + (q^2 - 1)(q + 1) = 2q^2 + 4q + 2 + q^3 + q^2 - q - 1 = q^3 + 3q^2 + 3q + 1 = (q + 1)^3$.

2) The maximum intersection of $\mathcal{V}$ by hyperplanes $H \in \mathfrak{H}$ happens when H contains one of the two subspaces, $\mathcal{S}'$, for instance, and $H \cap \mathcal{S} = H_{\mathcal{S}}$ is such that $H_{\mathcal{S}} \cap V$ is maximum. Note that in such a case $H_{\mathcal{S}}$ is a plane. A plane meets the regulus $\mathcal{R}$ in at most one line or in one conic, in any case in $q + 1$ points. Hence $s = q + 1$ and $v = \max\{|\mathcal{V} \cap H|\} = m + sq = (q + 1)^2 + (q + 1)q = 2q^2 + 3q + 1$.

3) The parameters n and k are clear. The minimum distance of $\mathcal{C}_{\mathcal{X}}$ is $d = n - v = (q + 1)^3 - (2q^2 + 3q + 1) = q^3 + q^2$. $\square$

Let $PG(9, q)$ be the 9-dimensional geometry over $F = GF(q)$. Denote $\mathfrak{H}$ the set of the hyperplanes.

Example 3 - Let $V = V_2^3$ and $V' = V_2^{3'}$ be two *celtic varieties* chosen and fixed in two complementary 4-dimensional subspaces $\mathcal{S}$ and $\mathcal{S}'$, respectively. It is $\overline{V} = \mathcal{S}$, $\overline{V'} = \mathcal{S}'$. Let $m := |V| = (q + 1)^2 = m' := |V'|$ (cf. [16], Lemma 7, and [17]).

Define a bijective rational mapping $\phi : V \rightarrow V'$ that maps a point $P_k \in V$ onto the point $\phi P_k = P'_k \in V'$.

Let $\mathcal{V} = \{\overline{l_k} = \overline{P_k, P'_k} | k \in GF(q)\}$ be the partial ruled set consisting of all lines connecting corresponding points. Such lines are the generatrices, V and V' are two directrices.

Denote $n = |\mathcal{V}|$ the number of the points of $\mathcal{V}$, $H_{\mathcal{S}} = H \cap \mathcal{S}$ for $H \in \mathfrak{H}$ and $s = \max\{|V \cap H_{\mathcal{S}}| \mid H \in \mathfrak{H}\}$, $v = \max\{|\mathcal{V} \cap H| \mid H \in \mathfrak{H}\}$. Denote $\mathcal{X}$ the projective system defined by $\mathcal{V}$, $\mathcal{C}_{\mathcal{X}}$ a linear code arising from $\mathcal{X}$.

Proposition 4 1) $n = |\mathcal{V}| = (q + 1)^3$.

2) $s = 3q + 1$, $v = m + qs = 4q^2 + 3q + 1$.

3) The linear code $\mathcal{C}_{\mathcal{X}}$ is an $[n, k, d]_q$ -code, with $n = (q + 1)^3$, $k = 10$, $d = q^3 - q^2$.

Proof. 1) Every two lines l_{k_1, λ_1} and l_{k_2, λ_2} are skew, so that it is enough to count the points on each line $l_{k, \lambda}$ times the number of the lines $l_{k, \lambda}$. The points of $V \cup V'$ are $2(q+1)^2$, the lines $l_{k, \lambda}$ are $(q+1)^2$, each of them with $q-1$ points, excluded the points of $V \cup V'$, that is $(q+1)^2(q-1)$ points, so that $n = |\mathcal{V}| = 2(q+1)^2 + (q-1)(q+1)^2 = 2q^2 + 4q + 2 + (q+1)(q^2 - 1) = q^3 + 3q^2 + 3q + 1 = (q+1)^3$.

2) The maximum intersection of $\mathcal{V}$ by hyperplanes $H \in \mathfrak{H}$ occurs when H contains one of the two subspaces, $\mathcal{S}'$, for instance, and $H \cap \mathcal{S} = H_{\mathcal{S}}$ is such that $H_{\mathcal{S}} \cap V$ is maximum. Note that in such a case $H_{\mathcal{S}}$ is a 3-dimensional subspace. A 3-dimensional subspace meets the variety V in at most two generatrix lines and the directrix line (cf. [16], Lemma 7), that is, in $3q+1$ points. Hence $s = 3q+1$ and $v = \max\{|\mathcal{V} \cap H|\} = m + sq = (q+1)^2 + (3q+1)q = 4q^2 + 3q + 1$.

3) The parameters n and k are clear. The minimum distance of $\mathcal{C}_{\mathcal{X}}$ is $d = n - v = (q+1)^3 - (4q^2 + 3q + 1) = q^3 + 3q^2 + 3q + 1 - (4q^2 + 3q + 1) = q^3 - q^2$. $\square$

Note - If in $PG(8, q)$ we choose two varieties, both with $(q+1)^2$ points, that is, a hyperbolic quadric $\mathcal{H}$ of a 3-dimensional subspace and a celtic variety V_2^3 in a complementary 4-dimensional subspace, it is easy to prove that the parameters n, s, v are the same of those of the Example 3 and Proposition 4, but $k = 9$, instead of $k = 10$ so that in such a case the related code is not as good as that one.

Further examples to study.

Let $\mathcal{S}$ and $\mathcal{S}'$ be two complementary r -subspaces in $PG(2r+1, q)$. Choose:

1) two rational normal curves, namely $\mathcal{C}_r \subset \mathcal{S}$, $\mathcal{C}'_r \subset \mathcal{S}'$. It is well known they are projectively equivalent (cf. [3], p.229).

2) two hyperbolic quadrics, namely $H \subset \mathcal{S}$, $H' \subset \mathcal{S}'$, two elliptic quadrics, namely $E \subset \mathcal{S}$, $E' \subset \mathcal{S}'$.

3) two varieties V_2^{r-1} , namely $V \subset \mathcal{S}$, $V' \subset \mathcal{S}'$, having the minimum directrix of the same order. From [1], p.292 follows that they are projectively equivalent.

4 Varieties from partial ruled set

In this section the varieties obtained in the three examples of the previous section are studied from the geometric point of view.

In $PG(7, q)$ choose a coordinate system in P^7 so that a point $P \approx (x_0, x_1, \dots, x_7) := F^*(x_0, x_1, \dots, x_7)$, $F^* = F \setminus \{0\}$.

Assume the two 3-dimensional subspaces $\mathcal{S}$ and $\mathcal{S}'$ are defined by $(x_5, x_6, x_7, x_8) = (0, 0, 0, 0)$ and $(x_0, x_1, x_3, x_4) = (0, 0, 0, 0)$, respectively.

1) Refer to the Example 1.

Without losing generality we can represent C_3 and C'_3 in $\mathcal{S}$ and in $\mathcal{S}'$, respectively, as follows:

$$C_3 = \{P_t = (1, t, t^2, t^3, 0, 0, 0, 0) | t \in GF(q)\},$$

$$C'_3 = \{P'_{t'} = (0, 0, 0, 0, 1, t', t'^2, t'^3) | t' \in GF(q)\}.$$

Because of the arbitrariness of the choice of the fundamental points on the two curves, we can assume that the corresponding points in the projectivity ϕ between C_3 and C'_3 are given by the same value of the parameter t , that is, $\phi : C_3 \rightarrow C'_3$ is defined by $\phi P_t = P'_t$ (cf. in [1], pp. 288–291, the analogous construction for the representation of a variety V_2^{r-1} of $PG(r, q)$).

The partial ruled set $\mathcal{V}$ consists of all lines (generatrices) connecting corresponding points, that is $\mathcal{V} = \{l_t = \overline{P_t, P'_t} | t \in GF(q)\}$.

The equations of a line l_t are the following:

$$x_0 = 1, x_1 = t, x_2 = t^2, x_3 = t^3, x_4 = \mu, x_5 = \mu t, x_6 = \mu t^2, x_7 = \mu t^3, \text{ for all } t, \mu \in GF(q).$$

Eliminating the two parameters, t, μ , we get the equations of the variety $\mathcal{V}$, which can be represented by the vanishing of the following matrix

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{vmatrix} = 0.$$

Proposition 5 *The variety $\mathcal{V}$ has dimension $\dim \mathcal{V} = 2$ and order $o\mathcal{V} = 6$.*

Proof. From Proposition 2 follows $|\mathcal{V}| = (q+1)^2$, therefore 2 is the dimension of $\mathcal{V}$. Moreover the order of the variety is $o\mathcal{V} = 6$, as the order of each directrix is 3 and they have no point in common (cf. [1], 7., p. 290). $\square$

2) Refer to the Example 2.

Without losing generality we can represent $\mathcal{R} = \{r_\infty, r_k | k \in GF(q)\}$ and $\mathcal{R}' = \{r'_\infty, r'_k | k \in GF(q)\}$ in $\mathcal{S}$ and $\mathcal{S}'$, respectively, with

$$r_\infty \begin{cases} x_0 = 0 \\ x_1 = 0 \end{cases} \quad r_k \begin{cases} x_2 = kx_0 \\ x_3 = kx_1 \end{cases},$$

$$r'_\infty \begin{cases} x_4 = 0 \\ x_5 = 0 \end{cases} \quad r'_k \begin{cases} x_6 = kx_4 \\ x_7 = kx_5 \end{cases}.$$

For a chosen and fixed $k \in GF(q)$, denote $\mathcal{P}_k$ and $\mathcal{P}'_k$ a point of the line r_k and of the line r'_k , respectively.

Let us define a correspondence $\phi : \mathcal{P}_k \rightarrow \mathcal{P}'_k$ that maps a point $P_k^\lambda \in \mathcal{P}_k$ onto the point $P'_k{}^\lambda \in \mathcal{P}'_k$, that is, $\phi P_k^\lambda = P'_k{}^\lambda$, with the rule

$$P_k^\lambda \approx (1, \lambda, k, k\lambda, 0, 0, 0, 0) \mapsto P'_k{}^\lambda \approx (0, 0, 0, 0, 1, \lambda, k, k\lambda).$$

Evidently ϕ is a *birational correspondence* or, a *projectivity* (cf. [1], pp. 288–291, the analogous construction for the representation of a variety V_2^{r-1} of $PG(r, q)$).

The partial ruled set $\mathcal{V}$ consists of all lines connecting corresponding points via ϕ , that is, $\mathcal{V} = \{l_{k,\lambda} = \overline{P_k^\lambda, P'_k{}^\lambda} | k, \lambda \in GF(q)\}$.

Note - Through such a projectivity to each of the $(q+1)$ conics partitionning the quadric $\mathcal{H}$ birationally corresponds one of the $(q+1)$ conics partitionning the quadric $\mathcal{H}'$. For example, the plane $\pi \subset \mathcal{S}$ of equation $x_0 = x_3$ meets $\mathcal{H}$ in the conic $\mathcal{C} = \{(k, 1, k^2, k, 0, 0, 0, 0) | k \in GF(q)\}$. To $\mathcal{C}$ corresponds the conic $\mathcal{C}' = \{(0, 0, 0, 0, k, 1, k^2, k) | k \in GF(q)\}$ of $\mathcal{H}'$. The ruled variety generated having $\mathcal{C}$ and $\mathcal{C}'$ as directrices, has $(q+1)^2$ points so that its dimension is 2 and its order 4, according [1], 7., p. 290, being $\mathcal{C} \cap \mathcal{C}' = \emptyset$.

The equations of a line $l_{k,\lambda}$ are the following:

$$x_0 = 1, x_1 = \lambda, x_2 = k, x_3 = k\lambda, x_4 = \mu, x_5 = \mu\lambda, x_6 = \mu k, x_7 = \mu k\lambda,$$

for all $\lambda, \mu \in GF(q)$. Eliminating the parameters, we get the equations of the variety $\mathcal{V}$

$$\frac{x_0}{x_1} = \frac{x_2}{x_3} = \frac{x_4}{x_5} = \frac{x_6}{x_7}, \quad \frac{x_0}{x_2} = \frac{x_1}{x_3} = \frac{x_4}{x_6} = \frac{x_5}{x_7}$$

where $x_0x_3 = x_1x_2$, $x_4x_7 = x_5x_6$ is the representation of $\mathcal{H}$ and of $\mathcal{H}'$, respectively in *internal* coordinates and appear twice. Therefore the equations of the variety $\mathcal{V}$ can be represented by the vanishing of the two matrices

$$\begin{vmatrix} x_0 & x_2 & x_4 & x_6 \\ x_1 & x_3 & x_5 & x_7 \end{vmatrix} = 0, \quad \begin{vmatrix} x_0 & x_1 & x_4 & x_5 \\ x_2 & x_3 & x_6 & x_7 \end{vmatrix} = 0.$$

Proposition 6 *The variety $\mathcal{V}$ has dimension $\dim \mathcal{V} = 3$ and order $o\mathcal{V} \geq 4$.*

Proof. From Proposition 3 follows $|\mathcal{V}| = (q+1)^3$, therefore 3 is the dimension of $\mathcal{V}$. Moreover, from the Note, one can deduced that the order of $\mathcal{V}$ is $o \geq 4$. If the conjecture of section 2 holds, then $o\mathcal{V} = 4$. $\square$

3) Refer to the Example 3.

In $PG(9, q)$ choose a coordinate system so that a point $P \approx (x_0, x_1, \dots, x_9) := F^*(x_0, x_1, \dots, x_9)$, $F^* = F \setminus \{0\}$.

Consider two complementary 4-dimensional subspaces $\mathcal{S}$ and $\mathcal{S}'$ in $PG(9, q)$ and two *celtic varieties* $V = V_2^3 \subset \mathcal{S}$, $V' = V_2^{3'} \subset \mathcal{S}'$, respectively.

Assume $\mathcal{S}$ and $\mathcal{S}'$ are defined by $(x_5, x_6, x_7, x_8, x_9) = (0, 0, 0, 0, 0)$ and $(x_0, x_1, x_3, x_4, x_5) = (0, 0, 0, 0, 0)$, respectively.

Without losing generality we can represent V and V' in $\mathcal{S}$ and in $\mathcal{S}'$, respectively, as follows:

$$V = \{P_{\lambda, \mu} = (s\lambda, s, \mu, \lambda\mu, s - \lambda^2, 0, 0, 0, 0, 0) | \lambda, \mu \in GF(q)\},$$

$$V' = \{P'_{\lambda', \mu'} = (0, 0, 0, 0, 0, s\lambda', s, \mu', \lambda'\mu', s - \lambda'^2) | \lambda', \mu' \in GF(q)\},$$

where $s \in GF(q)$ is a non-square (cf. [17], Section 3.5).

Because of the arbitrariness of the choice of the fundamental points on the two varieties, we can assume that the corresponding points in the projectivity ϕ between V and V' are given by the same value of the parameters λ, μ , that is, $\phi : V \rightarrow V'$ is defined by $\phi P_{\lambda, \mu} = P'_{\lambda, \mu}$ (cf. [1], pp. 288–291).

The partial ruled set $\mathcal{V}$ consists of all lines connecting corresponding points via ϕ , that is, $\mathcal{V} = \{l_{\lambda, \mu} = \overline{P_{\lambda, \mu}, P'_{\lambda, \mu}} | \lambda, \mu \in GF(q)\}$.

Note - From [16], Lemma 7, (d), follows that the maximum intersection of a hyperplane with a celtic variety is a singular cubic curve $\{g_1, C^2\}$ where g_1 is a line, C^2 is a conic. Through the projectivity ϕ , to each $\{g_1, C^2\}$ cut by a hyperplane of $\mathcal{S}$ on V corresponds one singular cubic curve $\{g'_1, C'^2\}$ cut by a hyperplane of $\mathcal{S}'$ on V' .

For example, the hyperplane $H \subset \mathcal{S}$ of equation $x_3 = 0$ meets V in the line $g_1 = \{(s, s, \mu, 0, s-1, 0, 0, 0, 0, 0) | \mu \in GF(q)\}$ and in the conic $C^2 = \{(s\lambda, s, 0, 0, s - \lambda^2, 0, 0, 0, 0, 0) | \lambda \in GF(q)\}$ of the plane $\pi \subset H$ having equations $x_3 = 0, x_2 = 0$. It is $g_1 \cap C^2 = \{(s, s, 0, 0, s-1, 0, 0, 0, 0, 0)\}$, $C^3 = g_1 \cup C^2$. To C^3 corresponds on V' the cubic $C'^3 = g'_1 \cup C'^2$ where $g'_1 = \{(0, 0, 0, 0, 0, s, s, \mu, 0, s-1) | \mu \in GF(q)\}$ and $C'^2 = \{(0, 0, 0, 0, 0, s\lambda, s, 0, 0, s - \lambda^2) | \lambda \in GF(q)\}$ of V' . The ruled variety generated having C^3 and C'^3 as directrices, has $2(2q+1) + (q-1)(q+1) = q^2 + 4q + 1$ points so that its dimension is 2 and its order 6, according [1], 7., p. 290, being $C^3 \cap C'^3 = \emptyset$.

The equations of a line $l_{\lambda,\mu}$ are the following

$$x_0 = s\lambda, x_1 = s, x_2 = \mu, x_3 = \lambda\mu, x_4 = s - \lambda^2, x_5 = \nu s\lambda, x_6 = \nu s, x_7 = \nu\mu, \\ x_8 = \nu\lambda\mu, x_9 = \nu(s - \lambda^2),$$

for all $\lambda, \mu, \nu \in GF(q)$. Eliminating the parameters, we get the equations of the variety $\mathcal{V}$

$$\frac{x_0}{x_1} = \frac{x_3}{x_4} = \frac{x_5}{x_6} = \frac{x_8}{x_7}, \quad \frac{x_2}{x_1} = \frac{x_3}{x_0} = \frac{x_7}{x_6} = \frac{x_8}{x_5}, \quad \frac{x_5}{x_0} = \frac{x_6}{x_1} = \frac{x_7}{x_2} = \frac{x_8}{x_3} = \frac{x_9}{x_4}.$$

The equations of $\mathcal{V}$ can be represented by the vanishing of the three matrices

$$\begin{vmatrix} x_0 & x_3 & x_5 & x_8 \\ x_1 & x_4 & x_6 & x_7 \end{vmatrix} = 0, \quad \begin{vmatrix} x_2 & x_3 & x_7 & x_8 \\ x_1 & x_0 & x_6 & x_5 \end{vmatrix} = 0, \quad \begin{vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 & x_9 \end{vmatrix} = 0.$$

Proposition 7 *The variety $\mathcal{V}$ has dimension $\dim \mathcal{V} = 3$ and order $o\mathcal{V} \geq 6$.*

Proof. From Proposition 4 follows $|\mathcal{V}| = (q+1)^3$, therefore 3 is the dimension of $\mathcal{V}$. Moreover, from the Note, one can deduced that the order of $\mathcal{V}$ is $o \geq 6$. If the conjecture of section 2 holds, then $o\mathcal{V} = 6$. $\square$

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